UNOBSERVED MECHANISM DESIGN: EQUAL PRIORITY AUCTIONS

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Abstract. We study the impact for mechanism design of the possibility that some participants are uninformed about the rules associated with a trading mechanism but otherwise rational. Since "deviations" by the mechanism designer are not observed by these uninformed participants the nature of the "equilibrium" of the design game changes, as do equilibrium mechanisms. We study the traditional independent private value auction environment and propose a method that makes it possible to characterize an interesting class of equilibrium outcomes for the game using standard reduced form direct mechanisms. We show that payoffs in the equilibrium where the seller's expected revenue is highest within this class can be characterized using a surprisingly simple mechanism called an *equal priority auction*. Informed bidders with intermediate valuations receive offers with the same probability as uninformed buyers, despite the fact the seller believes that the informed will accept the offers for sure, while uninformed buyers might not.

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1. Introduction

There is an acronym that floats around the internet - TL:DR - that explains why no one reads your email messages. It means "too long, didn't read". The long translation we adapt in this paper is "... there is undoubtedly information in your message, but it's value to me isn't likely to be as high as what I could get by reading something else". The spirit is similar to rational inattention except that we are interested the impact this has on revenue optimal trading mechanisms rather than on details of how information is acquired.

This type of inattention was noticed long ago. The marketing literature has documented buyers' tendency to ignore information when they make purchase decisions. The simplest commitment of all is a price commitment. Dickson and Sawyer (1990) asked buyers in supermarkets about their price knowledge as they were shopping. Only 50% of all respondents to their in store survey claimed to know the price of the object they had just taken off the supermarket shelf to put in their basket. Even when the item being placed in the basket had been specially marked down and heavily advertised, 25% of consumer did not even realize the good was on special.

Of course, having buyers be pleasantly surprised to learn that a price is lower than they expected isn't really a problem. The problem is the buyers who didn't know the price was on special, and went somewhere else to buy it. If prices can't influence buyer behavior, marketing has a problem.

The Dickson and Sawyer (1990) results sound somewhat behavioral. However uninformed buyers can exist when all traders are fully rational in the usual way economists understand that term. For example something as simple as having to create a user account on the seller's website is enough to create uninformed traders. There is a cost to filling out a web form and providing credit card details and other personal information. How big this cost is relative to the gains to trade on the website is hard to know. However, it is surely enough that some buyers just won't bother to create these accounts.

The seller could respond by giving up on buyers who won't create an account. However, our theorems below show that this is not as profitable for sellers as creating a trading algorithm that provides gains to trade to both types of buyers.

Creating these trading opportunities creates two problems. Both problems stem from the fact that uninformed buyers can't see what commitments the seller is making to informed account holders. In fact we'll show that equilibrium will require that the seller hide the details of the trading algorithm from buyers who don't have accounts. Since no commitments can be made to the uninformed, actual trading mechanisms will have to be ex post individually rational. This means the the seller can only make offers to the uninformed which may be rejected. What happens when there is a rejection depends on trading conventions that are exogenous to the seller. Here we assume that rejected offers lead to a failure to trade. We discuss alternatives later in the paper.

The second issue arises because the informed can mimic the uninformed but not conversely. Whatever treatment the algorithm specifies for the uninformed will create potential incentive constraints for the informed. Furthermore, since the uninformed can't see the commitments of the informed, there is a sequential rationality constraint for the seller in his treatment of the uninformed. Since the uninformed must correctly anticipate how their messages affect their treatment, this creates a potentially very complex fixed point problem.

As we are primarily interested in developing a method to characterize revenue maximizing mechanisms that we can adapt to more general problems, we focus here on equilibrium outcomes in which uninformed buyers convey no useful information to the seller before they trade.

We develop a solution concept that is a refinement of Bayesian equilibrium. In a finite environment our solution concept coincides with the seller optimal perfect Bayesian equilibrium. In continuous environments, our solution always exists while perfect Bayesian equilibrium never does.[1] In the case where uninformed buyers convey no information to sellers, we show that our solution is unique. One mechanism that implements our solution we call an *equal priority auction*. This auction treats informed buyers with intermediate valuations in exactly the same way as it treats uninformed buyers. When informed buyers have very high or very low valuations, the seller treats messages as bids. If the seller decides to sell to a buyer with one of these very high or low valuation, she will make an offer equal to the second highest bid she has received - much as she would in a standard auction.

When the highest bids of the informed are in an intermediate range, the best mechanism commits the seller to make an offer to the informed and uninformed with equal probability. The sense in which this is a commitment is that if the offer is made to the uninformed it will be rejected with positive probability, whereas an informed buyer who has submitted an intermediate bid will always accept it. Like committing to a reserve price in a standard auction, committing to an ex post unprofitable action is there to satisfy incentive constraints for informed bidders.

As a consequence, the seller's best revenue falls short of the revenues from a Myerson optimal auction by an amount that depends on how likely it is that buyers are uninformed. As the probability buyers are all informed approaches 1, revenues approach Myerson optimal revenues. If the probability is close to one that all buyers are uninformed, revenues converge to those of a fixed price trading mechanism.

Our numerical simulations suggest that even when the probability that buyers are uninformed is intermediate, trade occurs at the fixed equal priority price with high probability. The fixed price offer is triggered when the highest bid from the informed lies in a non-degenerate interval that is larger the more likely it is that bidders are uninformed. This may help to explain why auction like mechanisms are relatively uncommon for many commodities.

One well known trading platform on which auctions *are* used is eBay. The environment on eBay doesn't fit our model exactly because buyers arrive randomly. However, a seller on eBay can implement something very close to what we describe here by running an auction with a "buy it now" option. Since buy it now options disappear on eBay once a buyer submits a bid, both low and high value buyers will want to bid, while intermediate value buyers would be inclined to accept the buy it now price since they will pay it anyway in our mechanism if they win the auction.

1.1. **Heuristic.**

To understand our modeling approach it may help to make a short heuristic argument.

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[1] We thank a referee for pointing this out to us.

In an environment with uninformed buyers, the seller would really prefer that all buyers be informed since actual trades will be carried out by some kind of algorithm which can implement any kind of commitment. For the moment, assume that the informed and uninformed use different message spaces. A natural commitment in our environment would be for the seller to commit to a second price auction with optimal reserve and bids contained in the message space of the uninformed. Committing to an optimal reserve which would seemingly implement the Myerson optimal revenue. Acting on their expectations, the uninformed will bid their valuations anticipating being offered a price which is equal to the second highest bid.

Now the seller has a profitable deviation which will yield more revenue than the Myerson auction. He can change his mechanism to a first price auction in which bids must be submitted in the message space that is only known to informed bidders. The uninformed can't see this change, so they continue to bid their valuations in the message space they understand. Payoffs to the informed are unchanged by this deviation since they just alter their bids. The seller will extract the buyer's full surplus each time he chooses to trade with an uninformed bidder.

We'll show that in equilibrium seller's will always use mechanisms that require informed bidders to send messages in their own message space. The complication for sellers arises from the fact that informed buyers don't have to bid in their own message space which creates a new incentive constraint that the equilibrium mechanism must satisfy.

2. Unobserved Mechanism Design

There are n potential buyers of a single homogeneous good. Each buyer has a privately known valuation w that is independently drawn from the interval $[0,1]$. We assume that all valuations are distributed according to some distribution F with strictly positive density f. Buyer's payoff when they buy at price p is given by $w-p$. The seller's cost is zero, so the profit from selling at price p is just p.

Define

$$\pi(w) = (1-F(w))w$$

as the revenue function from a take-it-or-leave-it offer w to uninformed buyers. In what follows,[2] we restrict attention to distribution functions such that $\pi(w)$ is strictly concave. Following the standard auction literature, we also define

$$\phi(w) = w - \frac{1-F(w)}{f(w)}$$

as the virtual valuation function for informed buyers. We have $\phi(0) < 0$ and $\phi(1) = 1$, and so $\phi(w)$ crosses 0 at least once. Since $\pi'(w) = -\phi(w)f(w)$, concavity of $\pi(\cdot)$ implies that $\phi(w)$ crosses 0 only once. Let the crossing point be r^*; this is also the unique maximizer of $\pi(w)$. Furthermore, $\phi(w)$ is strictly increasing in v for $w \geq r^*$.[3] The valuation r^* represents the optimal reserve price in a standard

[2]The concavity assumption is used when we characterize the optimal equal priority auction and show that it achieves an equilibrium of our unobserved mechanism design game. It is not used in Theorem 1.

[3]At any $w \in (0,1)$, if $f(w)$ is non-decreasing, then by definition $\phi(w)$ is strictly increasing; if $f(w)$ is strictly decreasing at w and if $\phi(w) \geq 0$, then $\phi(w)$ is strictly increasing in w, because concavity of $\pi(w)$ implies that $\phi(w)f(w)$ is strictly increasing in w.

auction, regardless of the number of buyers.[4] That is, when $\alpha = 0$, the seller's outside option is always 0, so the reserve price is such that the virtual valuation of the buyer with w at the reserve price is equal to the seller's outside option.

Buyers are either *informed* or *uninformed*. We use τ_i as the "information type" of buyer i, and write $\tau_i = \epsilon$ if i is informed, and $\tau_i = \mu$ if i is uninformed. Uninformed buyers communicate with the seller using a message space $\mathcal{M}_\mu$ - assumed to be a compact metric space which embeds $[0,1]$, the set of values. Informed buyers have access to a distinct message space $\mathcal{M}_\epsilon$, also compact and metric. We'll assume that $\mathcal{M}_\epsilon$ embeds both $[0,1]$ and $\mathcal{M}_\mu$.

The most important assumption is that it is common knowledge that the seller can tell whether or not a message comes from $\mathcal{M}_\epsilon$, for example because they come from a different source. So when a buyer sends a message in $\mathcal{M}_\epsilon$, the seller knows that they are informed. If a buyer's messages comes from $\mathcal{M}_\mu$, the seller can't tell whether the buyer is informed or uninformed.

Uninformed buyers do not see the rules the seller is using to convert messages to offers. Informed buyers are fully aware of these rules. The seller and each of the buyers believes that each of the others is informed with probability $1-\alpha$ independent of their valuation.

The seller writes an algorithm that processes the messages sent by all the buyers, then chooses which buyer to make an offer to. All buyers know that if the seller makes them an offer p and they accept it, then the seller is committed to transact with them at price p.

Any offer can be rejected. We assume when an offer is rejected, the process ends without trade. We discuss this assumption later in the paper. This is quite different from standard mechanism design where a mechanism produces an allocation. This turns our problem into a game where the payoffs in the game are endogenously determined by the seller.

Let $\mathcal{M} = \mathcal{M}_\mu \cup \mathcal{M}_\epsilon$. Let $q_i : \mathcal{M}^n \to [0,1]$ be an integrable function that gives the probability with which an offer is made to bidder i for every possible profile of messages from buyers. A profile of these functions is feasible if

$$\sum_i q_i\left(b_1,\ldots,b_n\right) \leq 1$$

and $q_i\left(b\right) \geq 0$ for every profile $b \in \mathcal{M}$.

Let $\triangle\left[0,1\right]$ be a set of probability measures on the interval of values such that every bounded function is integrable. Let $P_i : \mathcal{M}^n \to \triangle\left[0,1\right]$ be an integrable function that describes the distribution of price offers buyer i receives conditional on being made an offer. If we use the notation $(P,q) = (\left\{P_i\right\}_{i=1}^n, \left\{q_i\right\}_{i=1}^n)$, then the seller's mechanism or algorithm is just a feasible pair (P,q).

