TRUNCATED HEDONIC EQUILIBRIUM

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Abstract. Workers and firms in a bilateral matching market choose wages and human capital investments then match assortatively. This paper studies Bayesian equilibria of finite matching games and shows that as the matching market becomes large, Bayesian equilibria can be approximated by a *truncated hedonic equilibrium* in the continuous matching market. We use the truncated hedonic equilibrium concept to study these limits. Under some conditions the worst types of workers and firms pool their investments at a level that exceeds the competitive level, while higher types choose bilaterally efficient wages and investments providing a potential explanation for the persistent skewed appearance of wage distributions. In these cases, we show that the lowest workers over invest in human capital. In particular, this means that under these conditions, Bayesian Nash equilibria of large finite investment matching games do not approximate simple hedonic equilibria, though they do approximate truncated hedonic equilibria.

The workhorse model for understanding endogenous characteristics in the labour market is something called the *hedonic pricing equilibrium.* The paper that formalized this idea was Rosen (1974). In his paper, the characteristics of firms are fixed exogenously, but their wages vary to compensate workers for different levels of education. In a recent paper, Peters and Siow (2002) showed how hedonic pricing equilibria could be used to explain the characteristics of firms as well even if wages aren't flexible, and even if both sides characteristics are endogenous.[1][2] A very general existence theorem for hedonic equilibria with endogenous characteristics on both sides of the market, and with quasi linear utility is given by Ekeland (2003).

The question this paper addresses is whether hedonic equilibrium can be supported as the limit of a sequence of Bayesian equilibria in

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[1]For a formal definition of hedonic equilibrium in the absence of quasi-linearity, see Peters (2004a).

[2]Similar competitive arguments have been proposed by Cole, Mailath, and Postlewaite (2001b), and Han (2006).

larger and larger finite matching investment games. The answer is that under a reasonable set of circumstances, it cannot be so supported. For this reason we also provide a concept that can be used to study these limits. We refer to as *truncated hedonic equilibrium.* The reason for the name will become apparent below. However, for the moment, the main idea is that a truncated hedonic equilibrium can be used to show the differences between the efficient outcomes associated with hedonic equilibrium and the outcomes supported as limits of large non-cooperative matching investment games.

Truncated hedonic equilibria involve bunching near the bottom of the distribution of wages and investments. Anyone who has seen a wage histogram is familiar with this phenomena. Most wages are low, with a long tail trailing out to the right. In a truncated hedonic equilibrium, the bunching occurs at a wage that is strictly above any minimum wage that might otherwise cause bunching in a simple hedonic equilibrium. As a result, this explains why wage distributions are skewed even in occupations whose wages are not constrained by minimum wages.

Furthermore, this bunching has welfare implications. We show that wages and investments are both too high at the bottom of the wage investment distribution in the sense that lower wages and investments would make both the worker and the firm who employs him better off. We also show that wages for all workers are higher than they would be in a simple hedonic equilibrium. Higher wages lead to lower firm profits (hence presumably inefficient entry, though we do not pursue this point analytically). Finally, we analyze the effect of a minimum wage on a truncated hedonic equilibrium. As in any hedonic equilibrium the minimum wage affects not only the lowest paid workers, but also all the more highly paid workers as well. This is necessary to support matching of the better workers with better firms. We can also show, using truncated hedonic equilibrium that a minimum wage will lead to additional bunching - effectively causing the wage distribution to become more highly skewed.

For some insight into why these results emerge, it helps to review how hedonic equilibrium works. Figure 0.1 illustrates. The picture is a variant of the one that appear in Rosen. In the figure, there are workers who make costly investments in human capital. These investments are measured along the horizontal axis. Firms pay wages that are measured along the vertical axis. Indifference curves (in (h, w) space) for workers are convex upward. Iso-profit curves for firms are convex downwards (or concave) with lower curves representing higher profits. Workers all have different costs of acquiring human capital, and human capital has a different marginal product at all firms. In a hedonic equilibrium, an

