

THE MAPINATOR CLASSIFICATION: THEORY

MICHAEL PETERS, JAMES YU (UBC)

ABSTRACT. The paper provides a frictional matching model to describe a high skill labor market. *High skill* here means valuable workers whose types are described primarily by qualitative data. We show how trading patterns identify firms (buyers) *perceptions* of worker (seller) quality. Assuming these perceptions are correct, we show how to use the model to estimate the value of the information embedded in these perceptions. This method provides a way to evaluate the impact of machine learning algorithms that further refine buyers information. We use the theoretical results to provide 'tier based' classification of economics phds based on their graduating university.

All trades are idiosyncratic. Matching theory often explains this with a model of assortative matching.¹ In such models every firm in a market is different and behaves that way. Similarly all workers are different. This is reasonable in an environment where wage offers are public and workers quality is determined by their easily measurable qualifications.

High skill markets don't fit this model every well. Workers value is measured almost entirely by qualitative data - reference letters and job market papers, for example. It is often easy to identify a good job market paper, or a bad job market paper. Yet to distinguish two potential workers who both have good job market papers is very difficult.

Ex ante, it is also hard to distinguish between hiring firms. Wage offers describe only part of the reward associated with a specific job. Qualitative aspects of a job are more important to workers with high skills. Wage offers, even when they are visible, often involve back loaded future wage increases that depend on ill specified ex post adjustments who nature depend on the firm's reputation.

This problem doesn't stem from any lack of public information. At least in an economics job market, qualitative data is there in abundance. Graduates have job market papers and letters of reference that provide quite detailed descriptions of graduates research. Hiring firms' ads convey little useful information beyond the fact that there is a job opening.

The downside is that the market is so large (over 3000 phd graduates each year) that no recruiting committee has the time or energy to process all that information, and no applicant has the time to carefully research the history of every hiring firm.

Our approach in this paper is to assume that when traders make decisions they classify. Classification leads to very coarsely partitioning the set of alternatives. An

Date: May 23, 2024.

The authors are grateful for comments and advice from Li Hao, Wei Li, Sergei Serverinov, Vitor Farinha Luz, Sam Gyetvay, Jesse Perla, Vadim Marmar, Kevin Song and Hiro Kasahara. Thanks to David Weaver for pointing out an error in a previous version.

¹The survey by Chade, Eeckhout, and Smith (2017) is illustrative of the approach. In matching theory it is common to assume that traders types are drawn from a continuum, as in Becker (1973); Hopkins (2005); Siow (2003), or Peters (2010); Eeckhout and Kircher (2010).

example of classification in the economics job market is the mapinator classification. The intention of the model presented here is to make it possible to estimate the value of trade with this kind of classification, then examine the distributional effects of machine learning techniques that refine this information partition.

We model market interactions using a frictional trading model that might best be described as *reverse directed search*. The 'reverse part' comes from the fact that potential employers make offers (or apply) to graduates.²

In this sense the approach resembles Julien, Kennes, and King (2000) except that here we will assume the bids are determined exogenously (as they effectively are in a second price auction for labor). This allows us to focus on trading patterns rather than the bidding process.

In this version of the paper we restrict attention to the theory. The model is intended to be one that can be directly estimated. to be effective it must apply to arbitrary information partitions. This makes it complex. So we begin by illustrating with an example in which the workers in the market are partitioned into just two tiers - maybe high and low productivity.

This example shows the basic logic and describes all the math that is used in the general case.

For the two tier case we can show that algorithmic information that refines the partition is always beneficial in aggregate. However, using two communities of hiring firms, we can show that these gains to trade are not enjoyed uniformly. Using a computational example, we show that refined information hurts hiring clusters with weaker offer distributions. Benefits of algorithmic information accrue primarily to richer hiring communities.

The Model. The model we use is a frictional matching model that combines the ideas in Julien, Kennes, and King (2000) with the mathematical approach in Peters (2010). Basically workers are separated into quality *tiers*. The value of graduates from a particular tier is the same for all other market participants. Universities make offers to applicants without knowing exactly what other offers the applicant might have. Offers are rejected with positive probability as a result. As a result, graduates from the top tier are in large demand. Making an offer to the top tier is more valuable if it is accepted, but since everyone wants graduates from this tier, it is less likely that such offers will be accepted.

This is the rationale for the directed search approach. The model makes it possible to quantify the flow of placements between tiers.

We start with a collection of hiring institutions, D (demand). We'll refer to these institutions as firms. There is also a collection of workers S (supply) who are looking for jobs. The workers have different skill levels from 1 to $k > 1$. We imagine these skill levels are easy to identify but imprecise in the sense that skill levels partition all workers into equivalence classes. Workers within the same equivalence class are difficult to rank in terms of value. Workers from different equivalence classes are not. For the rest of the paper, we'll refer to these skill levels or equivalence classes as *tiers*.

Firms in D try to hire workers by making them offers. Offer values are random variables lying in the interval $[0, 1]$. Firms are partitioned coarsely into communities

²Much of the literature on directed search is concerned with graduates or job seekers making applications. See for example Peters (2010).

indexed from 1 to K . Firms in community j have offer values x that are drawn using a distribution F_j .

A firm who hires a worker from tier t with an offer x receives a payoff v_t . We assume throughout that $v_t \geq v_{t+1}$, so that tier 1 workers are the ones that have the highest market value.

When a worker from tier t is hired by any firm with an offer x , the worker's payoff is just x .

When a worker gets no offers, or a firm's offer is rejected, payoffs are (normalized to) zero.

The key assumption is that the firm's offer value is drawn exogenously and can't be manipulated to influence the hiring process. In particular, the firm can't make different offers to different workers.

The game that market participants play is straightforward. Firms privately learn their offer values x . The only strategic part of the process is that each firm simultaneously decides which worker to make an offer to. Once offers have been made, each worker who receives an offer accepts the best one they get. This determines all the payoffs.

The number of workers in tier t is m_t , with $m = \sum_{t=1}^k m_t$. When needed we'll use the notation m to refer both to the set of workers and the number of workers.

Hiring firms know their own community, but they don't know the communities of their competitors. There are n firms in the market in total. Each potential competitor is assumed to be in community i with probability ρ_i . The probability that each competitor has an offer whose value is less than or equal to x is then

$$F(x) = \sum_{t=1}^K \rho_t F_t(x).$$

Offers and matching. A strategy rule for firm j is a mapping from $[0, 1] \rightarrow \Delta(m)$ describing the probability that an offer will be made to each worker. Let $\tilde{\pi}_j^i(x)$ be the probability with which firm j with an offer of value x makes an offer to applicant i . We'll restrict attention to equilibrium that satisfy two symmetry restrictions: first, since every graduate from tier t is viewed to have the same ex ante value v_t we'll assume that for each x , $\tilde{\pi}_j^i(x) = \tilde{\pi}_j^{i'}(x)$ for every pair i and i' in m_t ; and second, $\tilde{\pi}_j^i(x)$ does not depend on j .

Exploiting this symmetry we can again, with a slight abuse, simplify the strategy rule and write $\tilde{\pi}_t(x)$ to be the probability with which an firm with an offer of value x makes it to *some* worker from tier t . The probability the firm makes this offer to any particular worker in tier t is $\frac{\tilde{\pi}_t(x)}{m_t}$:

A symmetric equilibrium for this game is a vector valued rule $\{\tilde{\pi}_t(x)\}_{t \in T}$ such that $\sum_t \tilde{\pi}_t(x) = 1$ for all x and $\tilde{\pi}_t(x) > 0$ implies

$$v_t \left(1 - \int_x^1 \frac{\tilde{\pi}_t(\tilde{x})}{m_t} dF(\tilde{x}) \right)^{n-1} \geq v_{t'} \left(1 - \int_x^1 \frac{\tilde{\pi}_{t'}(\tilde{x})}{m_{t'}} dF(\tilde{x}) \right)^{n-1}$$

for every $t' \neq t$.