A strategy rule σ_i for buyer i is a pair of integrable functions $(\sigma_i^\epsilon, \sigma_i^\mu)$ with $\sigma_i^\epsilon : [0,1] \times \Gamma \to \triangle\left(\mathcal{M}\right)$ and $\sigma_i^\mu : [0,1] \to \triangle\left(\mathcal{M}_\mu\right)$ that specifies what messages the buyer will send for each of their valuations conditional on whether the buyer

[4]In much of the auction literature, the seller has the fixed outside option of keeping the good. The virtual valuation function $\phi(w)$ is assumed to be strictly increasing to simplify the analysis (the "regular case" in Myerson (1981)). In our model, the seller's outside option in an auction with informed buyers is to give it to an uninformed buyer with a take-it-or-leave-it offer, and is endogenous. We do not need to assume that $\phi(w)$ is strictly increasing for valuations below r^*.

observes the seller's mechanism.[5] Write $\sigma = (\sigma^{\epsilon}, \sigma^{\mu}) = \left(\{\sigma_i^{\epsilon}\}_{i=1}^{n}, \{\sigma_i^{\mu}\}_{i=1}^{n}\right)$. We'll use the usual notation $(\sigma_{-i}^{\epsilon}, \sigma_{-i}^{\mu})$ to refer to the strategy rules used by the other players. Note that these strategy rules depend on the other buyers' information types which buyer i doesn't know. Note that when taking expectations, it must be taken over both profiles of the other buyers' valuations v_{-i} and profiles of their information types τ_{-i}.

Let $\mathcal{R}(\gamma, \sigma)$ be the expected revenue for the seller from mechanism $\gamma = (P, q)$ when the buyers use strategy rules given by σ. This is given by

$$\mathcal{R}(\gamma, \sigma) = \mathbb{E}_{v,\tau} \left\{ \sum_{i=1}^{n} q_i(\sigma) \int_{p_i \leq v_i} p_i \; dP_i(p_i; \sigma) \right\},$$

where the expectation is taken over profiles of buyers' valuations v and their information types τ. Note that $\mathcal{R}(\gamma, \sigma)$ depends on σ^{ϵ} only through $\sigma^{\epsilon}(\cdot, \gamma)$.

Definition. The imperfect information game $\mathcal{G}$ is defined to be the extensive form game of imperfect information in which the seller first commits to some $\gamma \in \Gamma$, then the buyers send messages to the seller that depend on γ if and only if they are informed.

The game $\mathcal{G}$ implicitly depends on the probability α with which buyers are uninformed. When this is important, we'll sometimes write $\mathcal{G}(\alpha)$, but otherwise we'll omit the α.

Our solution concept uses a refinement of Bayesian Nash equilibrium. Neither perfect Bayesian nor sequential equilibrium work in our context because sellers can use mechanisms which preclude any kind of sequential rationality. For example, the seller could deviate to a mechanism in which all informed bidders are asked to submit bids with an offer with price 0 made to the buyer who submits the highest bid which is strictly less than 1. This would be a silly deviation. Yet no matter what beliefs the players hold about each other or what strategies they play, either some buyer will have a profitable deviation, or some buyers will not be able to find best replies.[6]

In order to describe the refinement, we need the following definitions:

Definition. The *continuation game* $\mathcal{G}(\gamma, \sigma^{\mu})$ is the Bayesian game played by all the informed buyers where the seller's uses mechanism $\gamma = (P, q)$ and the uninformed buyers use strategy σ^{μ}. A profile of strategies $\zeta_i : [0, 1] \to \triangle(\mathcal{M})$ used by each informed bidder i is called a *continuation equilibrium* of $\mathcal{G}(\gamma, \sigma^{\mu})$ if for all i, $v_i \in [0, 1]$, $b' \in \mathcal{M}$,

$$\mathbb{E}_{v_{-i},\tau_{-i}} \left\{ q_i \left(\zeta_i(v_i), \zeta_{-i}(v_{-i}), \sigma_{-i}^{\mu}(v_{-i})\right) \int \max\left[v_i - p_i, 0\right] dP_i \left(p_i; \zeta_i(v_i), \zeta_{-i}(v_{-i}), \sigma_{-i}^{\mu}(v_{-i})\right) \right\}$$

$$\geq \mathbb{E}_{v_{-i},\tau_{-i}} \left\{ q_i \left(b', \zeta_{-i}(v_{-i}), \sigma_{-i}^{\mu}(v_{-i})\right) \int \max\left[v_i - p_i, 0\right] dP_i \left(p_i; b', \zeta_{-i}(v_{-i}), \sigma_{-i}^{\mu}(v_{-i})\right) \right\}.$$

Using the above continuation idea, we can give a simple definition of Bayesian equilibrium.

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[5]For save notation, we consider only pure strategies by informed and uninformed buyers. The expressions and definitions introduced below can be easily extended to mixed strategies.

[6]We thank a referee for pointing this out to us.

Definition. The mechanism $\gamma = (P, q)$ along with strategies $\{\sigma^\epsilon, \sigma^\mu\}$ constitute a *Bayesian equilibrium* for the game $\mathcal{G}$, if $\mathcal{R}(\gamma, \sigma^\epsilon, \sigma^\mu) \geq \mathcal{R}(\gamma', \sigma^\epsilon, \sigma^\mu)$ for all $\gamma' \in \Gamma$; $\sigma^\epsilon(\cdot, \gamma)$ is a continuation equilibrium for $\mathcal{G}(\gamma, \sigma^\mu)$; and for all i, $v_i \in [0,1]$, $b' \in \mathcal{M}_\mu$,

$$\mathbb{E}_{v_{-i},\tau_{-i}}\left\{q_i\left(\sigma_i^\mu(v_i), \sigma_{-i}^\epsilon(v_{-i},\gamma), \sigma_{-i}^\mu(v_{-i})\right)\int \max[v_i - p_i, 0]\, dP_i\left(p_i; \sigma_i^\mu(v_i), \sigma_{-i}^\epsilon(v_{-i},\gamma), \sigma_{-i}^\mu(v_{-i})\right)\right\}$$

$$\geq \mathbb{E}_{v_{-i},\tau_{-i}}\left\{q_i\left(b', \sigma_{-i}^\epsilon(v_{-i},\gamma), \sigma_{-i}^\mu(v_{-i})\right)\int \max[v_i - p_i, 0]\, dP_i\left(p_i; b', \sigma_{-i}^\epsilon(v_{-i},\gamma), \sigma_{-i}^\mu(v_{-i})\right)\right\}$$

As usual this isn't a very restrictive solution concept since strategy rules used by the informed don't have to be a continuation equilibrium away from the equilibrium path after γ is offered. As we can't use solution concepts that impose sequential rationality off the equilibrium path, we use the following refinement:

Definition. The triple $\{\gamma, \sigma^\epsilon, \sigma^\mu\}$ is a U-equilibrium if it is a Bayesian equilibrium and in addition there does not exist an alternative mechanism γ' and a continuation equilibrium ζ for $\mathcal{G}(\gamma', \sigma^\mu)$ such that

$$\mathcal{R}(\gamma', \zeta, \sigma^\mu) > \mathcal{R}(\gamma, \sigma^\epsilon, \sigma^\mu) \tag{2.1}$$

Since this is an unusual equilibrium concept, a few comments are in order. First, observe that the concept of a continuation equilibrium depends on fixed behavior of the uninformed. Since the uninformed don't know the mechanism that is being used off the equilibrium path, no restrictions are imposed on their behavior in a continuation equilibrium.

Second, seller deviations to alternative mechanisms as described in (2.1) are restricted to deviations for which some continuation equilibrium exists. This avoids the problem when the seller offers a mechanism for which no continuation equilibrium exists. If it were the case that the seller could only choose prices from a finite set, then we could use perfect Bayesian equilibrium as part of our solution concept since seller's choice set would be finite. In this case, the on path strategies in a U-equilibrium would always be part of a perfect Bayesian equilibrium. The sense in which our solution concept is stronger is that it selects out the seller optimal perfect Bayesian equilibrium.

2.1. **Direct mechanisms.** We do not have a full characterization of all U-equilibria. However, we can characterize a special U-equilibrium called *babbling equilibrium*, where uninformed buyers send messages that are uninformative of their valuations, that is, $\sigma_i^\mu(w) = \sigma_i^\mu(w')$ for all i and valuation pair w, w'.[7] In any U-equilibrium, the behavior of the uninformed is known and fixed, and the rest of the equilibrium can be found by finding the seller's best reply to this behavior. Since the seller can fully commit to the informed buyers we can find this best reply using the revelation principle and solving for an optimal mechanism. The definition of U-equilibrium then requires the behavior of the uninformed to be a best reply to the optimal mechanism. This is generally a difficult fixed-point problem. For babbling equilibrium, however, the problem can be solved by restricting to direct mechanisms that ignore messages from uninformed buyers.

To do so we need to add some notation to describe a symmetric direct mechanism. In what follows the notation m always means the number of uninformed buyers (i.e.,

[7] We haven't been precise enough about the space of feasible mechanisms to prove existence. We partially address this issue in Theorem 1 below.

buyers who send messages in $\mathcal{M}^\mu$). We reorder n buyers such that the first $n-m$ of them are informed; the orders among the informed and among the uninformed are arbitrary. For each $v=(v_1,\ldots,v_n)\in[0,1]^n$, and for each $i=1,\ldots,n-m$, let

$$\rho_m^i(v)=(v_i,v_2,\ldots,v_{i-1},v_1,v_{i+1},\ldots,v_{n-m},v_{n-m+1},\ldots,v_n\};$$

that is, $\rho_m^i(v)$ switches the positions of v_1 and v_i. Now we have

Definition. A symmetric direct mechanism δ is a collection of functions

$$\left\{(q_m^\epsilon,p_m^\epsilon)_{m=0}^{n-1},(q_m^\mu,p_m^\mu)_{m=1}^{n}\right\}$$

where $q_m^\tau,p_m^\tau:[0,1]^n\to[0,1]$, $\tau=\epsilon,\mu$, satisfy

- $(q_m^\tau(v),p_m^\tau(v))$, $\tau=\epsilon,\mu$, are invariant to $(v_{n-m+1},\ldots,v_n)$;
- $(q_m^\epsilon,p_m^\epsilon)$ are invariant to permutations of $(v_2,\ldots,v_{n-m})$, and (q_m^μ,p_m^μ) are invariant to permutations of $(v_1,\ldots,v_{n-m})$;
- for all v and for all m,

$$\sum_{i=1}^{n-m}q_m^\epsilon\left(\rho_m^i(v)\right)+mq_m^\mu(v)\le 1. \tag{2.2}$$

The function $q_m^\mu(v)$ gives the probability with which an offer $p_m^\mu(v)$ is made to an uninformed buyer given that there are m uninformed buyers and the profile of valuations is $v=\{v_1,\ldots,v_n\}$. The function $q_m^\epsilon(v)$ gives the probability with which an offer $p_m^\epsilon(v)$ is made to buyer 1 given that there are m uninformed buyers and the valuation profile of buyers $i=2,\ldots,n$ is $v_{-1}=\{v_2,\ldots,v_n\}$.

Since uninformed buyers babble, we require the allocation and the offer functions of both the informed and the uninformed to be independent of the valuations of the latter. Symmetry requires the allocation and the offer functions of uninformed buyers to be invariant to permutations of the valuation profile of the informed, and the allocation and the offer functions of each informed buyer to be invariant to permutations of the valuation profile of the other informed buyers.