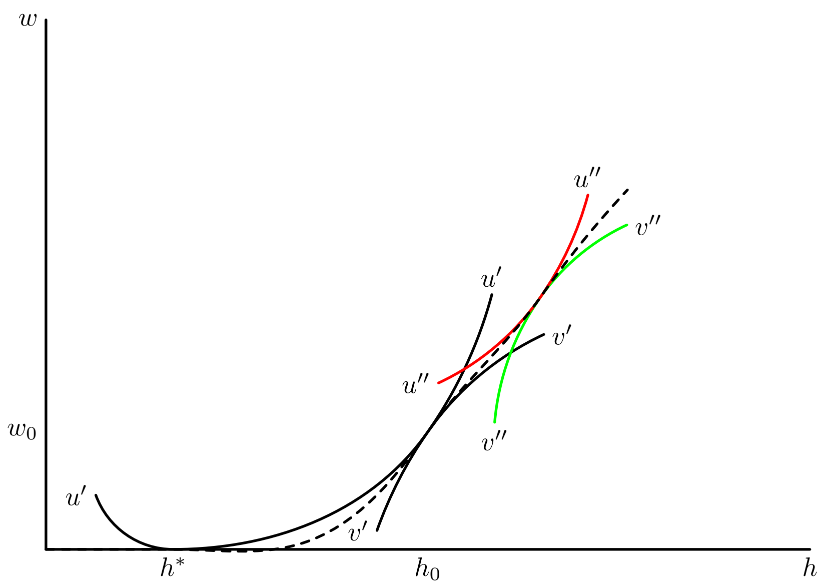

FIGURE 0.1. Competitive (Hedonic) Equilibrium

auctioneer announces a return function that describes the wage that will be paid for each level of human capital investment. This return function also specifies the wage that a firm must pay to acquire some level of human capital. This return function is the dashed line in the figure.

Firms and workers know this return function, and offer whatever wage or make whatever investment puts them on the highest indifference curve they can attain on this return function. The return function must be market clearing. So at every point along this function there is a worker firm pair whose indifference curve and iso-profit curve are tangent. The *marginal* worker is the lowest worker type who makes an investment above what is privately optimal for him. To ensure market clearing, this marginal worker must be just indifferent between remaining unmatched while making his privately optimal investment, and choosing his best investment along the market return function.

As should be evident from the diagram, a hedonic equilibrium is pareto optimal both in the sense that it optimally matches workers and firms (just assortatively in this example), and in the sense the workers have the right incentives to make the bilaterally optimal investment within their match. This model is widely used to explain how prices of goods vary with their underlying characteristics as in Rosen (1974), Sattinger (1975) or Epple (1987).

The difference between hedonic and truncated hedonic equilibrium revolves around what happens at the bottom of the wage investment distribution. In Figure 0.1, the lowest investment that actually exceeds a privately optimal investment is h_0. The lowest wage paid in equilibrium is w_0. The marginal worker who makes the corresponding hedonic investment h_0 has a lot more human capital than any lower quality worker, who never expect to match. The marginal worker ought then realize that he could cut his investment slightly and retain his existing match, since the firm with whom he is paired will still strictly prefer him to the next best alternative.

The hedonic equilibrium, instead disciplines the deviating worker by rewarding him with a lower wage when he cuts his investment. This lower wage is set low enough to deter the deviation, but not so low as to cause the lowest firm type to deviate. In Figure 0.1, the payoff to deviating below the support of the equilibrium distribution is given by the dashed line. It may be apparent from the Figure that there are many ways to construct this out of equilibrium reward function.

In any symmetric Bayesian equilibrium in which workers and firms use monotonic strategies, the lowest worker type must make whatever investment is privately optimal for him, while the lowest firm type must offer the lowest possible wage. The reason is that both understand this bottom of the distribution problem. They know they will end up with the lowest investment and wage with probability 1, so they have no incentive whatsoever to invest or offer positive wages. What we show below is that in the limit of Bayesian Nash equilibria, many workers and firms of the lowest types appear to pool at very low, but positive wages and investments. In very large games, the reason deviations are deterred is because there is a vanishingly small set of very low worker and firm types who do invest below the pool. As a result, when some worker decides to cut his investment, there is a strictly positive probability that some of these very low type workers will make investments higher than his, so that he will lose his match. The unpredictable behavior of these very low types supports the equilibrium.