Equilibrium with 2 Hiring communities and 2 Worker Tiers. The main logic of the paper is easy to explain with 2 Worker Tiers and 2 Firm Communities. The full characterization of equilibrium is provided in the appendix. To make the 2 tier example simpler we'll just imagine the firms' common strategy rule to be some

function $\tilde{\pi} : [0, 1] \rightarrow [0, 1]$ to describe the probability that an offer of value x will be made to a worker in tier 1. Then $1 - \tilde{\pi}(x)$ is the probability the offer is made to a tier 2 worker.³

The probability the offer x is made to any particular worker in tier 1 is $\frac{\tilde{\pi}(x)}{m_1}$. Since all offers are in $[0, 1]$ workers will always accept the highest offer they receive. We'll assume, to avoid trivialities, that $v_1 > v_2 > 0$.

Then an equilibrium for this 2 by 2 case is a strategy rule $\tilde{\pi}$ that satisfies $\tilde{\pi}(x) > 0$ ($1 - \tilde{\pi}(x) > 0$) only if

$$v_1 \left(1 - \int_x^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(\tilde{x}) \right)^{n-1} \geq (\leq) v_2 \left(1 - \int_x^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_2} dF(\tilde{x}) \right)^{n-1}.$$

If a firm has an offer of value x and offers it to some worker in tier 1, the probability that offer loses out to an offer \tilde{x} made by any particular competitor is equal the probability the competitor draws that offer value and decides to make it to the same worker. The probability the offer x is accepted is then equal to the probability that none of the firms' competitors actually do that. This gives the probability of acceptance functions

$$Q_1(x) = \left(1 - \int_x^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(\tilde{x}) \right)^{n-1}$$

for offers made to a tier 1 worker and

$$Q_2(x) = \left(1 - \int_x^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_1} dF(\tilde{x}) \right)^{n-1}$$

for a tier 2 worker. The equilibrium condition just says that if $\tilde{\pi}(x) > 0$ then it must be that

$$v_1 Q_1(x) \geq v_2 Q_2(x),$$

and similarly for tier 2.

The main characterization Theorem the says:

Theorem 1. *If*

$$v_1 \left(1 - \frac{1}{m_1} \right)^{n-1} < v_2$$

then there is a unique symmetric equilibrium characterized by a cutoff x^ and a constant*

$$(0.1) \quad \bar{\pi} = \frac{\left(\frac{v_2}{v_1} \right)^{\frac{1}{n-1}} \frac{m_1}{m_2}}{1 + \left(\frac{v_2}{v_1} \right)^{\frac{1}{n-1}} \frac{m_1}{m_2}}.$$

with x^ as the solution to*

$$(0.2) \quad v_1 \left(1 - \frac{1 - F(x^*)}{m_1} \right)^{n-1} = v_2$$

³This method of indexing the strategy rules is slightly different than the one used for the general model where the strategy rule is described recursively using the convention $\tilde{\pi}_1$. The logic of the argument, however, is identical.

such that

$$(0.3) \quad \tilde{\pi}(x) = \begin{cases} \bar{\pi} & x \leq x^* \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $x > x^*$. then for any strategy rule $\tilde{\pi}(\tilde{x})$

$$\begin{aligned} v_1 \left(1 - \int_x^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(x) \right)^{n-1} &\geq \\ v_1 \left(1 - \int_x^1 \frac{1}{m_1} dF(x) \right)^{n-1} &> v_2. \end{aligned}$$

So offering to a worker from tier 1 is a dominant strategy for any offer that exceeds x^* .

On the other hand, suppose that $x < x^*$. Suppose that there is a non-degenerate interval $[x_1, x_2]$ containing x with $x_2 \leq x^*$ such that $\tilde{\pi}(\tilde{x})$ is equal to zero for $x \in [x_1, x_2]$. Without loss we can take

$$v_1 \left(1 - \int_{x_2}^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(x) \right)^{n-1} \geq v_2 \left(1 - \int_{x_2}^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_2} dF(x) \right)^{n-1}$$

since this is true for x^* . Then

$$\begin{aligned} &v_2 \left(1 - \int_x^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_2} dF(x) \right)^{n-1} \\ < v_2 \left(1 - \int_{x_2}^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_2} dF(x) \right)^{n-1} &= \\ v_1 \left(1 - \int_{x_2}^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(x) \right)^{n-1} &= \\ v_1 \left(1 - \int_x^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(x) \right)^{n-1} & \end{aligned}$$

because $\tilde{\pi}(\tilde{x})$ is zero between x and x_2 . This is a profitable deviation. A similar argument shows that $\tilde{\pi}(x)$ can't be equal to 0 on any non-degenerate interval below x^* .

We conclude that

$$v_1 \left(1 - \int_x^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(x) \right)^{n-1} = v_2 \left(1 - \int_x^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_2} dF(x) \right)^{n-1}$$

for all x below x^* . Rewriting

$$1 - \int_x^1 \frac{\tilde{\pi}(\tilde{x})}{m_1} dF(x) \equiv \left(\frac{v_2}{v_1} \right)^{\frac{1}{n-1}} \left(1 - \int_x^1 \frac{1 - \tilde{\pi}(\tilde{x})}{m_2} dF(x) \right)$$

and differentiating gives

$$-\frac{\tilde{\pi}(x)}{m_1} \equiv - \left(\frac{v_2}{v_1} \right)^{\frac{1}{n-1}} \frac{1 - \tilde{\pi}(\tilde{x})}{m_2}$$

and solving for the constant gives the formula in the theorem.

Since the market is large, the formulas are simplified by taking limits. \square

Proposition 2. *Let $m_{1n} = \alpha_1(n-1)$ and $m_{2n} = \alpha_2(n-1)$. Then as $n \rightarrow \infty$ we have*

$$\lim_{n \rightarrow \infty} F(x_n^*) = 1 + \alpha_1 \log\left(\frac{v_2}{v_1}\right)$$

while

$$\lim_{n \rightarrow \infty} \bar{\pi}_n = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

Proof. The second equality is simplest:

$$\begin{aligned} \bar{\pi}_n &= \frac{\left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} \frac{m_{1n}}{m_{2n}}}{1 + \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} \frac{m_{1n}}{m_{2n}}} = \\ &= \frac{\left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} \frac{\alpha_1}{\alpha_2}}{1 + \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} \frac{\alpha_1}{\alpha_2}} = \\ &= \frac{\frac{\alpha_1}{\alpha_2}}{1 + \frac{\alpha_1}{\alpha_2}} = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \end{aligned}$$

For the second note that for all n , x_n^* satisfies

$$\begin{aligned} \left(1 - \frac{1 - F(x_n^*)}{m_{1n}}\right) &= \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} \implies \\ 1 - \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} &= \frac{1 - F(x_n^*)}{m_{1n}} \implies \\ \frac{1 - \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}}}{\frac{1}{m_{1n}}} &= 1 - F(x_n^*) \implies \\ F(x_n^*) &= 1 - \frac{1 - \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}}}{\frac{1}{m_{1n}}}. \end{aligned}$$

So

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n^* &= \lim_{n \rightarrow \infty} F^{-1}\left(1 - \frac{1 - \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}}}{\frac{1}{m_{1n}}}\right) = \\ &= F^{-1}\left(1 - \lim_{y \rightarrow 0} \frac{1 - \left(\frac{v_2}{v_1}\right)^y}{\frac{1}{\alpha_1} y}\right) = \\ &= F^{-1}\left(1 + \alpha_1 \log\left(\frac{v_2}{v_1}\right)\right) \end{aligned}$$

by L'Hopitals Rule and the fact the Assumption that F is monotonically differentiable. \square

Henceforth we just use the convention $x^* = F^{-1}\left(1 + \alpha_1 \log\left(\frac{v_2}{v_1}\right)\right)$ and $\bar{\pi} = \frac{\alpha_1}{\alpha_1 + \alpha_2}$. There are two other limit results that will be useful:

Proposition 3. *For the two probabilities*

$$\lim_{n \rightarrow \infty} Q_{1n}(x) = \begin{cases} e^{-\frac{1-F(x)}{\alpha_1}} & x > x^* \\ e^{-\frac{(F(x^*)-F(x))\bar{\pi}}{\alpha_1} - \frac{1-F(x^*)}{\alpha_1}} & \text{otherwise;} \end{cases}$$

$$\lim_{n \rightarrow \infty} Q_{2n}(x) = e^{-\frac{(F(x^*)-F(x))(1-\bar{\pi})}{\alpha_2}}.$$