Since $\rho_m^i(v)$ switches the positions of the first element of v and its i-th element, the sum $\sum_{i=1}^{n-m}q_m^\epsilon\left(\rho_m^i(v)\right)$ gives the probability that the offer is made to one of the first $n-m$ elements of v. Then (2.2) ensures that when the informed buyers have valuations given by the first $n-m$ valuations in v, the probability with which the good is offered to one of them plus the probability that it is offered to one of the uninformed buyers is less than or equal to 1.

We can use the above definitions to build something that looks exactly like a traditional reduced form mechanism. The probability with which an informed buyer whose valuation is w receives an offer when there are m uninformed is

$$Q_m^\epsilon(w)=\mathbb{E}_v\left\{q_m^\epsilon(v)|v_1=w\right\}.$$

Similarly

$$P_m^\epsilon(w)=\mathbb{E}_v\left\{q_m^\epsilon(v)p_m^\epsilon(v)|v_1=w\right\}$$

is the expected price the informed bidder with valuation w would pay. Note that we have assumed that in any direct mechanism an informed buyer accepts the offer he receives with probability one. The is no max operator for informed buyers. This assumption is justified because informed buyers know the mechanism.

For each $m = 0, \dots, n-1$, let $B(m; n-1, \alpha)$ be the probability that there are m uninformed buyers among the $n-1$ others. This probability is given by

$$B(m; n-1, \alpha) = \binom{n-1}{m} (1-\alpha)^{n-1-m} \alpha^m.$$

Now by taking expectations over m we have the usual reduced form functions:

$$Q^\epsilon(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q_m^\epsilon(w),$$
$$P^\epsilon(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) P_m^\epsilon(w).$$

We then have

$$U^\epsilon(w) = wQ^\epsilon(w) - P^\epsilon(w).$$

At this point, we inherit all the usual results from mechanism design in iid environments for each of the informed buyers. In particular, if the mechanism δ is incentive compatible *with respect to valuations*, the payoff to an informed buyer with valuation w can be written as

$$U^\epsilon(w) = \int_0^w Q^\epsilon(x)\, dx, \tag{2.3}$$

with $Q^\epsilon(\cdot)$ non-decreasing.[8]

The (interim) payoff to an uninformed bidder with valuation w is

$$U^\mu(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v \left\{ q_{m+1}^\mu(v) \max\left[w - p_{m+1}^\mu(v), 0 \right] \right\}.$$

Definition. The mechanism δ is incentive compatible for informed buyers if (2.3) holds, $Q^\epsilon(\cdot)$ is non-decreasing and

$$U^\epsilon(w) \geq U^\mu(w)$$

for every w.

From standard arguments and properties of the binomial distribution, it is straightforward to show that the seller's revenue from any incentive compatible direct mechanism δ is given by

$$R(\delta) = n(1-\alpha) \int_0^1 Q^\epsilon(w)\, \phi(w) f(w) dw + \sum_{m=1}^{n} B(m; n, \alpha) \mathbb{E}_v \left\{ m q_m^\mu(v) \pi\left(p_m^\mu(v)\right) \right\},$$

where the first term is the revenue from informed buyers, and the second term is the revenue from the uninformed buyers. The following result provides a two-way relationship between the optimal direct mechanism and a U-equilibrium of the unobserved mechanism design game with babbling by uninformed buyers.

[8]See, for example, Myerson (1981). We have assumed $U^\epsilon(0) = 0$ for simplicity. This is usually not part of requirement for incentive compatibility, but clearly necessary for any revenue maximizing direct mechanism.

Theorem 1. *Fix a game of unobserved mechanisms $\mathcal{G}(\alpha)$. For any babbling equilibrium (γ, σ), there is an incentive compatible and symmetric direct mechanism δ^*, with $R(\delta^*) = \mathcal{R}(\gamma, \sigma)$ and $R(\delta^*) \geq R(\delta)$ for every incentive compatible direct mechanism δ. Conversely, any incentive compatible and symmetric direct mechanism δ^* that maximizes $R(\delta)$ can be used to construct a babbling equilibrium (γ, σ) such that $\mathcal{R}(\gamma, \sigma) = R(\delta^*)$.*

The proof of this is relatively straightforward. We provide a sketch of the argument here. Fix a babbling equilibrium (γ, σ) in the game $\mathcal{G}(\alpha)$ with message spaces $\mathcal{M}_\epsilon$ and $\mathcal{M}_\mu$. The continuation equilibrium $\sigma(\cdot, \gamma)$ in the game (γ, σ^μ) on the equilibrium path is just an equilibrium of a standard Bayesian game among the informed buyers. The seller doesn't actually care what the uninformed buyers say in an equilibrium in which they don't convey information about their types - all he needs to keep track of is whether or not a buyer's message was in $\mathcal{M}_\mu$. So it doesn't matter here whether uninformed buyers use asymmetric strategies. By the standard revelation principle, there is an incentive compatible direct mechanism δ in which informed buyers report their information type and valuations truthfully, and gives the same expected revenue as γ. This direct mechanism δ might not be symmetric. However, it is well known that in the symmetric, independent private values environment, an asymmetric mechanism can't produce a higher expected revenue than a symmetric one. The definition of U-equilibrium allows the seller to choose the continuation equilibrium for the fixed strategy of uninformed buyers σ^μ. This means that there is a symmetric incentive compatible mechanism δ^* that achieves the equilibrium revenue $\mathcal{R}(\gamma, \sigma^\epsilon, \sigma^\mu)$ and is an optimal incentive compatible mechanism with respect to informed buyers.

The reverse direction follows by construction. Fix any message $b_\mu \in \mathcal{M}_\mu$. Let $\sigma_i^\mu(v_i) = b_\mu$ for all i and all $v_i \in [0,1]$. By assumption $\mathcal{M}_\epsilon$ embeds $[0,1]$ so we can find a subset of $\mathcal{M}_\epsilon$ and a bijection B_ϵ between this subset and $[0,1]$. For each i and $v_i \in [0,1]$, let $\sigma_i^\epsilon(v_i, \delta^*) = B_\epsilon(v_i)$, and $\sigma_i^\epsilon(v_i, \gamma) = b_\mu$ for all $\gamma \neq \delta^*$. Then, $(\beta^*, \sigma^\epsilon, \sigma^\mu)$ is a U-equilibrium of $\mathcal{G}(\alpha)$.

3. Equal-priority auctions

Our main result is that for distributions such that $\pi(\cdot)$ is concave, the outcome of a symmetric equilibrium of the game $\mathcal{G}(\alpha)$ where uninformed buyers babble corresponds to a revenue maximizing "equal priority auction." We'll show this in two parts. First we'll describe the set of equal priority auctions and describe one that gives the seller the highest expected revenue. Later we'll show how to verify this is the best for the seller among all direct mechanisms.

An equal priority auction is fully characterized by four numbers, a "reserve price" r, a price offer t, and the upper and lower bound v_+ and v_- of an interval of buyer types. We'll assume throughout that $r \leq t \leq v_- \leq v_+$.

In what follows, there is some message that is treated as if the buyer who sent that message is uninformed. Each of the informed buyers sends a bid. A realized profile of messages and bids will then have m messages saying uninformed, and $n - m$ bids. Denote the number of informed buyers who bid in the interval $[v_-, v_+]$ as k. The auction treats the k informed buyers and m uninformed buyers with the same allocation priority. Priorities of informed buyers who bid above v_+ and who bid below v_- are equal to their bids, with the former all higher than the $(m + k)$

buyers and the latter all lower than them. The allocation and offers in an equal priority auction are determined in the following way:

- If $m \geq 1$ and the highest bid received from the informed bidders is no larger than v_+, then the seller makes an offer t to each uninformed bidder and an offer v_- to each informed bidder who bid in the interval $[v_-, v_+]$ with probability $1/(m+k)$.
- Otherwise, the seller makes an offer to the informed buyer who made the highest bid. Let v' be the second highest bid by an informed buyer. The offer to the high bidder is

$$\begin{cases} v' & v' > v_+ \\ r & m = 0; v' < r \\ v' & m = 0; v' \in (r, v_-) \\ \frac{v_- + (m+k)v_+}{m+k+1} & \text{otherwise.} \end{cases}$$

These rules constitute an indirect mechanism and support some kind of Bayesian equilibrium in bidding strategies. Our main theorem is going to say that conditional on uninformative messages from the uninformed bidders, the revenue maximizing mechanism is going to be a special kind equal priority auction. To see what that means, and to understand how to find the optimal one, one bit of notation is required. Suppose for the moment, potentially counter factually, that informed buyers bid their true valuations Then using the allocation rule in the indirect mechanism, we can calculate the probability with which each type of informed buyer trades. This probability of trade function Q^ϵ for an informed buyer is given by

(3.1)

$$\begin{cases} 0 & \text{if} \quad w < r \\ (1-\alpha)^{n-1} F^{n-1}(w) & \text{if} \quad w \in [r, v_-) \\ \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+)/(m+k+1) & \end{cases}$$

where

$$B_k^{n-1-m}(v_-, v_+) = \binom{n-1-m}{k} (F(v_+) - F(v_-))^k F^{n-1-m-k}(v_-).$$

For informed buyers with valuation w between r and v_-, $Q^\epsilon(w)$ is such that trade occurs only when there are no uninformed buyers who have a higher priority. For $w > v_+$, we have

$$Q^\epsilon(w) = ((1-\alpha) F(w) + \alpha)^{n-1},$$

so informed buyers with valuation w above v_+ have a higher priority than uninformed buyers.

For convenience, we denote $Q^\epsilon(w)$ for $w \in [v_-, v_+]$ as $\chi(v_-, v_+)$. To provide a convenient formula, we re-do the double summations over m and k by first summing over k for fixed $l = m+k$, and then summing over l. We can rewrite $\chi(v_-, v_+)$ as

$$\sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \frac{1}{l+1} \sum_{k=0}^{l} \binom{l}{k} ((1-\alpha)(F(v_+) - F(v_-)))^k \alpha^{l-k}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \frac{1}{l+1} ((1-\alpha)(F(v_+) - F(v_-)) + \alpha)^l.$$

Thus,

$$\chi(v_-, v_+) = \frac{((1-\alpha)F(v_+) + \alpha)^n - ((1-\alpha)F(v_-))^n}{n((1-\alpha)(F(v_+) - F(v_-)) + \alpha)}. \tag{3.2}$$

The function χ gives the probability that a buyer whose valuation is in the pooling interval $[v_-, v_+]$ receives an offer. The logic in $\chi(v_-, v_+)$ is that an informed bidder has the same chance of receiving an offer as any of the uninformed buyers and informed buyers whose valuations are in the interval $[v_-, v_+]$ as long as none of the other informed bidders has valuation above v_+. This explains why in the formula (3.2) the denominator is the expected number of buyers who have the equal priority, and the numerator is the total probability that there is one with the priority.