There are many ways to model the interactions between firms and workers out of equilibrium. We mention some alternatives that have been discussed in the literature in the next section. The approach taken in this paper is fully non-cooperative. It is assumed that worker investments and firm wages are all set simultaneously. This approach treats wages much like any other characteristic of the firm - for example, size, or working environment. It fits with some labour markets in which wages must be paid to existing workers as well as newly hired workers, preventing firms from being flexible ex post with the wage they offer.

Many of the other details of the non-cooperative interaction could be modeled in different ways. The different payoff functions may be common knowledge as they are in Peters (2004b). The labour market is probably best approximated by a model in which the payoff functions of firms are common knowledge, while workers have privately known types. Firms might compete for workers after observing workers' human capital investments as they do in Bulow and Levin (2006). It would also be sensible to imagine workers making their education and investment decisions based on wage that had been established first by firms. In all these cases, the details of the limit arguments are slightly different. However, following the arguments in Kalai (2004), the allocations supported in equilibrium should look much the same under all these different assumptions when the number of traders is very large. For instance, the game in which workers invest first, then firms compete for the best workers, differs from the game discussed here because a deviating worker will affect the distribution of wages that prevails ex post. Again, following Kalai (2004), the effect that the worker has on wages vanishes as the game gets large. As a consequence limiting behavior should look exactly like it does in the simultaneous move game.

0.1. **Literature.** The argument for replacing the hedonic equilibrium concept with a truncated hedonic equilibrium arises from the way the auctioneer rewards investments that lie off the equilibrium path. The argument here suggests that these off equilibrium payoffs should be understood by considering the effects that these deviations would have on matching probability and expected partner quality in a large game. The exact motivation for the off equilibrium reward function was not discussed explicitly in Rosen (1974). Competitive theory typically doesn't worry about payoffs off equilibrium. For example, the price taking assumption can't be right in a competitive equilibrium in which demand equals supply. However, Rosen explicitly sets the wage associated with an out of equilibrium investment by a worker to be equal to the wage that would yield zero profit for the firm who hires this deviating worker. The motivation presumably is that there are many (identical) firms waiting in the wings to replace the firm who is there already. If the firm offers a wage to the deviator that yields positive profits, one of these potential firms will step in to offer more. Of course, no firm will offer the deviating worker a wage that is higher than the one that supports zero profit. We are interested here in an environment where these other potential firms are either non-existent, or are of considerably worse quality that the firms who offer wages in equilibrium.

A pair of papers that take this approach is Cole, Mailath, and Postlewaite (2001a) and Cole, Mailath, and Postlewaite (2001b). In their model, matches and transfers are determined after investments are realized to support a core allocation conditional on traders' investments. In the latter reference, the game is large, and as part of the specification, the payoffs of all types are fixed in the face of deviations. So if a worker, for example, cuts investment below the investment he is supposed to make in equilibrium, he hurts the firm with whom he is ultimately matched. To maintain this firm's core payoff, the transfer that this firm makes to the deviating worker is reduced accordingly. In this case, a necessary condition for equilibrium investment is that any change in investments and wage that maintains the payoff of the firm is detrimental to the worker. Bilateral efficiency is then ensured. In the finite version of the story (described in Cole, Mailath, and Postlewaite (2001a)) bilateral efficiency is guaranteed when every worker and firm type has a twin. In that event, the equal treatment property of the core guarantees that if a worker cuts investment, transfers must adjust so that the firm with whom the worker is matched is unaffected by the change. Again bilateral efficiency is guaranteed. In both cases, the hedonic equilibrium allocation is one allocation that can be supported as a cooperative equilibrium of the large pre-marital investment game. In both cases, the fact that matching and transfers are determined cooperatively post match ensures that transfers are adjusted to punish workers and firms who deviate from the hedonic investment.