Proof. Begin with $x < x^*$:

$$\begin{aligned} Q_1^n(x) &= \\ & \left(1 - \int_x^1 \frac{\tilde{\pi}_n(\tilde{x})}{m_{1n}} dF(\tilde{x}) \right)^{n-1} = \\ & \left(1 - \int_x^{x_n^*} \frac{\bar{\pi}_n}{\alpha_1(n-1)} dF(\tilde{x}) - \int_{x_n^*}^1 \frac{1}{\alpha_1(n-1)} dF(\tilde{x}) \right)^{n-1} = \\ & \lim_{n \rightarrow \infty} \left(1 - \frac{\bar{\pi}_n(F(x_n^*) - F(x))}{\alpha_1(n-1)} - \frac{1 - F(x_n^*)}{\alpha_1(n-1)} \right)^{n-1} \\ & \exp \left((n-1) \log \left(1 - \frac{\bar{\pi}_n(F(x_n^*) - F(x))}{\alpha_1(n-1)} - \frac{1 - F(x_n^*)}{\alpha_1(n-1)} \right) \right) = \\ & \exp \left(\frac{\log \left(1 - \frac{\bar{\pi}_n(F(x_n^*) - F(x))}{\alpha_1} y - \frac{1 - F(x_n^*)}{\alpha_1} y \right)}{y} \right). \end{aligned}$$

For any $\bar{\pi}$ and any x^* this function is decreasing in y or increasing in n and has a pointwise limit given by

$$= e^{-\frac{\bar{\pi}(F(x^*)-F(x))}{\alpha_1} - \frac{1-F(x^*)}{\alpha_1}}$$

by L'Hopital's Rule. Since the sequence of functions given by the second last line above is increasing in n for all of its arguments, and converges pointwise to a continuous function, then it converges uniformly by Dini's theorem. Then from the Moore-Osgoode theorem, the order with which limits are taken can be interchanged so we get the limit

$$e^{-\frac{\bar{\pi}(F(x^*)-F(x))}{\alpha_1} - \frac{1-F(x^*)}{\alpha_1}}.$$

The other limits in the statement of the Proposition are proved the same way. \square

The value of information with 2 tiers. We are now in a position to evaluate market information. We can do this in a fairly simple way with 2 tiers. We can also provide an analytical result.

We imagine that there are two tiers, but that initially, firms can't tell the tiers apart. An algorithm is invented that identifies the workers in each tier. The market then adjusts to this information by changing its hiring strategy. We want to evaluate the total surplus that is created in each case.

With the information provided by the algorithm, firms with high value offers can target them at tier 1 workers. As with all directed search, this additional competition means that high value offers are less likely to be accepted. Frictions rise and the overall number of successful offers falls.

As offers are targeted towards tier 1, it becomes easier to hire tier 2 workers, who produce less surplus.

These are the basic tradeoffs. It is possible to show that in the two tier case, the information provided by the algorithm will always increase the total surplus associated with trade, though not by much. Whether this true an arbitrary number of tiers and communities is not known.

However, the more interesting result is that total surplus created by different hiring communities will be different because their offers are drawn from different distributions. We show how to calculate these differences.

The focus of this paper is primarily the impact of information. We can address this in a straightforward way by comparing total surplus generated by trade when firms know the tier structure with the total surplus when they don't. Ex post surplus for a firm with offer value x is given by

$$\tilde{\pi}(x) v_1 Q_1(x) + (1 - \tilde{\pi}(x)) Q_2(x) v_2.$$

The more appropriate measure of surplus would be the interim surplus for a firm in community i , given by

$$V_i = \int_0^1 (\tilde{\pi}(\tilde{x}) Q_1(\tilde{x}) v_1 + (1 - \tilde{\pi}(\tilde{x})) Q_2(\tilde{x}) v_2) dF_i(\tilde{x}).$$

We'll use this integral to evaluate the differential impact of the algorithmic information across hiring communities.

The appropriate measure of the aggregate effect of the information is given by the average surplus generated by all communities is

$$V(v_1, v_2) \equiv \int_0^1 (\tilde{\pi}(\tilde{x}) Q_1(\tilde{x}) v_1 + (1 - \tilde{\pi}(\tilde{x})) Q_2(\tilde{x}) v_2) dF(\tilde{x}).$$

Then to estimate the value of information that is used to distinguish members of each tier we just need to compare the various surplus formulas in the case where $v_1 > v_2$ to the case where all workers generate expected surplus

$$\bar{v} = \frac{\alpha_1 v_1 + \alpha_2 v_2}{\alpha_1 + \alpha_2}.$$

This is the expected surplus generated from a randomly selected worker.

Proposition 4. *For the two tier model*

$$V(v_1, v_2) > V(\bar{v}, \bar{v})$$

Proof. Substituting the results of Propositions 1 to 3, average surplus is

$$\int_0^{x^*} \left\{ v_1 \bar{\pi} e^{-\frac{(F(x^*) - F(x))\bar{\pi}}{\alpha_1} - \frac{1 - F(x^*)}{\alpha_1}} + v_2 (1 - \bar{\pi}) e^{-\frac{(F(x^*) - F(x))(1 - \bar{\pi})}{\alpha_2}} \right\} dF(x) + \int_{x^*}^1 v_2 e^{-\frac{1 - F(x)}{\alpha_1}} dF(x).$$

Let $y^* = F(x^*)$ and do a change of variable to get

$$\int_0^{y^*} \left\{ v_1 \bar{\pi} e^{-\frac{(y^* - y)\bar{\pi}}{\alpha_1} - \frac{1 - y^*}{\alpha_1}} + v_2 (1 - \bar{\pi}) e^{-\frac{(y^* - y)(1 - \bar{\pi})}{\alpha_2}} \right\} dy + \int_{y^*}^1 v_1 e^{-\frac{1 - y}{\alpha_1}} dy.$$

$$\int_0^{y^*} \left\{ v_1 \bar{\pi} e^{-\frac{(y^* - y)\bar{\pi}}{\alpha_1} - \frac{1 - y^*}{\alpha_1}} + v_2 (1 - \bar{\pi}) e^{-\frac{(y^* - y)(1 - \bar{\pi})}{\alpha_2}} \right\} dy + \int_{y^*}^1 v_1 e^{-\frac{1 - y}{\alpha_1}} dy$$

$$\int_0^{y^*} \left\{ v_1 e^{-\frac{1 - y^*}{\alpha_1}} \frac{\alpha_1}{\alpha_1 + \alpha_2} e^{-\frac{(y^* - y)}{\alpha_1 + \alpha_2}} + v_2 \frac{\alpha_2}{\alpha_1 + \alpha_2} e^{-\frac{(y^* - y)}{\alpha_1 + \alpha_2}} \right\} dy + \int_{y^*}^1 v_1 e^{-\frac{1 - y}{\alpha_1}} dy =$$

$$\begin{aligned}
& \int_0^{y^*} \left\{ v_1 e^{-\frac{1-y^*}{\alpha_1}} e^{-\frac{y^*}{\alpha_1+\alpha_2}} \frac{\alpha_1}{\alpha_1+\alpha_2} e^{\frac{y}{\alpha_1+\alpha_2}} + v_2 \frac{\alpha_2}{\alpha_1+\alpha_2} e^{-\frac{y^*}{\alpha_1+\alpha_2}} e^{\frac{y}{\alpha_1+\alpha_2}} \right\} dy + \int_{y^*}^1 e^{-\frac{1}{\alpha_1}} v_1 e^{\frac{y}{\alpha_1}} dy \\
& \left\{ v_1 \alpha_1 e^{-\frac{1-y^*}{\alpha_1}} + v_2 \alpha_2 \right\} \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right] + v_1 \alpha_1 \left[1 - e^{-\frac{1-y^*}{\alpha_1}} \right] = \\
& v_2 \alpha_2 \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right] + v_1 \alpha_1 \left[1 - e^{-\frac{1-y^*}{\alpha_1}} \right] + \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right] v_1 \alpha_1 e^{-\frac{1-y^*}{\alpha_1}} = \\
& v_2 \alpha_2 \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right] + v_1 \alpha_1 \left(\left[1 - e^{-\frac{1-y^*}{\alpha_1}} \right] + \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right] e^{-\frac{1-y^*}{\alpha_1}} \right) > \\
& v_2 \alpha_2 \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right] + v_1 \alpha_1 \left[1 - e^{-\frac{y^*}{\alpha_1+\alpha_2}} \right].
\end{aligned}$$