The trading probability $Q^\epsilon(w)$ of an informed buyer with valuation w is weakly increasing. It is continuous except at three valuations. It jumps up at $w = r$ from 0 to $Q^\epsilon(r)$. Another upward jump occurs at $w = v_-$:

$$\chi(v_-, v_+) > B(0; n-1, \alpha) B_0^{n-1}(v_-, v_+) = (1-\alpha)^{n-1} F^{n-1}(v_-).$$

It jumps up for the third time at $w = v_+$:

$$\chi(v_-, v_+) < \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+) = ((1-\alpha)F(v_+) + \alpha)^{n-1}.$$

Then mimicking direct mechanisms we could define an expected payoff $U^\epsilon(w)$ to an informed buyer as follows:

$$U^\epsilon(w) = \int_0^w Q^\epsilon(x) dx. \tag{3.3}$$

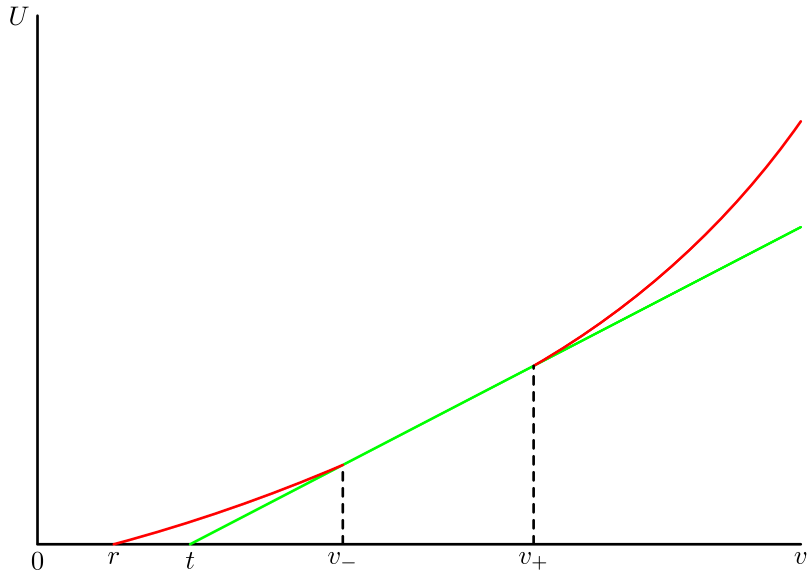
We have the following result.

Lemma 2. *There is a Bayesian equilibrium in truthful bidding strategies if*

$$\int_r^{v_-} (1-\alpha)^{n-1} F^{n-1}(w) dw \geq \chi(v_-, v_+)(v_- - t) \tag{3.4}$$

Two arguments are needed. The first is to show that the transfers defined above together with the allocation rule are the ones that make truthful bidding incentive compatible by informed buyers. Note when informed buyers bid their valuations truthfully, they accept their offers with probability one. Since the allocation rule is monotone, we accomplish the first step by showing that the payoff of informed buyers from truthful bidding matches the payoff defined by (3.3) and (3.1) (Myerson,1981). The second is to show that when t satisfies condition (3.4) no informed buyer can improve their payoff by pretending to be uninformed. This just follows from the observation that the right hand side of (3.4) is the expected payoff for an informed buyer with valuation v_- pretending to be uninformed. By construction, uninformed buyers have the same allocation priority as informed buyers whose valuations are in $[v_-, v_+]$. The expected payoff of informed buyers given by (3.3) and (3.1) is strictly convex between r and v_- and above v_+. Thus, the incentive condition for informed buyers not to pretend to be uninformed is satisfied if and only if it holds for an informed buyer with valuation v_-.

The following figure shows the Bayesian equilibrium payoffs to bidders with various valuations in an equal priority auction with a binding incentive compatibility constraint (3.4). The green line represents the payoff each buyer type achieves by acting as an uninformed bidder. The red curve represents the payoff to informed bidders - except that the payoff to informed bidders who bid in the interval $[v_-, v_+]$ coincides with the green line.



In an equal priority auction with a binding incentive compatibility constraint (3.4), it is a matter of indifference for informed buyers with valuations in $[v_-, v_+]$ whether they participate in the auction by truthfully reporting their valuations, or wait for the take-it-or-leave-it offer t just like an uninformed buyer. Indeed, the same truth telling equilibrium among informed buyers is implemented if we change the transfer rule, so that an informed buyer with valuations in the pooling interval $[v_-, v_+]$ receives the offer t, instead of the maximum of the second highest bid and reserve price r when there are no other buyers in the equal priority pool, or v_- when there is at least one buyer in the pool. Informed buyers with low valuations, between r and v_-, and those with high valuations, above v_+, have strict incentives to participate in the auction.

3.1. **Revenue Maximizing Equal Priority Auction.** This already looks like a direct mechanism, albeit one with very specific allocation rules. The seller's expected revenue from informed buyers is given by

$$n(1-\alpha)\int_r^1 Q^\epsilon(w)\phi(w)f(w)dw, \tag{3.5}$$

and the revenue from uninformed buyers is given by

$$\sum_{m=1}^{n} B(m;n,\alpha)\sum_{k=0}^{n-m} B_k^{n-m}(v_-, v_+)\frac{m}{m+k}\pi(t) = n\alpha\chi(v_-, v_+)\pi(t). \tag{3.6}$$

The revenue maximizing equal-priority auction $\{r, t, v_-, v_+\}$ maximizes the sum of (3.5) and (3.6) subject to

$$r \le t \le v_- \le v_+;$$

and (3.4). The following lemma characterizes optimal equal-priority auctions.

Lemma 3. *If (r, t, v_-, v_+) is an optimal equal-priority auction, then*

$$0 < r < r^* < t < v_- < v_+ < 1.$$

Further, (3.4) holds with equality, and

$$\alpha(\pi(t)-\phi(v_+)) = (1-\alpha)\left((v_- - t)(\phi(v_+) - \phi(v_-))f(v_-) + \int_{v_-}^{v_+} f(w)(\phi(v_+) - \phi(w))dw\right); \tag{3.7}$$

$$-\alpha\pi'(t) = (1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-); \tag{3.8}$$

$$-\phi(r)f(r) = (\phi(v_+) - \phi(v_-))f(v_-). \tag{3.9}$$

The three conditions (3.7), (3.8) and (3.9) are just the first order conditions for an interior optimum. To establish that the optimal auction is indeed interior, satisfying $0 < r < t < v_- < v_+ < 1$, our proof (in the appendix) uses a variational argument.

In a revenue maximizing equal priority auction, the reserve price r for selling to informed buyers with low valuations (below v_-) is set below the standard optimal reserve price r^* in the absence of uninformed buyers, as can be seen from (3.9). This sacrifices revenue when all informed buyers have low valuations and there are no uninformed buyers, but provides incentives for informed buyers whose valuations are low but close to v_- to participate in the auction instead of pretending to be uninformed. Correspondingly, (3.8) implies that the take-it-or-leave-it price t to uninformed buyers is raised above the optimal monopoly price r^* in the absence of informed buyers. This reduces the revenue when all buyers are uninformed, but provides disincentive for informed buyers to pretend to be uninformed.

If the seller does not give the good to an informed buyer, he can always make a take-it-or-leave-it offer to an uninformed buyer if there is one. Absent of incentives, the seller would set the reserve price $\overline{r}(t)$ for informed buyers so that the virtual valuation is equal to the expected profit $\pi(t)$ of making the offer t to an uninformed buyer:

$$\phi(\overline{r}(t)) = \pi(t).$$

However, by condition (3.7), the optimal equal priority auction has $\phi(v_+) < \pi(t)$. This means that the seller gives the good to informed buyers even though their virtual valuations are lower than the value of the seller's "outside option" $\pi(t)$. This reason for doing this is to provide incentives for informed buyers with valuations just above v_+ to participate in the auction rather than wait for the take-it-or-leave-it offer.

The interval $[v_-, v_+]$ is non-degenerate as long as uninformed buyers are present in the model, i.e., $\alpha > 0$. Briefly if the interval is degenerate, the seller can raise expected revenue by cutting the price t that he offers to the uninformed. The downside is that he loses revenue from the informed who are pooled together with the uninformed. A variational argument can be used more generally to show that the cutting the price offer to the uninformed has a first order impact on profits, while the loss from the informed is second order.

When all bidders are surely informed the revenue from the optimal equal priority auction converges to the revenue from the standard auction with reserve price r^*, as it becomes optimal for the seller not to distort the reserve price r at all to provide incentives (equation 3.9). The pooling 3.8 participating in the auction and receiving a take-it-or-leave-it offer t_0 when all other buyers have valuations below v_0,[9]

$$\int_{r^*}^{v_0} F^{n-1}(w)dw = F^{n-1}(v_0)(v_0 - t).$$

The limit values of v_0 and t_0 satisfy the above indifference condition and the limit version of first order conditions (3.7) and (3.8), given by

$$\pi'(t_0)(v_0 - t) + \pi(t_0) - \phi(v_0) = 0.$$

We have $t_0 > r^*$ and $\pi(t_0) > \phi(v_0)$. When α is arbitrarily close to 0, the incentives for informed buyers not to pretend to be uninformed are provided by raising the take-it-leave-it offer to an unlikely uninformed buyer above r^*, and not selling to uninformed buyers even when the profit from doing so exceeds virtual valuations of informed buyers.

In the opposite limit of $\alpha = 1$, bidders are surely uninformed, and the revenue from the optimal equal priority auction converges to the revenue from a take-it-or-leave-it offer r^*. By (3.8), the seller no longer distorts t to provide incentives for informed buyers. From (3.7), the upper-bound of the pooling interval converges to $\overline{r}(r^*)$, satisfying

$$\phi(\overline{r}(r^*)) = \pi(r^*),$$

as the need for the seller to provide incentives for informed buyers with valuations just above the upper-bound becomes second order. From the binding constraint (3.4), the lower-bound of the pooling interval becomes r^*.[10] This is to prevent an unlikely informed buyer with a valuation equal to the lower bound from pretending to be uninformed, as the buyer has almost zero chance of winning the auction with the limit reserve price r_1 satisfying (3.9)

$$-\phi(r_1)f(r_1) = \pi(r^*)f(r^*).$$

As long as α is strictly less than 1, however, the auction is what provides incentives for informed buyers with valuations just below the lower bound of the interval not to pretend to be uninformed.

3.2. **Equilibrium mechanisms.** We use Lagrangian relaxation to show that an optimal equal-priority auction provides the seller the highest expected revenue among all direct mechanisms.

Recall that a direct mechanism δ consists of a series of functions $(q^{\epsilon}_m, p^{\epsilon}_m)_{m=0}^{n-1}$ and $(q^{\mu}_m, p^{\mu}_m)_{m=1}^{n}$. We first use the assumption that $\pi(\cdot)$ is strictly concave to simplify the optimal design problem. We show that replacing offers $p^{\mu}_m(v)$ to uninformed buyers with a single offer reduces the deviation payoff for informed from pretending to be uninformed, and improves the seller's revenue from uninformed buyers due to concavity of $\pi(\cdot)$.

Lemma 4. *If $\pi(\cdot)$ is strictly concave, then in any optimal direct mechanism, $p^{\mu}_m(v)$ is independent of m and v.*

[9] The limit of $\chi(v_-, v_+)$ as α goes to 0 and v_- and v_+ shrink to the same point of v_0 is $F^{n-1}(v_0)$. That is, when all other bidders are almost surely informed, a deviating informed bidder will be the only buyer in the equal priority pool and will get the good with probability one if all other bidders (who are informed) have valuation below v_0.

[10] The limit of $\chi(v_-, v_+)$ as α goes to 1 is $1/n$, as an unlikely informed buyer will surely face $n - 1$ uninformed buyers in the equal priority pool after pretending to be uninformed.