The objective in this paper is to provide a solution concept that is more non-cooperative and doesn't rely on the coordination implicit in the definition of the core. One way to motivate the core 'non-coooperatively' is to observe that the ex-post competitive solution is in the core. The ex-post competitive solution is one that assigns a price to each worker after the workers' investments are realized, such that there is no excess demand or supply for any worker. This competitive solution can punish a deviating worker by lowering his ex post wage. Of course, this has the same basic flaw as the hedonic solution. The firm who is supposed to pay the lowest wage to the marginal worker should realize that he can offer something lower before the marginal worker will prefer unemployment. In fact, the non-cooperative procedure that the competitive solution is supposed to approximate is the one discussed above in which workers invest first, then first compete non-cooperatively for these workers ex post. As explained above in conjunction with the discussion of Kalai's large game paper, the limit properties of such a two stage model will look exactly like the results discussed here.

The paper Peters (2004b) is similar in structure to this one. It assumes that all workers have the same preferences, and that all firms have the same profit functions, then shows that when there are more workers than firms, Nash equilibria lead to inefficient investments in large games. The primary difference here is that workers and firms have private types that determine their strategies. So the analysis is based on pure strategy equilibria rather than mixed equilibria. However, the important difference is that the distribution of private types makes it possible to compare the equilibrium allocations directly with hedonic allocations. In particular, the Bayesian analysis provides the truncated hedonic equilibrium concept that can be used to analyze large games. We exploit this concept here to do number of comparative static exercises.

A paper that uses a non-cooperative approach is Felli and Roberts (2000). They design a bidding game that occurs after firms and workers initial investments are realized. A worker at the bottom of the distribution who cuts his or her investment understands that this will affect the outcome of the bidding game. Workers Bertrand compete their surplus away by making wage offers to firms. So each worker earns a surplus that is equal to what they could attain at their next best alternative. For example, the marginal worker ends up with the surplus that he or she could earn by being unemployed. When she considers raising her investment ex ante, she expects to receive that value of that investment to an unemployed worker. Since the marginal value of the investment is much higher with the firm with whom she is ultimately employed, she will end up under investing in equilibrium.

Finally, an approach that seems at first glance to be reasonable (but for which there is no literature of which I am aware) is to imagine that firms commit themselves ex ante to wage schedules that reward investments differentially ex post. The lowest firm type could then commit itself to a schedule that punishes deviations by a worker, thus supporting the hedonic outcome. Of course, the firm might regret ex post having committed itself to cutting wages, or refusing to hire the otherwise profitable worker just because the worker was not investing as the hedonic equilibrium requires that they do. However, since deviations would no longer be profitable for workers, the firm would be happy to offer such a commitment. This approach could certainly be used to support the hedonic outcome. Apart from the inherent implausibility of firms offering to commit themselves simply because it will support an equilibrium, the difficulty with this approach is that these commitments can be used to support many different outcomes, not just the hedonic equilibrium. To see how this might work in the simultaneous move game, pick some arbitrary investment level and wage which is profitable for the marginal worker and lowest firm type. Suppose the firm writes a contract that commits it to pay this arbitrary wage if the worker has this arbitrary investment level, but commits to pay the worker nothing if he or she has a lower investment. This is a sensible contract for the firm as long as no worker actually invests less than this arbitrary level. No worker wants to invest less because it won't be paid.

Finally, we make a distinction between the model here in which worker investments are valuable to firms and signaling models with two sided investment, similar to Hopkins (2005) or Hoppe, Moldovanu, and Sela (2005) in which worker investments serve only to signal the unobservable productive type of the worker. The hedonic approach cannot be directly applied to signaling, and signaling is always wasteful[3] so different issue are involved in that literature.

1. Fundamentals

The market consists of m firms and n workers with $n > m$. Each firm has a privately known characteristic x. If is commonly believed that these are independently drawn from a distribution F on a closed connected interval $X = [\underline{x}, \overline{x}] \subset \mathbb{R}^+$. This characteristic measures the value of worker investment to the firm. Firms with higher types have higher marginal value for worker human capital. Similarly, each worker has a type y that affects his or her investment cost. Again it is assumed that these are independently drawn from a distribution G on a closed connected interval $Y = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}^+$. The distributions F and G are both assumed to be differentiable, with both F' and G' uniformly bounded above.

Each firm has a single job that it wants to fill with one worker. Each worker wants to fill one job. In order to match, firm i chooses a wage $w_i \in W \subset \mathbb{R}^+$. Each worker j chooses a human capital investment $h_j \in H \subset \mathbb{R}^+$. Workers and firms are then matched assortatively, with the most skilled worker (the worker with the highest h_j) being hired by the firm with the highest wage, and similarly for lower wages and investments. Ties are resolved by flipping coins.[4]

[3]Though one might ask as Hoppe, Moldovanu, and Sela (2005) do whether market participants would be better off if signaling were abandoned and participants randomly matched.