To show that this last line is $V(\bar{v}, \bar{v})$, observe by substitution

$$V(\bar{v}, \bar{v}) = \bar{v} \int_0^1 \left\{ e^{-\frac{(1-y)}{\alpha_1+\alpha_2}} \right\} dy$$

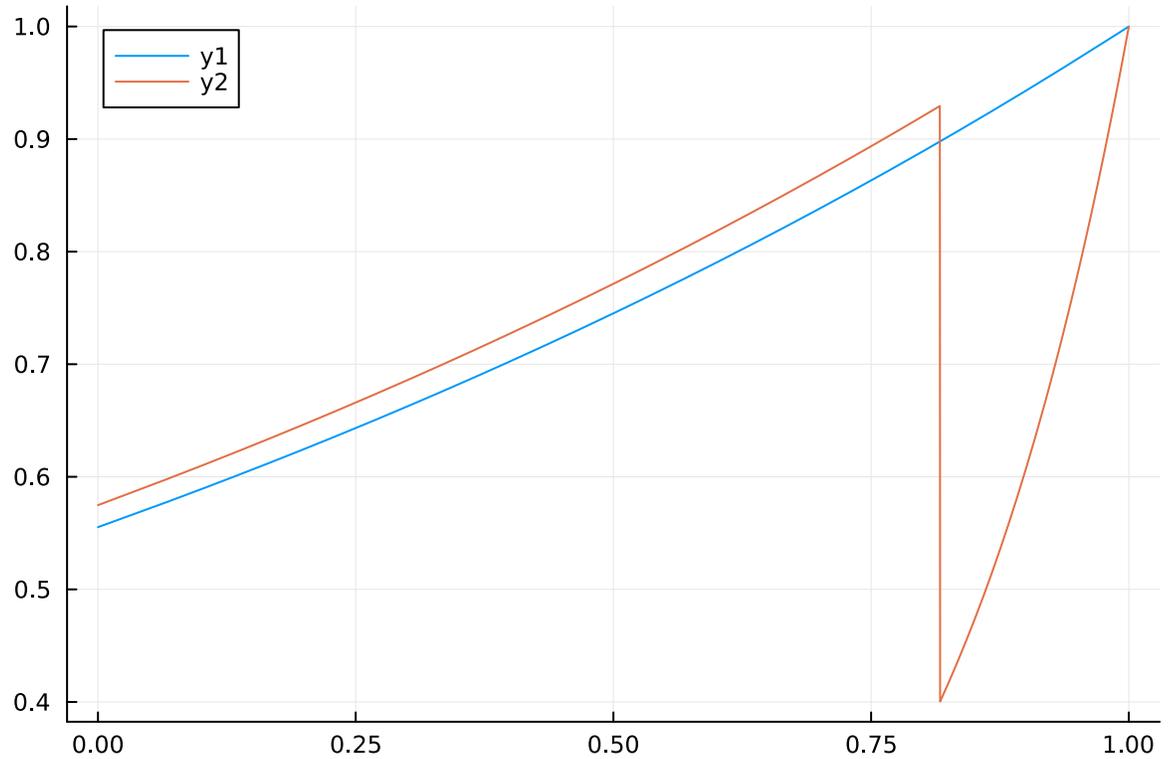
$$\bar{v} e^{-\frac{1}{\alpha_1+\alpha_2}} \int_0^1 \left\{ e^{\frac{y}{\alpha_1+\alpha_2}} \right\} dy$$

$$\bar{v} e^{-\frac{1}{\alpha_1+\alpha_2}} (\alpha_1 + \alpha_2) \left(e^{\frac{1}{\alpha_1+\alpha_2}} - 1 \right)$$

$$\bar{v} (\alpha_1 + \alpha_2) \left(1 - e^{-\frac{1}{\alpha_1+\alpha_2}} \right)$$

□

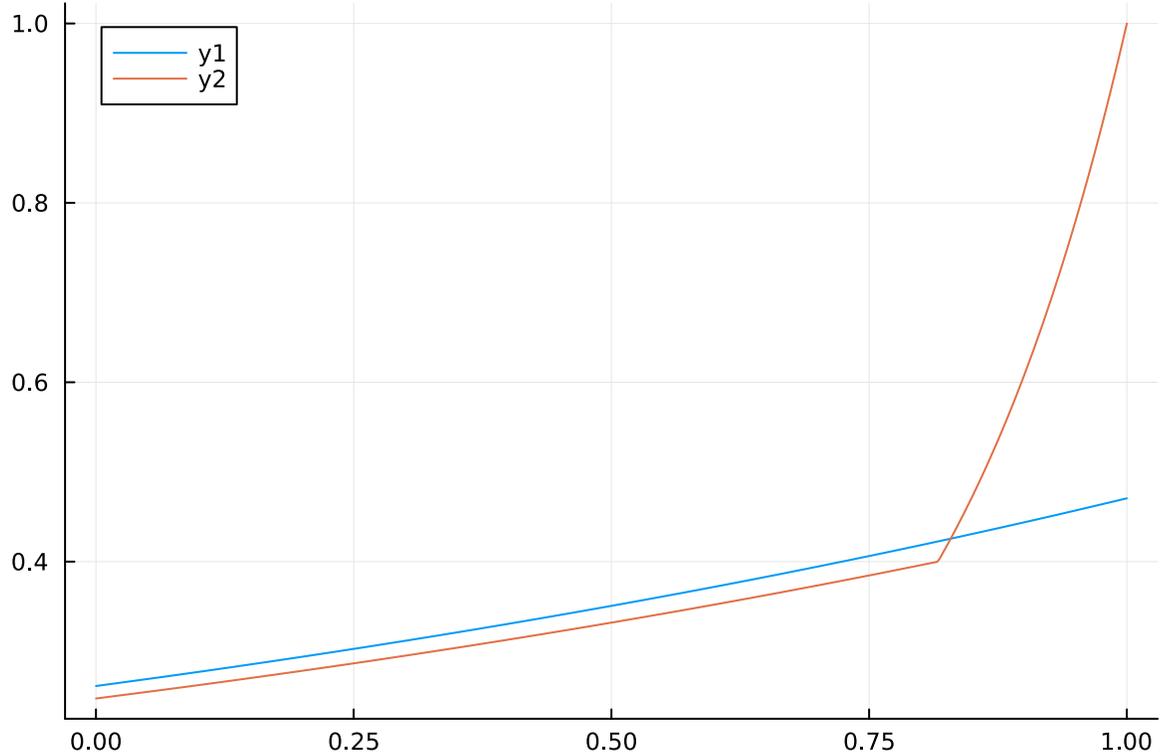
To see what is going it is probably easier to look at some computations and figures. The first figure shows the overall probability of trade for a firm as a function of its offer value.



The blue line shows the trading probability for each offer value when firms cannot tell which workers belong to each tier. The red line shows what happens when firms have access to information identifying the tiers. Firms make offers to top tier workers with positive probability no matter what their offer value. Their success probability rises a bit when tier members are identified because their offers are more likely to be accepted when they make them to tier 2 workers.

The calculations were done using values $v_1 = 1$, and $v_2 = .4$, with values for $\alpha_1 = .2$ and $\alpha_2 = 1.2$. These values coincide roughly with estimates we did using the mapinator classification described below. However, the calculations are just for illustration.

The corresponding average surplus calculations are given in the following:



Unsurprisingly firms with very high offers would like to know how to direct them so they are better off when tier members can be identified. Most of the time (the picture is based on a uniform distribution as explained in the theory) firms would rather not know the workers' types. This isn't because firms don't like information, they are hurt by the strategic changes that take place when information is made available.