Using Lemma 4, we denote the constant price offered to the uninformed as p^μ. Define

$$Q^\mu = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v \left\{ q^\mu_{m+1}(v) \right\}$$

to be the total probability of an offer expected by an uninformed buyer (or a deviating informed bidder). As we have shown in Theorem 1 the revenue maximizing direct mechanism can be found by choosing a feasible mechanism δ that supports a trading probability for the uninformed Q^μ and a non-decreasing trading probability function $Q^\epsilon(\cdot)$ that maximizes

$$n(1-\alpha) \int_0^1 \left\{ Q^\epsilon(w)\, \phi(w) f(w) dw \right\} + n\alpha Q^\mu \pi(p^\mu)$$

subject to

$$\int_0^w Q^\epsilon(x)\, dx \geq Q^\mu \max\left[w - p^\mu, 0\right] \tag{3.10}$$

for all w.

Let $\lambda(\cdot)$ be an arbitrary non-negative Lagrangian function from $[0,1]$ into $\mathbb{R}$. The relaxed problem is to maximize

$$n(1-\alpha) \int_0^1 \left\{ Q^\epsilon(w)\, \phi(w) f(w) dw \right\} + n\alpha Q^\mu \pi(p^\mu)$$
$$+ \int_0^1 \lambda(w) \left\{ \int_0^w Q^\epsilon(x)\, dx - Q^\mu \max\left[w - p^\mu, 0\right] \right\} dw,$$

again by choosing $(q^\epsilon_m, p^\epsilon_m)_{m=0}^{n-1}$, $(q^\mu_m)_{m=1}^{n}$, and p^μ such that the feasibility constraint (2.2) is satisfied, and $Q^\epsilon(\cdot)$ is non-decreasing.

The above problem can have different solutions depending on the choice of $\lambda(\cdot)$. It is well known that the solution to the relaxed problem is an upper bound on the solution to the full problem no matter what the Lagrangian function.[11] The method of proof is to try to find a function $\lambda(\cdot)$ such that the solution to the relaxed problem is an equal priority auction. Since the equal priority auction yields an upper bound on the seller's payoff in the full problem, and since it satisfies all the constraints in the full problem, it must be a solution to the full problem.

To see how we came up with the multiplier function $\lambda(\cdot)$, use integration by parts and rewrite the Lagrangian as

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha) \int_0^1 \left\{ n(1-\alpha)\phi(w) f(w) + \int_w^1 \lambda(x) dx \right\} Q^\epsilon_m(w) dw$$
$$+ \sum_{m=0}^{n-1} B(m; n-1, \alpha) \left(n\alpha\pi(p^\mu) - \int_0^1 \lambda(w) \max[w - p^\mu, 0] dw \right) Q^\mu_{m+1}.$$

We want to choose $\lambda(\cdot)$ to have the following properties: (i) It takes value of 0 outside of $[v_-, v_+]$ so that the constraint (3.10) is slack. (ii) It takes non-negative values on $[v_-, v_+]$ such that the value of the expression in the first bracket in the

[11]The solution to the original problem is feasible, so the integral in the payoff to the relaxed problem is non-negative. In turn, the solution to the original problem gives a lower payoff in the relaxed problem than the solution to the relaxed problem itself.

above Lagrangian is constant, so that it is point wise maximizing to have constant $Q_m^\epsilon(w)$ for all $w \in [v_-, v_+]$. (iv) The constant value of the expression in the first bracket in the above Lagrangian matches the constant value of the expression in the second bracket, so that it is point wise maximizing to treat informed buyers with valuations in the pooling interval the same as uninformed buyers in terms of allocation. (iv) The value of the expression in the first bracket is greater than that in the second bracket for $w > v_+$ and smaller for $w < v_-$, so that informed buyers have higher priorities than uninformed buyers if their valuations are higher than v_+ and lower priorities if their valuations are lower than v_-.

Theorem 5. *Suppose that $\pi(\cdot)$ is strictly concave. Then, a revenue maximizing equal priority auction is a revenue maximizing direct mechanism.*

Putting together Theorems 5 and 1, we have shown that when $\pi(\cdot)$ is concave, the outcome of a symmetric equilibrium of the game $\mathcal{G}(\alpha)$ where uninformed buyers babble corresponds to an optimal equal priority auction. Conversely, once we solve for the revenue maximizing equal priority auction, we can construct a password mechanism to support a symmetric equilibrium of the game. Since equal priority auctions are relatively straightforward to describe and optimize over, we believe our result provides a simple characterization of equilibrium outcomes of the unobserved mechanism design game in the important class of uncommunicative messaging by uninformed buyers.

The relative simplicity of optimal equal priority auctions also allows us to understand welfare implications of unobserved mechanism design. The seller is of course worse off compared to when all buyers are informed, as unobservability reduces the power of commitment necessary for standard optimal auctions. This means that the seller has incentives to "educate" buyers about the mechanism being offered. But such attempt would be thwarted so long as the commitments in the mechanism remain unverifiable.

When all n buyers are informed, they face the standard optimal reserve price of r^*. In a symmetric uncommunicative equilibrium of the unobserved mechanism design game $\mathcal{G}(\alpha)$, the seller sets $r < r^*$, so an informed buyer with a valuation between r and r^* is better off than when there are no uninformed buyers around. Informed buyers with higher valuations are affected by the presence of uninformed and uncommunicative buyers in two opposing ways: they can win the auction even though some uninformed buyer has a higher valuation, but they may also lose to an uninformed with a lower valuation. The net effect is generally ambiguous, but we can show that informed buyers with sufficiently high valuations benefit from having uninformed buyers around if the number of buyers is sufficiently large.[12]

For uninformed buyers, the relevant welfare comparison question is how they are affected by the presence of informed buyers. If there are no informed buyers,

[12] To see this, note that

$$U^\epsilon(1) = \int_r^1 Q^\epsilon(w)dw > \int_{v_+}^1 ((1-\alpha)F(w)+\alpha)^{n-1}dw.$$

The above is greater than $\int_{r^*}^1 F^{n-1}(w)dw$ when n is sufficiently large, because by integration by parts, it is implied by

$$(1-\alpha)\int_{v_+}^1 ((1-\alpha)F(w)+\alpha)^{n-2}f(w)wdw < \int_{r^*}^1 F^{n-2}(w)f(w)wdw,$$

which is true for large enough n by using another integration by parts.

uninformed buyers have an equal chance of receiving a take-it-or-leave-it offer equal to r^*. Since in a symmetric uncommunicative equilibrium of $\mathcal{G}(\alpha)$ the seller sets the take-it-or-leave-it offer t strictly above r^*, an uninformed buyer with a valuation w just above r^* is worse off in equilibrium than when there are no informed buyers around. For uninformed buyers with higher valuations, they have a higher priority than informed buyers with valuations below v_-, which makes them better off in equilibrium, but lose out to informed buyers with valuations above v_+. The net effect is again ambiguous, but we can show that uninformed buyers are all worse off in equilibrium than when there are no informed buyers if the number of buyers is sufficiently large.[13]

4. Different Message spaces

So far we assumed that informed and uninformed have access to different message spaces. In equilibrium the seller requires the informed to send messages that the uninformed do not know how to send. This reduces notation and makes it easier to explain the building blocks like $U-$equilibrium. However this separation of message spaces isn't necessary. Buyers 'learn' the mechanism in one of two ways: either they pay a cost to learn which messages lead to outcomes they want; or they guess (for free) which messages work.

As we argued heuristically in the introduction, the seller will deliberately prevent those who are guessing from getting it right. It isn't that the seller doesn't want them to know how to bid, it is that he can't help exploiting them when they do.

One way to support this is to have the seller randomize over mechanisms. Those who pay the cost of seeing the mechanism learn the result of the randomization and so learn the seller's commitments.

Generally, mixed strategy equilibrium in mechanisms could allow the uninformed to correctly guess some properties of the seller's mechanism. For example, the seller might randomize with equal probability between 2 reserve prices in an auction. The uninformed might send informative message in such an equilibrium. If they did, then apart from optimality conditions, the seller would have to be indifferent between which of the two reserve prices he uses. More generally, creates a complicated fixed point problem.

In this paper we focus on equilibria in which uninformed buyers do not communicate at all. This can be supported with a single message space for both types if the seller's randomization (over mechanisms) induces the uninformed to believe that there is no relationship between their messages and their trading probability.

Suppose that the uninformed do know how to bid in $\mathcal{M}_\epsilon$. This can be accomplished by using something called a *password mechanism.* The seller chooses any pure mechanism with a message space which is a subset of $\mathcal{M}_\epsilon$ then requires that each such message be submitted along with a number chosen randomly from $[0,1]$ (which must also be embedded in $\mathcal{M}_\epsilon$).

For example, the seller could choose a mechanism that requires that users create user accounts before they submit their bids. The uninformed don't know this.[14]

[13]To see this, note that

$$U^\mu(1) = \chi(v_-, v_+)(1-t) < ((1-\alpha)F(v_+)+\alpha)^{n-1}(1-r^*).$$

The above is less than $(1-r*)/n$ when n is sufficiently large. The payoff functions are piece wise linear, an uninformed buyer with any valuation is worse off in equilibrium.

[14]More precisely, they don't know how to create a user account.

An informed user who creates an account then submits a bid is also submitting a user id. The assignment of user ids, of course, is randomized.

Assuming the equilibrium is such that the uninformed babble, then any mixed equilibrium must be payoff equivalent to a $U-$equilibrium. The reason is that every mechanism that lies in the support of the randomization must provide the seller with the same expected revenue. At the point where the realization of the mechanism is revealed to the informed, the distribution of messages from the uninformed is unchanged. Revenues can then be no higher than revenues in a $U-$equilibrium, a consequence of our Theorem 5 above. Notice that Theorem 5 uses direct mechanisms, so this will be true even if the seller's commitment to the informed also involves randomization.

5. Further Discussion

We have assumed that the seller chooses an offer, not an outcome. By construction the mechanism is ex post individually rational for this reason. If an offer is made to an uninformed buyer in our equilibrium, it will be rejected with positive probability. If it is, we assume the game ends without trade. For the auction among the informed buyers this assumption has no impact since the winner of the auction always wants to accept the offer when they win. For the uninformed this assumption is unrealistic. Once the seller learns who the uninformed buyers are, the seller may want to approach them in sequence with offers. One question is how this might change if the seller could follow up a rejection by making an offer to one of the other bidders.

Our results generalize to other trading technologies. The key property we use in most of the proofs is the assumption that when buyers are uncommunicative, buyers who trade with positive probability will trade with probability that is independent of their value. Other technologies will also satisfy this property. For example, if the seller can make a take it or leave it offer to each of a group of buyers in turn until one of them accepts it, then provided each buyer in the group is approached with equal probability, the buyer' trading probability will be independent of their type. In this case the equilibrium mechanism will be an equal priority auction like the one we described in Figure 3. Of course, the parameters of this auction will be different because an offer to the uninformed is more likely to be accepted making fixed price trade more profitable.

On the other hand, there are trading technologies for which our results won't hold. For example, if the trading technology is such that the seller can make as many offers as the seller wants after a rejection, the seller could start with a very high offer which is made to each buyer in turn. If the offer is rejected by all the buyers, then the seller could lower it slightly and repeat the process. Continuing with this until a bidder accepts implements a kind of descending clock auction which will generate revenue close to the Myerson optimal auction. In this mechanism there is no communication at all yet all buyers trading probability increase with their types.

Since the seller can't make commitments to the uninformed, whatever trading process occurs once the seller chooses which buyers he wants to approach must be something that uninformed buyers understand and expect beforehand. If they expect a take it or leave it offer immediately and don't get one, a natural continuation equilibrium would be for them to leave. We assume here this negotiation process is beyond the seller's control.