[4]Since attention is focused on symmetric equilibrium in which all traders use monotonic strategies, ties occur with zero probability. The tie breaking rule is then inconsequential.

Payoffs for firms and workers depend on their characteristic, their investment or wage, and on the investment or wage of the partner with whom they are eventually matched. The payoff of a firm who offers wage w_i and is matched with a worker of type h_j is

$$v\left(x_j\right) h_j - w_i \tag{1.1}$$

where $v\left(x_j\right)$is a monotonically increasing function of x_j that is bounded away from 0 on X. The corresponding payoff for a worker whose investment is h_j who finds a job at wage w_i with a firm of type x is

$$w_i - c(h_j)\tau\left(y_j\right) \tag{1.2}$$

where c is a non-negative strictly increasing convex function of h_j with bounded derivative and $\tau\left(\cdot\right)$ is a decreasing function. Marginal costs are assumed to be uniformly bounded in the sense that $0 \leq \frac{\partial c(h_j)}{\partial h_j} < B$ for all $y_j \in Y$, all $h_j \in H$, where B is some finite positive number.

We assume that each worker begins the game with what is effectively an endowment $h^*\left(y_j\right) \geq 0$ for each y_j. This is an accumulation of human capital the worker makes no matter what he expects to happen in the matching game. Assume that $h^*\left(\underline{y}\right) = 0$ and that $\frac{\partial h^*(y_j)}{\partial y_j} > 0$. The assumption that $h^*\left(\underline{y}\right) = 0$ is used simply to shorten some of the formulas below. The assumption that different worker types have different endowments rules out trivial self fulfilling equilibria in which no one invests because all firms are expected to offer the same wage, which they do because they don't expect anyone to invest.

Finally;

Assumption 1. *The sets H and W are bounded intervals and $H \times W$ contains all pairs (h, w) such that $v\left(x_i\right) h - y \geq 0$ and $w - c\left(h\right)\tau\left(y\right) \geq 0$ for some pair (x, y).*

The matching game is a simultaneous move game among all workers and firms. Each firm picks a non-negative wage, each worker pick a non-negative investment. Then workers and firms are matched assortatively. For simplicity, we assume the firm hires the best available worker ex post even if this involves a loss.

2. Bayesian Equilibrium

We begin with an analysis of the Bayesian game.[5] In what follows, h and w refer to strategy rules, $\tilde{h}$ and $\tilde{w}$ refer to vectors of size n and

[5]An alternative and very sensible model would involve workers choosing human capital in the first stage, while firms compete for realized human capital levels ex post. The main difference would be that in the latter case workers should realize that their own investments would affect the equilibrium of the subgame among

m respectively of realized investments and wages. Similarly $\tilde{y}$ and $\tilde{x}$ are vectors of realized types for workers and firms respectively, while y_j and x_i refer to specific types. For any vector $x \in \mathbb{R}_n$, $x_{k:n}$ means the k^{th} lowest element of the vector.

Attention is restricted to symmetric equilibrium in which individuals on the same side of the market all use the same strategy rule. For the moment assume that these strategy rules are monotonic, so that ties occur with probability 0. Then the outcome a trader enjoys depends on his or her rank in the distribution of realized types. Suppose this rank is k, so that $k-1$ of the others have lower types. Then $k-1$ of the others should have lower investments if all workers are using the same monotonic strategy. For future reference, note that if a worker has rank $k \leq n-m$, then there are at least m workers with higher rank, and they will take up all the positions offered by the m firms. Otherwise, the worker will be matched with the firm whose rank is $k-(n-m)$,