Consistent with Proposition 4, the gain in average surplus associated with information about the tiers is 6% of the surplus without information.

However the gain is not shared equally. For example, if there is a community of firms that draws its values from the distribution x^2 (the rich tier - because it is a convex distribution function), it gains 15% more surplus once the tiers are identified. On the other hand if there is a community of firms that draws its offer values from the distribution function $2x - x^2$ (a poor tier with a concave distribution function) their surplus falls by 3% once tier membership is revealed.

Tier discovery. Part of the estimation problem is tier discovery - which workers belong to which tiers and which firms belong to each hiring community. It is possible to illustrate the method using the two tier example we developed above. The outcome data for the market can be summarized in an adjacency table. For example

| | Tier 1 | Tier 2 |
|-------------|------------------|------------------|
| Community A | \tilde{p}_{11} | \tilde{p}_{12} |
| Community B | \tilde{p}_{21} | \tilde{p}_{22} |

The rows represent the hiring communities. In this exercise row i represents the number of successful from each tier by firms in Community A. So for example, \tilde{p}_{22} represents the number of tier 2 workers hired by firms in Community B.

The columns represent the number of tier 1 and tier 2 workers who got jobs. For example, \tilde{p}_{11} is the number of workers from tier 1 who got jobs in Community B.

The tilde over \tilde{p}_{ij} is there to indicate that whatever this number is, it is a random outcome of some stochastic process.

In this case we can use the theory to describe the stochastic process. In the matching model, the probability that firm in community i hires from tier 1 is

$$q_{i1} \equiv \int_0^{x^*} \bar{\pi} e^{-\frac{(F(x^*)-F(x))\bar{\pi}}{\alpha_1}} dF_t + \int_{x^*}^1 e^{-\frac{1-F(x)}{\alpha_1}} dF_t(x)$$

while for tier 2, the hiring probability it is

$$q_{i2} = \int_0^{x^*} (1 - \bar{\pi}) e^{-\frac{(F(x^*)-F(x))\bar{\pi}}{\alpha_1}} dF_t.$$

Use ρ_A to be the probability that a randomly drawn placement will involve a firm in community A. Let n be the total number of placements in the sample data. Then a randomly drawn placement will end up in cell ij with probability $\rho_i \frac{q_{ij}}{q_{i1} + q_{i2}}$. The denominator comes from the fact that the adjacency matrix can't record trade failures.

The implication is that the adjacency matrix is a random vector that has a multinomial distribution with parameters n and

$$p = \left\{ \rho_1 \frac{q_{11}}{q_{11} + q_{12}}, \rho_1 \frac{q_{12}}{q_{11} + q_{12}}, \rho_2 \frac{q_{21}}{q_{21} + q_{22}}, \rho_2 \frac{q_{22}}{q_{21} + q_{22}} \right\}.$$

Maximum likelihood can then be used to estimate the parameters and, in principal, the tier structure.

Equilibrium with many tiers and communities. The problem is now to characterize equilibrium for the 'general' case. This problem is effectively the same problem as the one discussed in Peters (2010), except that the distribution of types across graduates has only finite support.

Recall that tiers are ordered such that $v_1 > v_2 > \dots v_k$. One thing to keep in mind is that in this section, the constants that define the offer strategy are ordered in such a way that $\bar{\pi}_1 = 1$. In the two tier example we discussed in the main paper the constant $\bar{\pi}$ was defined as the probability an offer is made to a top tier worker. With the indexing used in this section, the probability an offer is made to a top tier worker is $1 - \bar{\pi}_2$ so that $\bar{\pi}$ is equal to $1 - \bar{\pi}_2$.

Theorem. *There is a finite collection of cutoffs $\{x_0, x_1, \dots, x_k\}$ and a set of positive constants $\{\pi_j\}_{j=1,k}$ such that $x_0 = 1$, $\pi_1 = 1$, and for each $t > 0$ if $x \in [x_t, x_{t-1})$*

$$(0.4) \quad \tilde{\pi}_j(x) = \begin{cases} \pi_j \prod_{i < j} (1 - \pi_i) & j \leq t \\ 0 & \text{otherwise.} \end{cases}$$

Any firm who has an offer x will randomize when choosing where to make its offer, except when $x \in [x_1, 1)$. In the latter case the firm will make its offer for sure to some applicant in tier 1. For offers whose value exceeds x_t offers are made with positive probability to each tier whose applicants have a value that is at least v_t .

In the general case the proofs all use the methods we established in the 2 tier example. For instance the cutoff x_1 (which is the same as x^* in the example) is determined in exactly the same way. The proof that making an offer to a lower tier is dominated.

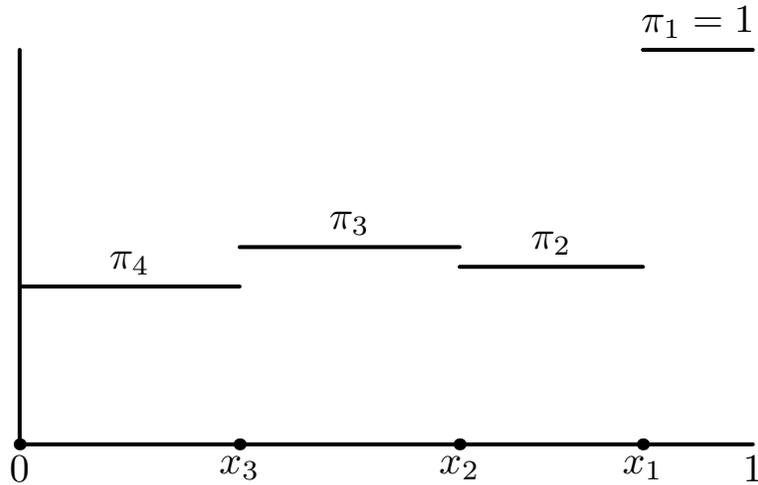
The same technique is used to show that there is an interval of offer values below x_1 such that making offers to any tier below tier 2 is dominated. Since we know from the 2 tier case that the only symmetric equilibrium must have the property that firms whose offer values are in a non-degenerate interval below x_1 , the probability that these offers are made to tier 1 is

$$\frac{1}{1 + \left(\frac{v_2}{v_1}\right)^{\frac{1}{n-1}} \frac{m_1}{m_2}}$$

just as it was in the two tier case.

This recursive construction arises because firms never care what other firms with lower offer values do. So the entire proof in the general case just involves extending those arguments to lower tiers.

The following figure illustrates what the general theorem says:



The horizontal axis represents the various values for x , the value of an institution's offer. The vertical axis measures a probability between 0 and 1.

When $x \in (x_1, 1]$ the value of the institution's offer is very high, the offer is made with probability 1 to a graduate from tier 1. Within the tier, the offer is made with equal probability to each graduate from that tier.

If the institution has an offer x in the interval $(x_2, x_1]$ then the offer is made to a tier 2 graduate with probability π_2 and to a tier 1 graduate with probability $(1 - \pi_2)$. It is worth noting that by construction, conditional of choosing to make an offer to a tier one applicant, the institution behaves exactly as if it had an offer whose value were in the interval $(x_1, 1]$.

This same logic applies to the all other offers. For example if, as in the diagram, there are four tiers and an offer is in the interval $[0, x_3]$, the offer goes to a tier 4 applicant with probability π_4 . However, conditional on choosing to make an offer to an applicant in one of the higher tiers, say tier 2, the institution behaves exactly the same way as it would have if its offer value were in the interval $(x_2, x_1]$, that is, it would make an offer to a tier 2 graduate with probability π_2 .

So the unconditional probability with which such an offer goes to a tier 2 graduate is $\pi_2(1 - \pi_3)(1 - \pi_4)$.

Theorem is proved in full in the appendix.