One other important assumption is that whether or not a buyer is informed is exogenous and does not depend on their type. On the surface it might seem that higher valuation buyers should be more likely to be informed. As the model now stands, allowing buyers to decide whether they want to be informed based on some cost and knowledge of their value leads to unraveling. No matter what strategies buyers use some subset of valuations will induce buyers to learn the rules. Once these choices have been made, sellers will choose revenue maximizing mechanisms for the informed as a best reply to buyer strategies. Every such mechanism will leave some buyer valuations with zero surplus. In a U-equilibrium buyers with these low values must understand this before they choose to become informed, so they will be unwilling to pay any cost to learn the rules.

This isn't an issue for us. No seller would be willing to pay the considerable cost of designing, coding and advertising an algorithm for a one off auction unless the stakes in the auction are very high. A better assumption is that buyers learn rules in anticipation of repeated interactions for a variety of products for which they only know the distribution of their values. It could plausibly be that informed and uninformed buyers draw their values from different distributions. This is readily incorporated into our existing methodology at the cost of more notation. Again the equilibrium will an equal priority auction provided the uniformed are uncommunicative.

Finally, our assumption that the uninformed don't communicate any information about their values is important. We have constructed examples of U-equilibrium in which uninformed buyers can communicate the fact that their values are too low for them to trade. We believe this is the only kind of informative equilibrium that will survive a very reasonable equilibrium refinement. However we do not yet have a proof of this. In order to focus on the auction design implications we focus on the simplest case here.

5.1. **Literature.** As mentioned above, the idea that consumers might not notice prices is an old one in the marketing literature, as in Dickson and Sawyer (1990) and references therein. The approach had been used earlier in economics, as in, say Butters (1977), in which buyers randomly observe price offers in a competitive environment. In that literature, firms advertise prices which some buyers see, while others do not.[15] These papers considered the same problem that we do, which is how this unobservability would affect the prices that firms offer. The difference here is that we are interested in mechanisms, not prices.

What ignorant buyers do is to provide type dependent outside options to informed buyers. This is one of the most basic problems in the literature on competing mechanisms. One example is the paper by McAfee (1993). His model had buyers whose outside option involved waiting until next period to purchase in a competing auction market just like the one in the current period. He imposed large market assumptions to ensure that the value of these outside options was independent of the reserve price that any seller in the existing market chose.

In our paper, the value of this outside option depends on the nature of the mechanism the seller chooses for the informed. This makes it resemble the later

[15] See also Varian (1980), or Stahl (1994). Marian calls buyers informed if they see prices of all firms, and uninformed if they do not.

papers on competing mechanisms (at least in terms of outside options) like Virag (2010) who studies finite competing auction models where a seller who raises her reserve price increases congestion in other auctions, or Hendricks and Wiseman (2020) who study the same problem in a sequential auction environment.

With buyers potentially uninformed of the selling mechanism but nonetheless having rational expectations, the seller's commitment power is limited. There is an extensive literature on limited commitment (for example Bester and Strausz (2001), Kolotilin et al. (2013), Liu et al. (2019), or Skreta (2015)). To our knowledge, our model is the first to study commitment with respect to a subset of traders involved in the same transaction.

A recent paper by Akbarpour and Li (2020) provides another model of limited commitment. They assume that each individual buyer only observes the part of the seller's commitment in relation to the buyer's own report, and impose a "credibility" constraint that the seller does not wish to secretly alter other parts of the commitment. The logic we described above explaining why the second price auction can't survive as an equilibrium is used in a similar way in their paper. The difference between their approach and ours is that they assume the credibility constraint applies to all buyers and describe mechanisms that are immune to this constraint. Here we assume that credibility is an issue only for some buyers and find optimal mechanisms.

Our informed buyers can "prove" they are informed in the same sense as Ben-Porath et al. (2014). The main difference is that they assume that the social choice function is known by all the players, while in our model the driving force is the presence of buyers who are uninformed of the seller's mechanism. They also assume players have complete information about the state, but in our model only buyers know their own valuations.

Finally, our informed buyers can pretend they are uninformed but not the other way around. The one-sidedness of this incentive condition is similar to Denekere and Severinov (2006), who study an optimal non linear pricing problem with a fraction of consumers constrained to reporting their valuations truthfully. As in our paper, a "password" mechanism could be used to separate 'honest' consumers from "strategic" consumers who can misrepresent their valuations costlessly. In our model, there aren't truthful bidders in their sense of the term since uninformed buyers can still misrepresent their values in our model.

5.2. **Concluding remarks.** In this paper we have considered a traditional mechanism design problem and modified it by assuming some buyers do not know the mechanism the seller is using. We show that, assuming uninformed buyers don't communicate any useful information, the seller's revenue optimal equilibrium can be implemented with an equal priority auction. This mechanism is new as far as we know. It lies nicely between the extremes of pure auction, which is best when the seller is sure everyone is informed, and a simple take it or leave price offer to a buyer chosen randomly.

One of the nice advantages of the equal priority auction is that it is parametric - all equal priority auctions can be described using only 4 parameters, which makes is easy to show existence. The parameters all lie in a compact set, and the payoff functions are integrals which depend continuously on the parameters.

The parametric representation makes it possible to do computations, and in principal, do empirical work. As we mentioned above, one of the implications of the the equal priority auctions is that the distribution of bids in the auction will be endogenous. In particular, it will be bi-modal with high and low bids while intermediate valuation bidders trade at a fixed price. This is something like what happens on eBay, though eBay auctions differ in many ways from what we have modeled here.

Perhaps a restrictive assumption we use is that buyers are either fully informed or fully uninformed. A more reasonable assumption might be that buyers have partial information about commitments. For example, we could assume that some buyers may only be able to understand commitments to actions based on their own messages, but not commitments that depend on the messages of others. If all buyers have this type of partial information, then there is an equilibrium in which the seller implements the optimal auction of Myerson (1981) through a first-price sealed bid auction. This corresponds to the main result of Akbarpour and Li (2018), who frame the issue of partial observability in terms of limited commitment by the seller. When buyers have differential information about the seller's commitments - for example, if buyers either fully observe the seller's commitment or only observe the part based on their own message - we nonetheless believe that our basic insight could be extended to this kind of assumption. Yet we are reluctant to pursue without a better model of what buyers can and cannot understand.

6. Appendix: Omitted Proofs

Proof of Lemma 2.

Proof. First, an informed buyer with $v < r$ never wins the auction, and thus the expected payoff is 0, matching $U^\epsilon(v)$ in (3.1) and (3.3) for $v < r$.

Second, an informed buyer with $v \in [r, v_-)$ wins the auction only when $m = 0$ and all $n-1$ other informed buyers have valuation at most v, pays the maximum of r and the second highest valuation. Thus, the expected payoff is

$$v(1-\alpha)^{n-1}F^{n-1}(v) - \left(r(1-\alpha)^{n-1}F^{n-1}(r) + \int_r^v w \ d\left((1-\alpha)^{n-1}F^{n-1}(w)\right) \right).$$

By integration by parts, the above matches $U^\epsilon(v)$ in (3.1) and (3.3) for $v \in [r, v_-)$.

Third, an informed buyer with $v \in [v_-, v_+]$ wins the auction when $m = 0$ and all $n-1$ other informed buyers have valuation at most v_-, and pays the maximum of r and the second highest valuation. The contribution of this event to the buyer's expected payoff is

$$\begin{aligned} & v(1-\alpha)^{n-1}F^{n-1}(v_-) - \left(r(1-\alpha)^{n-1}F^{n-1}(r) + \int_r^{v_-} w \ d\left((1-\alpha)^{n-1}F^{n-1}(w)\right) \right) \\ =& U^\epsilon(v_-) + (v - v_-)(1-\alpha)^{n-1}F^{n-1}(v_-). \end{aligned}$$

The buyer also wins the auction with probability $1/(m+k+1)$ when there are m uninformed buyers, all $n-m-1$ other informed buyers have valuation at most v_+, and $m+k$ is at least 1 (where k is the number of informed buyers with valuation on $[v_-, v_+]$), and pays v_-. The contribution of this event to the buyer's expected payoff is

$$(v - v_-)\left(\chi(v_-, v_+) - (1-\alpha)^{n-1}F^{n-1}(v_-)\right).$$

The sum of the above two expressions matches $U^\epsilon(v)$ in (3.1) and (3.3) for $v \in [v_-, v_+]$.

Fourth, for an informed buyer with $v > v_+$ who wins the auction, he pays the maximum of the second highest bid and the reserve price. When the second highest bid is below v_-, which implies that $m = k = 0$, the reserve price is r, and the contribution to the expected payoff is

$$U^\epsilon(v_-) + (v - v_-)(1-\alpha)^{n-1} F^{n-1}(v_-).$$

When the second highest bid is between v_- and v_+, which implies that $m + k \geq 1$, the reserve price is $(v_- + v_+(m+k))/(m+k+1)$, and the contribution to the expected payoff is

$$\begin{aligned}&\sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+) \left(v - \frac{v_- + v_+(m+k)}{m+k+1} \right) \\ &\qquad - (1-\alpha)^{n-1} F^{n-1}(v_-)(v - v_-) \\ =&(v - v_+)((1-\alpha)F(v_+) + \alpha)^{n-1} + (v_+ - v_-)\chi(v_-, v_+) - (1-\alpha)^{n-1} F^{n-1}(v_-)(v - v_-).\end{aligned}$$

When the second highest bid w is above v_+, which occurs with probability

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)(F^{n-1-m}(w) - F^{n-1-m}(v_+)),$$

the buyer pays this bid, and so by integration by parts the contribution to the expected payoff is

$$\begin{aligned}&\int_{v_+}^{v} \sum_{m=0}^{n-1} B(m; n-1, \alpha)(F^{n-1-m}(w) - F^{n-1-m}(v_+)) dw \\ =&\int_{v_+}^{v} \sum_{m=0}^{n-1} B(m; n-1, \alpha) F^{n-1-m}(w) dw - (v - v_+)((1-\alpha)F(v_+) + \alpha)^{n-1}.\end{aligned}$$

The sum of the three expressions for the contributions to the expected payoff matches $U^\epsilon(v)$ in (3.1) and (3.3) for $v > v_+$. □

Proof of Lemma 3.