Let $h : Y \to H$ be the investment strategy used by workers. A Bayesian equilibrium for the workers' investment game is a strategy rule h such that $h\left(\underline{y}\right) = h^*\left(\underline{y}\right) = 0$, and for almost every $y_j, y' \in Y$,

$$\sum_{t=n-m}^{n} \begin{pmatrix} n-1 \\ t \end{pmatrix} G^t\left(y_j\right)\left(1-G\left(y_j\right)\right)^{n-1-t} \mathbb{E}\tilde{w}_{t+1-(n-m):m} - c\left(h\left(y_j\right)\right)\tau\left(y_j\right) \geq$$

$$\sum_{t=n-m}^{n} \begin{pmatrix} n-1 \\ t \end{pmatrix} G^t\left(y'\right)\left(1-G\left(y'\right)\right)^{n-1-t} \mathbb{E}\tilde{w}_{t+1-(n-m):m} - c\left(h\left(y'\right)\right)\tau\left(y_j\right).$$

The initial condition follows from the fact that if $h\left(\underline{y}\right) > h^*\left(\underline{y}\right)$, then the worst worker type could cut investment without affecting his rank in for any array of types of the other workers.

The first order condition necessary condition for maximization yields the differential equation

$$h'\left(y_i\right) c'\left(h\left(y_i\right)\right)\tau\left(y_j\right) = \sum_{t=n-m}^{n-1} \gamma_t'\left(y_i\right)\mathbb{E}\tilde{w}_{t-(n-m)+1:m} \tag{2.1}$$

where

$$\gamma_t\left(y_i\right) = \begin{pmatrix} n-1 \\ t \end{pmatrix} G^t\left(y_i\right)\left(1-G\left(y_i\right)\right)^{n-1-t}.$$

Under our assumptions, the Lipschitz condition holds, and the differential equation has a unique solution. Since each of the γ' functions

firms. Since we are mostly interested in large games where this effect is small, it seems sensible to stick the simpler simultaneous game form.

integrates to γ, integrate both sides to get

$$\int_{\underline{y}}^{y} h'(y_i)\, c'(h(y_i))\, \tau(y_i)\, dy_i = \sum_{t=n-m}^{n-1} \gamma_t(y_i)\, \mathbb{E}\tilde{w}_{t-(n-m)+1:m}. \tag{2.2}$$

Integrate the left hand side by parts and use the boundary condition to get

$$\tau(y_i)\, c(h(y_i)) - \int_{\underline{y}}^{y_i} c(h(\tilde{y}))\, \tau'(\tilde{y})\, d\tilde{y} = \overline{w}(y_i), \tag{2.3}$$

where $\overline{w}(y_i)$ is the right hand side of (2.2).

A similar argument applies for firms. The equilibrium condition is

$$v(x_i) \sum_{t=0}^{m-1} \binom{m-1}{t} F^t(x_i)\,(1-F(x_i))^{m-1-t}\, \mathbb{E}\tilde{h}_{t+(n-m+1):n} - w(x_i) \geq$$

$$v(x_i) \sum_{t=0}^{m-1} \binom{m-1}{t} F^t(x_i')\,(1-F(x_i'))^{m-1-t}\, \mathbb{E}\tilde{h}_{t+(n-m+1):n} - w(x_i') \tag{2.4}$$

The first order conditions for this problem yield the differential equation

$$w'(x_i) = v(x_i) \sum_{t=0}^{m-1} \phi_t'(x_i)\, \mathbb{E}\tilde{h}_{t+(n-m+1):n} \tag{2.5}$$

where

$$\phi_t(x_i) = \binom{m-1}{t} F^t(x_i')\,(1-F(x_i'))^{m-1-t}.$$

Integrating by parts and using the boundary condition $w(\underline{x}) = 0$ gives the formula

$$w(x_i) = v(x_i)\, \overline{h}(x_i) - v(\underline{x})\, \overline{h}(\underline{x}) - \int_{\underline{x}}^{x_i} v'(\tilde{x})\, \overline{h}(\tilde{x})\, d\tilde{x} \tag{2.6}$$

where

$$\overline{h}(\tilde{x}) = \sum_{t=0}^{m-1} \phi_t(\tilde{x})\, \mathbb{E}\tilde{h}_{t+(n-m+1):n}$$

is the expected quality of the firm's partner when the firm follows its equilibrium strategy.