Using the main theorem, we get the following corollary which is proved in Lemma as part of the proof of Theorem :

Corollary 5. *The constants $\{\pi_t\}_{t=1, K}$ and $\{x_t\}_{t=0, K-1}$ can be found by solving the following system of equations with $\pi_1 = 1$, $x_k = 0$ and $x_0 = 1$.*

$$\pi_t = \frac{\pi_{t-1}}{\pi_{t-1} + \left(\frac{v_t}{v_{t-1}}\right)^{\frac{1}{n-1}} \frac{m_{t-1}}{m_t}}$$

and

$$v_{t+1} = v_t \left(1 - \frac{\pi_t}{m_t} (F(x_{t-1}) - F(x_t))\right)^{n-1}$$

for each $t = 1, K - 1$.

The proof of Lemma solves this system recursively which verifies existence and uniqueness.⁴

Large Markets. As in the two tier case, we can take advantage of the fact that there is a lot of placement data. Parallel to the two tier case, we'll assume that $m_t = \alpha_t(n - 1)$ where the coefficient α_t is constant while n becomes large.

Theorem 6. *The limit values of π_t and x_t exist for each $t = 1, k - 1$ and are given by*

$$(0.5) \quad \lim_{n \rightarrow \infty} \pi_t^{(n)} = \frac{\alpha_t}{\sum_{s=1}^t \alpha_s}$$

for $t = 1, k - 1$, while $\lim_{n \rightarrow \infty} x_t^{(n)} = x_t^\infty$ is given by the solution to

$$(0.6) \quad F(x_t^\infty) = 1 - \sum_{j=1}^{t-1} \left(-\log\left(\frac{v_{j+1}}{v_j}\right)\right) \sum_{s=1}^j \alpha_s$$

for $t = 1, k - 1$, while $x_K^\infty = 0$.

The next result is a shortcut that is used in the trading probability functions below. Recall that the strategy rule of a firm when making an offer is a piecewise constant function. The probability that an offer is made to a worker from some tier t is some constant π_t multiplied by the probability that the firm doesn't make an offer to a lower tier worker. In the limit this probability has the same kind of structure as described above.

⁴Here the term 'active tier' means tiers which graduate students who are hired with some probability.

Corollary 7. If $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$ and $t \leq i$, $\tilde{\pi}_t^{(n)}(x) = \pi_t^{(n)} \prod_{s=t+1}^i (1 - \pi_s^{(n)})$. Then for any $x \in [x_{i-1}^\infty, x_i^\infty]$

$$\lim_{n \rightarrow \infty} \pi_t^{(n)} \prod_{s=t+1}^i (1 - \pi_s^{(n)}) = \frac{\alpha_t}{\sum_{s=1}^i \alpha_s}.$$

Theorem 8. For $x \in [x_{i-1}, x_i]$

$$Q_t(x) = e^{-(G_i^t(x) + \kappa_i^t)}$$

where

$$G_i^t(x) = (F(x_{i-1}) - F(x)) \frac{1}{\sum_{s=1}^i \alpha_s}$$

and

$$\begin{aligned} \kappa_i^t &= \sum_{j=t}^{i-1} (F(x_{j-1}) - F(x_j)) \frac{1}{\sum_{s=1}^j \alpha_s} = \\ &= \sum_{j=t}^{i-1} \left(-\log \left(\frac{v_j}{v_{j-1}} \right) \right) \sum_{s=1}^{j-1} \alpha_s \end{aligned}$$

The Sampling Distribution of the Adjacency Matrix. From Corollary 5, the cross tier placement rates can be calculated from knowledge of the various distributions F_i .

The probability an institution who makes an offer of value x to an applicant in tier t successfully hires the applicant is given by

$$(0.7) \quad Q_t(x) = \left(1 - \int_x^1 \frac{\tilde{\pi}_t(\tilde{x})}{m_t} dF(\tilde{x}) \right)^{n-1}.$$

Using Theorem and Corollary 5 we can write this as

$$Q_t(x) =$$

$$(0.8) \quad \left(1 - (F(x_{i-1}) - F(x)) \frac{\pi_t}{m_t} \prod_{l=t+1}^i (1 - \pi_l) - \sum_{j=t}^{i-1} (F(x_{j-1}) - F(x_j)) \frac{\pi_t}{m_t} \prod_{l=t+1}^j (1 - \pi_l) \right)^{n-1}$$

when $x \in [x_i, x_{i-1}]$

The probability that an institution from tier i hires an applicant from tier t is given by the following formula:

$$(0.9) \quad q_i^t = \int_0^1 \tilde{\pi}_t(x') Q_t(x') dF_i(x')$$

which can be written as

$$(0.10) \quad q_i^t = \sum_{s=t}^k \int_{x_s}^{x_{s-1}} \pi_t \prod_{l=s+1}^k (1 - \pi_l) Q_t(\tilde{x}) dF_i(\tilde{x}).$$

A randomly drawn offer made by a university from tier i will result in the offer being accepted with probability q_i^t which, of course depends on n . When this occurs, the hire is recorded in the adjacency matrix by adding 1 to the matrix

in cell (i, t) . Ultimately the sum recorded in each cell of the adjacency matrix is a random variable. It is tempting to assume that this number has a binomial distribution. However q_i^t is just the probability with which the university expects to hire from tier t . In the finite game, the probability a 1 is added to a cell depends on the number currently in the cell, since a previous hire will remove one of the available applicants from tier t .

Similarly the probability that a 1 is added to cell (i, t') depends on how many hires have been added to cell (i, t) , since a previous hire removes a competitor for the next university. As a consequence, the best we can do to understand the empirical distribution of outcomes in the adjacency matrix is to use a large market approximation since the impact of outcomes of others has a vanishing impact on the probability of any other offer being accepted.

Then using our previous limit results we get:

Corollary 9. *The limit*

$$\lim_{n \rightarrow \infty} q_i^t = \sum_{i=t}^k \int_{x_i}^{x_{i-1}} \frac{\alpha_t}{\sum_{s=1}^i \alpha_s} e^{-(G_i^t(x) + \kappa_i^t)} dF_\tau(\tilde{x})$$

At this point the results are such that we can begin to provide a description of the adjacency matrix. Assuming we could sample a finite number of M_i outcomes associated with universities from tier i , as they appear in the adjacency matrix above, then each outcome will be independently placed in one of the k cells representing the k different tiers that produce graduates. The outcome will be added to cell t with probability q_i^t . Failures aren't observable, so they aren't recorded in the adjacency matrix. The random vector for row i that would result would have a multinomial distribution with parameters M_i, k , and

$$\left\{ \frac{q_i^t}{\sum_{s=1}^k q_i^s} \right\}_{t=1, k}$$

While sampling from the distribution we can't control the tier from which the observation is drawn, with each draw being an institution from tier i with probability ρ_i . Then after M draws from this distribution the adjacency matrix becomes a random vector of dimension $K \times k$ distributed multinomial with parameters $M, K \times k$ and

$$(0.11) \quad \left\{ \frac{\rho_i q_i^t}{\sum_{j=1}^K \sum_{s=1}^k \rho_j q_j^s} \right\}_{i=1, K; t=1, k}.$$

Conclusion. The paper develops a frictional matching model that can be used to estimate the impact of improving information in the job market.

The model leaves open two important theoretical questions. The first is to develop necessary and sufficient conditions for which improving information on worker quality is beneficial, more specifically how improved information will differentially benefit different hiring communities.

The second is how information impacts pre-market incentives. For example, how are worker incentives to acquire information impacted, and how are firms offer distributions and investments affected.

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Appendix - Proof of Theorem.

Proof of Theorem :

Lemma 10. *Suppose F is monotonically increasing. There is a type x_1 such that if an institution has an offer of value $x > x_1$, then the only symmetric equilibrium strategy has them making the offer to one of the tier 1 applicants and choosing each applicant with equal probability.*

Proof. The payoff to making an offer to a tier 1 applicant is v_1 times the probability that it is accepted. It is accepted by the candidate if the candidate has no other offers, or if it is the highest value offer the candidate receives. So the payoff when an institution with an offer of value x makes an offer to a top tier applicant is

$$v_1 \left(1 - \int_x^1 \tilde{\pi}_1(\tilde{x}) dF(\tilde{x}) \right)^{n-1} \geq v_1 \left(1 - \frac{(1 - F(x))}{m_1} \right)^{n-1}.$$

This inequality follows from symmetry. If $\pi(\tilde{x})$ is the probability with which any institution with a value of \tilde{x} makes an offer to *some* tier 1 applicant, and $\frac{1}{m_1}$ is the symmetric probability with which an offer is made to any particular tier 1 applicant, then the left hand side of the equation attains a maximum if $\pi(x)$ is uniformly 1.