Proof. Define

$$D(r, t, v_-, v_+) = U^\epsilon(v_-) - U^\mu(v_-) = \int_r^{v_-} (1-\alpha)^{n-1} F^{n-1}(w) dw - \chi(v_-, v_+)(v_- - t),$$

and let R be the revenue from the equal-priority auction. We have

$$\begin{aligned}\frac{\partial D}{\partial r} &= -(1-\alpha)^{n-1} F^{n-1}(r); \\ \frac{\partial R}{\partial r} &= -n(1-\alpha)^n F^{n-1}(r) \phi(r) f(r)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial D}{\partial t} &= \chi(v_-, v_+); \\ \frac{\partial R}{\partial t} &= n\alpha\chi(v_-, v_+)\pi'(t).\end{aligned}$$

If $v_- < v_+$, or if $v_- = v_+$ and $dv_- < 0$, we have

$$\frac{\partial \chi(v_-, v_+)}{\partial v_-} = \frac{(1-\alpha) f(v_-)}{(1-\alpha)(F(v_+) - F(v_-)) + \alpha} \left(\chi(v_-, v_+) - ((1-\alpha)F(v_-))^{n-1}\right);$$

$$\frac{\partial D}{\partial v_-} = (1-\alpha)^{n-1} F^{n-1}(v_-) - \chi(v_-, v_+) - \frac{\partial \chi(v_-, v_+)}{\partial v_-}(v_- - t);$$

$$\frac{\partial R}{\partial v_-} = n(1-\alpha)((1-\alpha)^{n-1} F^{n-1}(v_-) - \chi(v_-, v_+))\phi(v_-) f(v_-) + n((1-\alpha)(\pi(v_-) - \pi(v_+)) + \alpha\pi(t)) \frac{\partial \chi(v_-, v_+)}{\partial v_-}.$$

If $v_- < v_+$, or if $v_- = v_+$ and $dv_+ > 0$, we have

$$\frac{\partial \chi(v_-, v_+)}{\partial v_+} = \frac{(1-\alpha) f(v_+)}{(1-\alpha)(F(v_+) - F(v_-)) + \alpha} \left(((1-\alpha)F(v_+) + \alpha)^{n-1} - \chi(v_-, v_+)\right);$$

$$\frac{\partial D}{\partial v_+} = -\frac{\partial \chi(v_-, v_+)}{\partial v_+}(v_- - t);$$

$$\frac{\partial R}{\partial v_+} = n(1-\alpha)\left(\chi(v_-, v_+) - ((1-\alpha)F(v_+) + \alpha)^{n-1}\right)\phi(v_+) f(v_+) + n((1-\alpha)(\pi(v_-) - \pi(v_+)) + \alpha\pi(t)) \frac{\partial \chi(v_-, v_+)}{\partial v_+}.$$

If $v_- = v_+ = \hat{v}$, we have

$$\frac{d\chi(\hat{v}, \hat{v})}{d\hat{v}} = \frac{(1-\alpha) f(\hat{v})}{\alpha} \left(((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - ((1-\alpha)F(\hat{v}))^{n-1}\right);$$

$$\frac{\partial D}{\partial \hat{v}} = (1-\alpha)^{n-1} F^{n-1}(\hat{v}) - \chi(\hat{v}, \hat{v}) - \frac{d\chi(\hat{v}, \hat{v})}{d\hat{v}}(\hat{v} - t);$$

$$\begin{aligned}\frac{\partial R}{\partial \hat{v}} &= n(1-\alpha)\left((1-\alpha)^{n-1} F^{n-1}(\hat{v}) - ((1-\alpha)F(\hat{v}) + \alpha)^{n-1}\right)\phi(\hat{v}) f(\hat{v}) + n\alpha\pi(t) \frac{d\chi(\hat{v}, \hat{v})}{d\hat{v}} \\ &= n(1-\alpha) f(\hat{v}) \left(((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - (1-\alpha)^{n-1} F^{n-1}(\hat{v})\right)(\pi(t) - \phi(\hat{v})).\end{aligned}$$

Let (r, t, v_-, v_+) be an optimal equal-priority auction. We first show that it is interior.

Suppose that $r = t < v_- \le v_+$. Recall that $U^\epsilon(v)$ is strictly convex for $v \in (r, v_-)$ while $U^\mu(v)$ is linear for $v \in (t, v_-)$. Since $Q^\epsilon(v)$ has an upward jump at $v = v_-$, we have $U^\epsilon(v_-) < U^\mu(v_-)$, violating the critical bidding condition (3.4).

Suppose that $r < t = v_- \le v_+$. We have $U^\epsilon(v_-) > U^\mu(v_-) = 0$, and so the critical bidding condition (3.4) is slack. Since $r < t$, we have $r < r^*$ or $t > r^*$, or both. If $r < r^*$, then by raising r marginally, the seller could increase the revenue because $\phi(r) < 0$ implies $\partial R/\partial r > 0$. If $t > r^*$, then by lowering t marginally, the seller could increase the revenue because $\pi'(t) < 0$ implies $\partial R/\partial t < 0$. With the critical bidding condition (3.4) slack, we have a contradiction to the assumption of optimality.

Suppose that $r = t = v_- \le v_+$. If $r = t < r^*$, then by raising t marginally, the seller relaxes the critical bidding condition (3.4) because $\partial D/\partial t > 0$, and increases the revenue because $\pi'(t) > 0$ implies $\partial R/\partial t > 0$. If $r = t > r^*$, then by lowering r marginally, the seller relaxes the critical bidding condition (3.4) because $\partial D/\partial r < 0$, and increases the revenue because $\phi(r) > 0$ implies $\partial R/\partial r < 0$. If $r = t = r^* = v_-$, then by lowering r marginally, the seller relaxes the critical bidding condition (3.4) because $\partial D/\partial r < 0$, without changing the revenue because $\partial R/\partial r = 0$. With (3.4) slack, the seller could then increase the revenue by either further raising v_- marginally if $v_- = r^* < v_+$, because $\phi(v_-) = 0$ implies $\partial R/\partial v_- > 0$, or by raising both v_- and v_+ by the same infinitesimal amount if $v_- = v_+ = r^*$, because $\partial R/\partial \hat{v} > 0$ when $\hat{v} = r^*$. In each case, we have a contradiction to the assumption of optimality.

Suppose that $r < t < v_- = v_+ = \hat{v}$. We have $\partial D/\partial \hat{v} < 0$ and $\partial R/\partial \hat{v} < 0$ has the same sign as $\pi(t) - \phi(\hat{v})$. Thus, $\pi(t) > \phi(\hat{v})$: otherwise, by decreasing v_- and v_+ by the same marginal amount, the seller relaxes the critical bidding condition (3.4) without decreasing the revenue, which would then allow the seller to increase the revenue by either raising r or lowering t, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. Since $\phi(1) = 1$, this implies that $\hat{v} < 1$. Now, consider perturbing the equal priority auction by reducing v_- from $\hat{v}$ and raising v_+ from $\hat{v}$ such that

$$-(\chi(\hat{v},\hat{v}) - (1-\alpha)^{n-1}F^{n-1}(\hat{v}))dv_- = (((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - \chi(\hat{v},\hat{v}))dv_+.$$

By construction,

$$\frac{\partial \chi(\hat{v},\hat{v})}{\partial v_-} = \frac{\partial \chi(\hat{v},\hat{v})}{\partial v_+}.$$

This implies that the critical bidding condition (3.4) is relaxed, because

$$\frac{\partial D}{\partial v_-}dv_- + \frac{\partial D(\hat{v})}{\partial v_+}dv_+ = ((1-\alpha)^{n-1}F^{n-1}(\hat{v}) - \chi(\hat{v},\hat{v}))dv,$$

which is strictly negative. The seller's revenue is unchanged, because

$$\begin{aligned}
&\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+ \\
=&n(1-\alpha)f(\hat{v})\left(\chi(\hat{v},\hat{v}) - (1-\alpha)^{n-1}F^{n-1}(\hat{v})\right)(\pi(t) - \phi(\hat{v}))dv_- \\
&+ n(1-\alpha)f(\hat{v})\left(((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - \chi(\hat{v},\hat{v})\right)(\pi(t) - \phi(\hat{v}))dv_+,
\end{aligned}$$

which is equal to 0 by construction. The seller could now increase the revenue by either raising r or lowering t, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. This contradicts the assumption of optimality.

Now, we establish the first-order conditions stated in the lemma. To begin, the critical bidding condition (3.4) binds at any optimal equal-priority auction. Otherwise, since $r < t$ implies that $r < r^*$ or $t > r^*$, or both, the seller could increase the revenue by either raising r or lowering t, a contradiction to the assumed optimality. Further, $r < r^* < t$. Otherwise, if $r^* \le r < t$, the seller could relax (3.4) by lowering r marginally without decreasing the revenue, which then would allow the seller to increase the revenue by lowering t. Similarly, if $r < t \le r^*$, the seller could relax (3.4) by raising t marginally without decreasing the revenue, which then would allow the seller to increase the revenue by raising r. Finally, $\pi(t) > \phi(v_+)$. Otherwise, by lowering v_+ marginally, the seller relaxes (3.4) because $\partial D/\partial v_+ < 0$, and increases the revenue, as $\partial R/\partial v_+$ has the same sign as

$$\begin{aligned}
&\alpha(\pi(t) - \phi(v_+)) + (1-\alpha)(\pi(v_-) - \pi(v_+)) - \phi(v_+)(F(v_+) - F(v_-)) \\
=&\alpha(\pi(t) - \phi(v_+)) - \int_{v_-}^{v_+}(\phi(v_+) - \phi(w))f(w)dw \\
<&\alpha(\pi(t) - \phi(v_+)),
\end{aligned}$$

contradicting the assumed optimality. Note that $\pi(t) > \phi(v_+)$ implies $v_+ < 1$.

To obtain (3.7), consider perturbations dv_- and dv_+, while keeping r and t unchanged. An optimality condition is that

$$\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+ = 0,$$

for all perturbations dv_- and dv_+ satisfying

$$\frac{\partial D}{\partial v_-}dv_- + \frac{\partial D}{\partial v_+}dv_+ = 0.$$

Thus we have

$$\frac{\partial R/\partial v_-}{\partial D/\partial v_-} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

Using the expressions for $\chi(v_-, v_+)$, $\partial\chi(v_-, v_+)/\partial v_-$ and $\partial\chi(v_-, v_+)/\partial v_+$, straightforward algebra lead us to the first-order condition (3.7) for an optimal equal-priority auction with respect to v_- and v_+. Note that (3.7) implies that

$$\frac{\partial R/\partial v_+}{\partial D/\partial v_+} = -n(1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-).$$

Next, to obtain (3.8), consider perturbations dt and dv_+. The resulting optimality condition is

$$\frac{\partial R/\partial t}{\partial D/\partial t} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

This gives the first order condition (3.8) with respect to t and v_+.

Lastly, to obtain (3.9), consider perturbations dr and dv_+, while keeping t and v_- unchanged. The resulting optimality condition is

$$\frac{\partial R/\partial r}{\partial D/\partial r} \geq \frac{\partial R/\partial v_+}{\partial D/\partial v_+},$$

and $r \geq 0$, with complementary slackness. This gives the first-order condition

$$-\phi(r)f(r) \leq (\phi(v_+) - \phi(v_-))f(v_-),$$

and $r \geq 0$, with complementary slackness. Note that $-\phi(0)f(0) = 1$. Since $\phi(v_+) < \pi(t)$ and $t > r^*$, we have $\phi(v_+) < \pi(r^*) < r^*$, while $v_- > t > r^*$. Thus,

$$(\phi(v_+) - \phi(v_-))f(v_-) = (\phi(v_+) - v_-)f(v_-) + 1 - F(v_-) < 1.$$

It follows that the optimal r is interior and so (3.9) holds. □

Proof of Lemma 4.