Since the equilibrium strategies are increasing, the worst firm will offer the lowest wage with probability 1. It will then match with the worker who has the $n-m+1^{st}$ lowest investment among workers. So $\overline{h}(\underline{x}) = \mathbb{E}\tilde{h}_{(n-m+1):n}$, which gives the equilibrium wage function as

$$w(x_i) = v(x_i)\,\overline{h}(x_i) - v(\underline{x})\,\mathbb{E}\tilde{h}_{(n-m+1):n} - \int_{\underline{x}}^{x} v'(\tilde{x})\,\overline{h}(\tilde{x})\,d\tilde{x}. \tag{2.7}$$

Since the functions $\overline{w}(y_i)$ and $\overline{h}(x_j)$ are respectively the expected wage paid to a worker of type y_i when he plays his equilibrium strategy, and the expected quality of the worker with whom a firm of type x_j matches when it plays its equilibrium strategy. Simple rearrangement gives the equilibrium payoff of the same worker and firm as $-\int_{\underline{y}}^{y_i} c(h(\tilde{y}))\,\tau'(\tilde{y})\,d\tilde{y}$ and $v(\underline{x})\,\mathbb{E}\tilde{h}_{(n-m+1):n} + \int_{\underline{x}}^{x} v'(\tilde{x})\,\overline{h}(\tilde{x})\,d\tilde{x}$ respectively.

Theorem 2. *A symmetric Bayesian equilibrium (h, w) exists for which both h and w are monotonically increasing differentiable strategy rules.*

There are two parts to the proof. The first is to show that a pair of strategy rules exist that simultaneously satisfy the differential equations (2.1) and (2.5). The proof follows the method in Peters (2004b) so we simply sketch the argument. Begin with arbitrary vectors (h^e, w^e) of expected order statistics for investment and wages and use these to compute solutions to (2.1) and (2.5). The Lipschitz conditions guarantee that these solutions exist. These solutions vary (sup norm) continuously as the parameters (h^e, w^e) change. Recompute the vectors of expected order statistics using the solutions to (2.1) and (2.5). The mapping from vectors of expected order statistics to expected order statistics defined this way is continuous in the usual sense. The boundedness of the set of feasible investments and wages ensures that this is a continuous mapping from a compact set into itself. The existence of a fixed point is then ensured by the Brouwer fixed point theorem.

The second part of the proof is to show that the first order conditions are also sufficient. This follows in a straightforward way from the monotonicity conditions on preferences.

We are interested in the hedonic relationships

$$\hat{w}(h') = \{\overline{w}(y') : h(y') = h'\}$$

and

$$\hat{h}(w') = \left\{\overline{h}(x') : w(x') = w'\right\}$$

that are traced out by the Bayesian equilibrium of the matching investment game. The basic idea is to check whether these resemble the hedonic relationship associated with competitive equilibrium. Of course, this is only likely to be true when the number of workers and firms is very large. For this reason we consider the 'limits' of the various functionals as the number of workers and firms becomes large.

Index the game by the number of firms, m, and suppose that the number of workers $n > m$. We want to consider sequences of games like this chosen such that the ratio of workers to firms converges to $\gamma > 1$ as m goes to infinity. By Theorem 2, there is for each m a pair of monotonic and differentiable Bayesian equilibrium strategies w_m and h_m. The equilibria define hedonic return functions $\hat{w}_m(\cdot)$ and $\hat{h}_m(\cdot)$ as defined above. They are both differentiable with derivatives bounded between 0 and the slope of the steepest worker or firm indifference curve on the set $H \times W$. This follows from the fact that the hedonic return function is tangent to some worker or firm indifference curve at every point in the support of equilibrium distributions. The slope of the workers' indifference curves are all equal to the marginal investment costs $\frac{\partial c(h)\tau(y_j)}{\partial h}$ which we have assumed to be bounded on $H \times W$. The implication is that the family of functions $\hat{w}_m$ is equi-continuous. Then each sub-sequence of function $\hat{w}_m$ that has a pointwise limit has a continuous pointwise limit. The same argument applies to the sequence $\hat{h}_m$. We are interested in the degree to which the pointwise limits of these function resemble standard hedonic return functions.