Now choose x_1 such that

$$v_2 = v_1 \left(1 - \frac{(1 - F(x_1))}{m_1} \right)^{n-1}$$

then any offer whose value exceeds x_1 will earn a strictly higher expected payoff when it is offered to a tier 1 applicant than it would if it were offered to a tier 2 (or lower) applicant. \square

An immediate corollary is that for x in $(x_1, 1]$, $\tilde{\pi}(x) = \frac{1}{m_1}$ is the probability with which an institution with an offer of value x makes an offer to any one of the

m_1 tier 1 applicants. Note that this mixing probability $\tilde{\pi}(x)$ is independent of x on this interval. Furthermore, x_1 is the solution to

$$(0.12) \quad 1 + m_1 \left(\left(\frac{v_2}{v_1} \right)^{\frac{1}{n-1}} - 1 \right) = F(x_1)$$

if a positive solution exists. If no solution exists with x_1 in $[0, 1]$ then $x_1 = 0$ and all offers go to candidates in the highest tier.

The next Lemma extends the argument to all other active tiers.

Lemma. *Suppose that there is a tier $t > 1$ and cutoff $x_{t-1} > 0$ such that every symmetric equilibrium has the property that there is a sequence of pairs $\{(x_k, \pi_k)\}_{k=1, t-1}$ with $\pi_1 = 1$, and $0 < \pi_k < 1$ such that for any tier $k \leq t-1$*

- *the probability that any offer x goes to an applicant in tier k conditional on the offer going to an applicant in tier k or higher is π_k ;*
- *the expected payoff associated with making an offer x to any tier $k \geq 1$ is the same for every $x \leq x_{k-1}$;*
- *the expected payoff associated with making an offer to tier k is strictly higher than the expected payoff associated with making an offer to tier $k+1$ for every $x > x_k$.*

Then there is a cutoff $x_t \geq 0$ such that (i) to (iii) hold when π_t satisfies

$$(0.13) \quad \pi_t = \frac{\pi_{t-1}}{\pi_{t-1} + \left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} \frac{m_{t-1}}{m_t}}$$

and x_t satisfies

$$(0.14) \quad v_{t+1} = v_t \left(1 - \frac{\pi_t}{m_t} (F(x_{t-1}) - F(x_t)) \right)^{n-1}$$

or

$$F(x_{t-1}) - \frac{m_t}{\pi_t} \left(1 - \left(\frac{v_{t+1}}{v_t} \right)^{\frac{1}{n-1}} \right) = F(x_t)$$

Proof. From Lemma 10, institutions whose offer x lies in $[x_1, 1)$ make an offer to an applicant in tier 1 with probability $\pi_1 = 1$ independent of their type in any symmetric equilibrium because it is a dominant strategy. So (i), (ii) and (iii) hold for tier 1 and cutoff x_1 .

For $x < x_{t-1}$, let $\tilde{\pi}_t(x)$ be the probability with which the offer is made to a tier t applicant and $1 - \tilde{\pi}_t(x)$ be the probability the offer is made to an applicant from one of the higher tiers. By (i), institutions with offers of value $\tilde{x} > x_{t-1}$ do not make offers to any applicants from tier below tier $t-1$, for example, from tier t . Then the payoff associated with an offer to any tier t applicant is

$$(0.15) \quad v_t \left(1 - \frac{\int_x^{x_{t-1}} \tilde{\pi}_t(\tilde{x}) dF(\tilde{x})}{m_t} \right)^{n-1}.$$

The expected payoff to an offer of value $x < x_{t-1}$ made to any higher tier $k \leq t-1$ is the same by (ii). the payoff to making the offer to a tier $t-1$ applicant is

$$(0.16) \quad v_{t-1} \left(1 - \frac{\int_x^{x_{t-1}} (1 - \tilde{\pi}_t(\tilde{x})) \pi_{t-1} dF(\tilde{x})}{m_{t-1}} - \frac{\pi_{t-1} (F(x_{t-1}) - F(x_{t-2}))}{m_{t-1}} \right)^{n-1}.$$

If there is an open interval below x_{t-1} where other institutions make their offers to v_t for sure. Let \underline{x} and \bar{x} be the greatest lower and least upper bounds on this region. Then for $x \in (\underline{x}, \bar{x})$, the payoff from making the offer to v_{t-1} is given by

$$\begin{aligned} v_{t-1} \left(1 - \frac{\int_{\bar{x}}^{x_{t-1}} (1 - \tilde{\pi}_t(\tilde{x})) \pi_{t-1} dF(\tilde{x})}{m_{t-1}} - \frac{\pi_{t-1} (F(x_{t-1}) - F(x_{t-2}))}{m_{t-1}} \right)^{n-1} &\geq \\ v_t \left(1 - \frac{\int_{\bar{x}}^{x_{t-1}} \tilde{\pi}_t(\tilde{x}) dF(\tilde{x})}{m_t} \right)^{n-1} &> \\ v_t \left(1 - \frac{\int_x^{x_{t-1}} \tilde{\pi}_t(\tilde{x}) dF(\tilde{x})}{m_t} \right)^{n-1} &. \end{aligned}$$

The first inequality follows because $x \geq \bar{x}$ makes an offer to an applicant in tier $t-1$ with positive probability. The second follows since $x < \bar{x}$ and a set $x' \in (\underline{x}, \bar{x})$ of positive measure make an offer to an applicant in tier t with probability 1.

This gives a profitable deviation.

A similar argument when $\tilde{\pi}_t(x)$ is zero on any open interval, implies that in any symmetric equilibrium

$$(0.17) \quad \begin{aligned} (v_t)^{\frac{1}{n-1}} \left(1 - \frac{\int_x^{x_{t-1}} \tilde{\pi}_t(\tilde{x}) dF(\tilde{x})}{m_1} \right) &= \\ (v_{t-1})^{\frac{1}{n-1}} \left(1 - \frac{\int_x^{x_{t-1}} (1 - \tilde{\pi}_t(\tilde{x})) \pi_{t-1} dF(\tilde{x})}{m_{t-1}} - \frac{\pi_{t-1} (F(x_{t-1}) - F(x_{t-2}))}{m_{t-1}} \right) & \end{aligned}$$

for almost all x .

Uniform equality requires the derivatives of these two functions to be equal, or

$$(v_t)^{\frac{1}{n-1}} \frac{\tilde{\pi}_t(x)}{m_1} = (v_{t-1})^{\frac{1}{n-1}} \frac{(1 - \tilde{\pi}_t(x)) \pi_{t-1}}{m_{t-1}}$$

which gives

$$\begin{aligned} \left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} \frac{m_{t-1}}{m_t} \tilde{\pi}_t(x) &= (1 - \tilde{\pi}_t(x)) \pi_{t-1} \\ \left(\left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} \frac{m_{t-1}}{m_t} + \pi_{t-1} \right) \tilde{\pi}_t(x) &= \pi_{t-1} \end{aligned}$$

or

$$\tilde{\pi}_t(x) = \frac{\pi_{t-1}}{\pi_{t-1} + \left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} \frac{m_{t-1}}{m_t}}.$$

This verifies properties () for tier t .

Finally, define x_t to be the solution to

$$v_t \left(1 - \frac{\pi_t (F(x_{t-1}) - F(x_t))}{m_t} \right)^{n-1} = v_{t+1}$$

to get (0.14). □

Combining Lemmas 1 and 2 gives the proof of Theorem .

Proof of Theorem 6.