Proof. Fix a direct mechanism $\{q_m^\epsilon, p_m^\epsilon\}_{m=0}^{n-1}$ and $\{q_m^\mu, p_m^\mu\}_{m=1}^{n}$. Define $p^\mu \in [0,1]$ to be the expected offer to uninformed buyers, given by

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\{q_{m+1}^\mu(v)(p^\mu - p_{m+1}^\mu(v))\} = 0.$$

Since $p_m^\mu(v) \in [0,1]$ for all v,

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\{q_{m+1}^\mu(v)\} \max[w - p^\mu, 0]$$

$$= \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\left\{q_{m+1}^\mu(v) \max[w - p_{m+1}^\mu(v), 0]\right\}$$

for all $w \le \min p^{\mu}_m(v)$ and for all $w \ge \max p^{\mu}_m(v)$. Since $U^{\mu}(w)$ is convex in w, we have

$$U^{\mu}(w) \ge \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\{q^{\mu}_{m+1}(v)\} \max[w - p^{\mu}, 0]$$

for all w. Thus, replacing each all functions $\{p^{\mu}_m(\cdot)\}_{m=1}^{n}$ with p^{μ} reduces the deviation payoff of an informed buyer from pretending to be uninformed. The seller's revenue from uninformed buyers is

$$\sum_{m=1}^{n} B(m; n, \alpha)\mathbb{E}_v \left\{m q^{\mu}_m(v)\pi\left(p^{\mu}_m(v)\right)\right\} = n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v \left\{q^{\mu}_{m+1}(v)\pi\left(p^{\mu}_{m+1}(v)\right)\right\}.$$

The lemma then follows from the strict concavity of $\pi(\cdot)$. □

Proof of Theorem 5.

Proof. Suppose that $\{r, t, v_-, v_+\}$ is a revenue maximizing equal priority auction. By Lemma 3, the first order conditions (3.7)-(3.9) are satisfied. We construct a non-negatively valued multiplier function $\lambda(w)$ for all $w \in [0, 1]$ such that the allocative rule $(q^{\epsilon}_m)_{m=0}^{n-1}$ and $(q^{\mu}_m)_{m=1}^{n}$, together with the offer to uninformed p^{μ} defined by $\{r, t, v_-, v_+\}$ solves the Lagrangian relaxation. By Lemma 2, the transfer rule we have specified for an equal priority auction supports a truthful bidding equilibrium among informed buyers. Thus we have found a direct mechanism $\left\{(q^{\epsilon}_m, p^{\epsilon}_m)_{m=0}^{n-1}, (q^{\mu}_m, p^{\mu}_m)_{m=1}^{n}\right\}$ that point-wise maximizes the Lagrangian.

For each $w \in [0, 1]$, denote

$$K^{\epsilon}(w) = n(1-\alpha)\phi(w) + \int_w^1 \lambda(x)dx/f(w);$$
$$K^{\mu} = n\alpha\pi(p^{\mu}) - \int_0^1 \lambda(x)\max[x - p^{\mu}, 0]dx.$$

We can then rewrite the Lagrangian as

$$(1-\alpha)^{n-1}\int_0^1 K^{\epsilon}(w)Q^{\epsilon}_0(w)f(w)dw + \alpha^{n-1}K^{\mu}q^{\mu}_n$$
$$+ \sum_{m=1}^{n-1}\left(\int_0^1 B(m; n-1, \alpha)K^{\epsilon}(w)Q^{\epsilon}_m(w)f(w)dw + B(m-1; n-1, \alpha)K^{\mu}Q^{\mu}_m\right),$$

where $Q^{\epsilon}_0(w)$ is the probability that an informed buyer with valuation w gets the good when all buyers are informed, and q^{μ}_n is the probability that each uninformed buyer gets the good when all buyers are uninformed.

Now we construct $\lambda(\cdot)$ as follows. Let $\lambda(w) = 0$ for all $w \notin [v_-, v_+]$, and let

$$\begin{aligned}\lambda(w) =& n(1-\alpha)\frac{d}{dw}(f(w)(\phi(w) - \phi(v_+)))\\ =& n(1-\alpha)(2f(w) + f'(w)(w - \phi(v_+)))\end{aligned}$$

for all $w \in (v_-, v_+)$, with $\lambda(v_-)$ and $\lambda(v_+)$ given by the corresponding limit from above and from below. Since by assumption $\pi(\cdot)$ is strictly concave, $f(w)\phi(w)$ is strictly increasing in w, and thus $\lambda(w) > 0$ at any $w \in [v_-, v_+]$ such that $f'(w) \le 0$. By (3.7) we have $\phi(v_+) < \pi(t) < \pi(r^*) < r^*$. Since $w \ge v_- > t > r^*$, we have $\lambda(w) > 0$ at any $w \in [v_-, v_+]$ such that $f'(w) > 0$. Thus, $\lambda(w)$ as constructed is non-negative for any w.

We will first show that $p^\mu = t$ maximizes the Lagrangian. For any $w \in [v_-, v_+]$, by construction

$$\int_w^1 \lambda(x)dx = n(1-\alpha)f(w)(\phi(v_+) - \phi(w)).$$

Using integration by parts, we have

$$\begin{aligned}
&\int_0^1 \lambda(w)\max[w - p^\mu, 0]dw \\
=& -\int_{v_-}^{v_+} (w - p^\mu)\ d\left(\int_w^1 \lambda(x)dx\right) \\
=& n(1-\alpha)\left((v_- - p^\mu)f(v_-)(\phi(v_+) - \phi(v_-)) + \int_{v_-}^{v_+} f(w)(\phi(v_+) - \phi(w))dw\right) \\
=& n(1-\alpha)\left((v_- - p^\mu)f(v_-)(\phi(v_+) - \phi(v_-)) + \phi(v_+)(F(v_+) - F(v_-)) - (\pi(v_-) - \pi(v_+))\right).
\end{aligned}$$

By (3.7), we have

$$K^\mu = n\alpha\phi(v_+) + n\alpha(\pi(p^\mu) - \pi(t)) + (p^\mu - t)n(1-\alpha)f(v_-)(\phi(v_+) - \phi(v_-)).$$

The above is strictly concave in p^μ. By (3.8), it is maximized at $p^\mu = t$, and thus

$$K^\mu = n\alpha\phi(v_+).$$

The remainder of the proof establishes that the direct mechanism $(q_m^\epsilon)_{m=0}^{n-1}$, $(q_m^\mu)_{m=1}^n$, and $p^\mu = t$ defined by $\{r, t, v_-, v_+\}$ is a point-wise maximizer of the Lagrangian relaxation. For $w \in [v_-, v_+]$, we have

$$\frac{B(m; n-1, \alpha)}{n-m}K^\epsilon(w) = \frac{B(m-1; n-1, \alpha)}{m}K^\mu.$$

For all $w > v_+$, since $\pi(\cdot)$ is strictly concave,

$$K^\epsilon(w) = n(1-\alpha)\phi(w) > n(1-\alpha)\phi(v_+) = K^\epsilon(v_+),$$

and so

$$\frac{B(m; n-1, \alpha)}{n-m}K^\epsilon(w) > \frac{B(m-1; n-1, \alpha)}{m}K^\mu.$$

For all $w < v_-$,

$$\begin{aligned}
K^\epsilon(w) &= n(1-\alpha)\phi(w) + \int_{v_-}^{v_+} \lambda(x)dx/f(w) \\
&= n(1-\alpha)(\phi(w) + f(v_-)(\phi(v_+) - \phi(v_-))/f(w)).
\end{aligned}$$

We claim that

$$\phi(w) + \frac{f(v_-)(\phi(v_+) - \phi(v_-))}{f(w)} < \phi(v_+)$$

for all $w < v_-$, and thus $K^\epsilon(w) < K^\epsilon(v_+)$ and

$$\frac{B(m; n-1, \alpha)}{n-m}K^\epsilon(w) \le \frac{B(m-1; n-1, \alpha)}{m}K^\mu.$$

To establish the claim, recall that in showing that the constructed multiplier function $\lambda(w)$ is positive for $w \in [v_-, v_+]$, we have proved that $f(w)(\phi(w) - \phi(v_+))$ is strictly increasing in w for all $w \ge \phi(v_+)$. This immediately implies that the claim holds for any $w \in [\phi(v_+), v_-)$. For $w < \phi(v_+)$, we have

$$f(w)(\phi(w) - \phi(v_+)) = f(w)(w - \phi(v_+)) - (1 - F(w)) < -(1 - F(w)) < -(1 - F(r^*)),$$

where the last inequality follows because $\phi(v_+) < \pi(t) < \pi(r^*) < r^*$, while

$$f(v_-)(\phi(v_+) - \phi(v_-)) < f(r^*)\phi(v_+) < f(r^*)r^*,$$

where the first equality comes from $f(w)(\phi(w) - \phi(v_+))$ being strictly increasing in w for all $w \geq \phi(v_+)$. The claim then follows from the definition of r^*.

To show that the direct mechanism defined by the equal-priority auction $\{r, v_-, v_+, t\}$ is a point-wise maximizer of the Lagrangian, we dis aggregate $Q_m(w)$ and write the Lagrangian as

$$(1-\alpha)^{n-1} \int_0^1 K^\epsilon(w) Q_0^\epsilon(w) f(w) dw + \alpha^{n-1} K^\mu q_n^\mu +$$

$$\sum_{m=1}^{n-1} \mathbb{E}_v \left\{ \frac{B(m; n-1, \alpha)}{n-m} \sum_{i=1}^{n-m} K^\epsilon(v_i) q_m^\epsilon(\rho_m^i(v)) + B(m-1; n-1, \alpha) K^\mu q_m^\mu(v) \right\}.$$

Fix any realized number m of uninformed buyers such that $1 \leq m \leq n-1$, and consider the last term in the above objective function. Suppose that for some realized valuation profile v we have $v_i > v_+$ for some $i = 1, \ldots, n-m$, but $q_m^\mu(v) > 0$. By (2.2), we can decrease $q_m^\mu(v)$ marginally by $dq_m^\mu(v) > 0$ and increase $q_m^\epsilon(\rho_m^i(v))$ by $m dq_m^\mu(v)$. Since

$$\frac{m}{n-m} B(m; n-1, \alpha) K^\epsilon(v_i) > B(m-1; n-1, \alpha) K^\mu,$$

the effect on the seller's revenue is strictly positive. Therefore, $q_m^\mu(v) = 0$ for any v such that $v_i > v_+$ for some $i = 1, \ldots, n-m$. Further, since $K^\epsilon(w)$ is strictly increasing for $w > v_+$, we have $q_m^\epsilon(\rho_m^i(v)) = 1$ for $v_i = \max[v_1, \ldots, v_{n-m}]$. Finally, since

$$\frac{B(m; n-1, \alpha)}{n-m} K^\epsilon(w) \leq \frac{B(m-1; n-1, \alpha)}{m} K^\mu.$$

for all $w \leq v_+$, with equality if $w \in [v_-, v_+]$, if v is such that $\max[v_1, \ldots, v_{n-m}] \leq v_+$, there is a maximizer of the Lagrangian such that $q_m^\epsilon(\rho_m^i(v)) = 0$ whenever $v_i < v_-$, and $q_m^\epsilon(\rho_m^i(v)) = q_m^\mu(v)$ if $v_i \in [v_-, v_+]$.

For $m = 0$ and the first term in the Lagrangian, the strict concavity of $\pi(\cdot)$ implies $K^\epsilon(w)$ for $w < v_-$ crosses 0 at most once and only from below. Thus, for r that satisfies (3.9), it is point-wise maximizing to set $q_0^\epsilon(\rho_0^i(v)) = 1$ if $v_i = \max[v_1, \ldots, v_n]$ and $v_i > v_+$, or if $v_i = \max[v_1, \ldots, v_n]$ and $v_i \in [r, v_-)$; set $q_0^\epsilon(\rho_0^i(v)) = 1/k$ if $v_i \in [v_-, v_+]$, $\max[v_1, \ldots, v_n] \in [v_-, v_+]$ and $\#\{j : v_j \in [v_-, v_+]\} = k$; and set $q_0^\epsilon(\rho_0^i(v)) = 0$ otherwise.

For $m = n$ and the second term in the Lagrangian, we have $q_n^\mu = 1/n$ because $K^\mu > 0$. □

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