Proof. From Corollary 5, $\pi_1^{(n)} = 1$ for all n , so that

$$\pi_2^{(n)} = \frac{1}{1 + \left(\frac{v_t}{v_{t-1}}\right)^{\frac{1}{n-1}} \frac{\alpha_1}{\alpha_2}}.$$

Then since $v_t < v_{t-1}$

$$(0.18) \quad \pi_2^\infty = \frac{1}{1 + \frac{\alpha_1}{\alpha_2}} = \frac{\alpha_2}{\alpha_1 + \alpha_2}.$$

So suppose that for $t' < t$

$$\pi_{t'}^\infty = \frac{\alpha_{t'}}{\sum_{s=1}^{t'} \alpha_s}.$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \pi_t^{(n)} &= \frac{\pi_{t-1}^\infty}{\pi_{t-1}^\infty + \frac{\alpha_{t-1}}{\alpha_t}} = \frac{\frac{\alpha_{t-1}}{\sum_{s=1}^{t-1} \alpha_s}}{\frac{\alpha_{t-1}}{\sum_{s=1}^{t-1} \alpha_s} + \frac{\alpha_{t-1}}{\alpha_t}} = \\ &= \frac{\frac{1}{\sum_{s=1}^{t-1} \alpha_s}}{\frac{1}{\sum_{s=1}^{t-1} \alpha_s} + \frac{1}{\alpha_t}} = \\ &= \frac{\frac{1}{\sum_{s=1}^{t-1} \alpha_s}}{\frac{\alpha_t}{(\sum_{s=1}^{t-1} \alpha_s) \alpha_t} + \frac{(\sum_{s=1}^{t-1} \alpha_s)}{(\sum_{s=1}^{t-1} \alpha_s) \alpha_t}} = \\ &= \frac{\alpha_t}{\sum_{s=1}^{\alpha_t} \alpha_s}. \end{aligned}$$

This gives (0.5).

For (0.6), start with the basic formula that determines each of the cutoffs:

$$v_t = v_{t-1} \left(1 - \frac{\pi_{t-1}^{(n)}}{\alpha_{t-1} (n-1)} \left(F(x_{t-1}^{(n)}) - F(x_t^{(n)}) \right) \right)^{n-1}$$

The right hand side of this equation is continuous in x_t and attains its maximum value v_{t-1} when $x_t^{(n)} = x_{t-1}^{(n)}$. It attains its minimum at $x_t^{(n)} = 0$. So $x_t^{(n)}$ either has a unique solution or is 0, in which case the right hand side exceeds the left.

Now for each case in which a solution exists, take roots of both sides of this equation to get

$$\left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} = 1 - \frac{\pi_{t-1}^{(n)}}{\alpha_{t-1} (n-1)} \left(F(x_{t-1}^{(n)}) - F(x_t^{(n)}) \right),$$

or

$$\left(1 - \left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} \right) \frac{\alpha_{t-1} (n-1)}{\pi_{t-1}^{(n)}} = \left(F(x_{t-1}^{(n)}) - F(x_t^{(n)}) \right).$$

This solution for $F(x_t^{(n)})$ is given by

$$F(x_t^{(n)}) = F(x_{t-1}^{(n)}) - \left(1 - \left(\frac{v_t}{v_{t-1}} \right)^{\frac{1}{n-1}} \right) \frac{\alpha_{t-1} (n-1)}{\pi_{t-1}^{(n)}}.$$

The right hand side is jointly continuous in $x_{t-1}^{(n)}$, $\pi_t^{(n)}$ and $\left(\frac{1}{n-1}\right)$ for $n > 1$. From our previous result,

$$\lim_{n \rightarrow \infty} \pi_t^{(n)} = \frac{\alpha_{t-1}}{\sum_{s=1}^{t-1} \alpha_s}$$

Now take limits.

$$\begin{aligned} \lim_{(n-1) \rightarrow \infty} \left(1 - \left(\frac{v_t}{v_{t-1}}\right)^{\frac{1}{n-1}}\right) \frac{\alpha_{t-1} (n-1)}{\pi_{t-1}^{(n)}} &= \\ \lim_{\tau \rightarrow 0} \frac{\left(1 - \left(\frac{v_t}{v_{t-1}}\right)^\tau\right)}{\tau} \lim_{\tau \rightarrow 0} \frac{\alpha_{t-1}}{\pi_{t-1}^{(n)}} &= \\ -\log \left(\frac{v_t}{v_{t-1}}\right) \sum_{s=1}^{t-1} \alpha_s & \end{aligned}$$

by L'Hopitals rule.

This along with continuity implies

$$F(x_t^\infty) = F(x_{t-1}^\infty) - \left(-\log \left(\frac{v_t}{v_{t-1}}\right)\right) \sum_{s=1}^{t-1} \alpha_s.$$

The result (0.6) follows by substituting for $F(x_{t-1}^\infty)$, $F(x_{t-2}^\infty)$ and so on. \square

Proof of 7.

Proof. From Lemma 6

$$\begin{aligned} \pi_t \prod_{s=t+1}^i (1 - \pi_s) &= \\ \frac{\alpha_t}{\sum_{s=1}^t \alpha_s} \prod_{l=t+1}^i \frac{\sum_{s=1}^{l-1} \alpha_s}{\sum_{s=1}^l \alpha_s} &= \\ \frac{\alpha_t}{\sum_{s=1}^t \alpha_s} \frac{\sum_{s=1}^t \alpha_s}{\sum_{s=1}^{t+1} \alpha_s} \dots \frac{\sum_{s=1}^{i-1} \alpha_s}{\sum_{s=1}^i \alpha_s} &= \\ \frac{\alpha_t}{\sum_{s=1}^i \alpha_s} & \end{aligned}$$

\square

Proof of Theorem 8:

Proof. From (0.8) whenever $x \in [x_i, x_{i-1}]$,

$$Q_t(x) = \left(1 - (F(x_{i-1}) - F(x)) \frac{\pi_t}{m_t} \prod_{l=t+1}^i (1 - \pi_l) - \sum_{j=t}^{i-1} (F(x_{j-1}) - F(x_j)) \frac{\pi_t}{m_t} \prod_{l=t+1}^j (1 - \pi_l)\right)^{n-1}.$$

We can use the fact that the x_i and π_i all have finite non 0 limits and treat them as if they were constants to simplify the notation and take limits in this expression. It can be rewritten as

$$\left(1 - (F(x_{i-1}) - F(x)) \frac{\pi_t}{\alpha_t(n-1)} \prod_{l=t+1}^i (1 - \pi_l) - \sum_{j=t}^{i-1} (F(x_{j-1}) - F(x_j)) \frac{\pi_t}{\alpha_t(n-1)} \prod_{l=t+1}^j (1 - \pi_l) \right)^{n-1}.$$

Using the previous limits define

$$\kappa_i^t = \sum_{j=t}^{i-1} (F(x_{j-1}) - F(x_j)) \frac{\pi_t}{\alpha_t} \prod_{s=t+1}^j (1 - \pi_s)$$

and

$$G_i^t(x) = (F(x_{i-1}) - F(x)) \frac{\pi_t}{\alpha_t} \prod_{s=t+1}^i (1 - \pi_s)$$

so that we can write

$$Q_t(x) = \left(1 - \frac{(G_i^t(x) + \kappa_i^t)}{(n-1)} \right)^{n-1}$$

and we want to calculate

$$\lim_{(n-1) \rightarrow \infty} \left(1 - \frac{(G_i^t(x) + \kappa_i^t)}{(n-1)} \right)^{n-1}.$$

This limit can be evaluated by taking the exponential of the limit of its log, so we want:

$$\begin{aligned} \lim_{(n-1) \rightarrow \infty} (n-1) \log \left(1 - \frac{(G_i^t(x) + \kappa_i^t)}{(n-1)} \right) &= \\ \lim_{y \rightarrow 0} \frac{\log(1 - y(G_i^t(x) + \kappa_i^t))}{y} &= \\ \lim_{y \rightarrow 0} \frac{- (G_i^t(x) + \kappa_i^t)}{1 - y(G_i^t(x) + \kappa_i^t)} &= - (G_i^t(x) + \kappa_i^t) \end{aligned}$$

Then we have

$$Q_t(x) = e^{-(G_i^t(x) + \kappa_i^t)}$$

□