To emphasize how the limits of these return functions are unusual, it is useful to define them with reference to the weak limits of the Bayesian equilibrium strategy rules. In finite matching games, these strategy rules are continuous, differentiable, strictly increasing and bounded above for every m from (2.5) and (2.1). By Helly's compactness theorem, there are then sub-sequences of these strategy rules that converge weakly to right continuous non-decreasing functions. We will proceed with a theorem that characterizes the strategy rules h and w that are candidates to be weak limits of sequences of equilibrium strategy rules.

3. Properties of the Limits of the Bayesian Equilibrium

In this section, we describe some of the properties of the functions $\hat{w}_\infty$, $\hat{h}_\infty$, w_∞ and h_∞ where $\hat{w}_\infty$ and $\hat{h}_\infty$ are the continuous limits of some subsequence of equilibrium payoff functions, while w_∞ and h_∞ are weak limits of some sub-sequence of equilibrium strategy rules. Proofs that are not explicitly included in the text are in Appendix 2.

The ratio γ of workers to firms that prevails in the limit is greater than 1, so there will be unemployed workers. Definey_0 to be the worker type such that $\gamma(1 - G(y_0)) = 1$. In the limit, the measure of the set of workers with types at least y_0 will be the same as the measure of the set of firms. Hence we expect workers with types higher than y_0

to match, while workers with types below this will remain unmatched. Define $h_0 \equiv h_\infty(y_0)$ and $w_0 \equiv w_\infty(\underline{x})$ and refer to these as the lowest investment and wage that are *realized in the limit*.[6] When it is convenient, we refer to the worker of type y_0 are the *marginal worker type.* We begin with some intuitively straightforward results. The proofs are in the appendix.

Lemma 3. *$h_\infty(y') = h^*(y')$ for each $y' < y_0$.*

Only the highest m worker types will match. The probability that a worker with type $y' < y_0$ will be one of the highest m worker types goes to zero with m. Since the worker shouldn't expect to match, there is no point making an investment beyond that which is privately optimal.

The next Lemma is a consequence of the fact that the equilibrium return function must separate the types below and above y_0.

Lemma 4. *For each $h' \in [h^*, h_0]$, $\hat{w}_\infty(h') - c(h')\tau(y_0) = -c(h^*(y_0))\tau(y_0)$*

Each of the payoff functions $\hat{w}_m$ and $\hat{h}_m$ is a hedonic return function. One obvious reason that they don't look like Rosen's hedonic return function is that there are two such functions instead of one. The next two theorems show how these two are reconciled in the limit. They show how a Rosen like hedonic return function can be constructed from limits of finite payoff functions.

Lemma 5. *For each $y' \geq y_0$, $\lim_{m\to\infty} \hat{w}_m(h_m(y')) - c(h_m(y'))\tau(y')$ is equal to $\max_h \hat{w}_\infty(h) - c(h)\tau(y')$.*

This Lemma says that if we knew the limit of the return function that the worker faced in the Bayesian equilibrium, then we could reconstruct the weak limit of his strategy rule by finding his best outcome on this return function.

For the next Lemma, define for each $y \geq y_0$,

$$\pi(y) = \{x : \gamma(1 - G(y)) = F(x)\}$$

We refer to a seller of type $\pi(y)$ as the *hedonic partner* of a worker of type y. The next Lemma shows how to construct the limit return function from the weak limits of the strategy rules and the hedonic partner function.

Lemma 6. *$\hat{w}_\infty(h_\infty(y')) = w_\infty(\pi(y'))$ for each $y' \geq y_0$.*

[6] This terminology is slightly misleading, primarily because of the nature of weak convergence. First, when there are finitely many players, equilibrium strategy rules are monotonic and continuous, so all wages and investments above 0 are in the support of the equilibrium strategies. Second, by weak convergence, the limit of type y_0's equilibrium investment will be less than or equal to h_0.

Parallel arguments establish exactly the same results for firms:

Lemma 7. *(i) for each* $w' \in [w^*, w_0]$*,* $v(\underline{x})\,\hat{h}_\infty(w') - w' = v(\underline{x})\,\hat{h}_\infty(w_0) - w_0$*;*
(ii) for each $x' \geq \underline{x}$*,* $\lim_{m\to\infty} v(x')\,\hat{h}_m(w_m(x')) - w_m(x')$ *is equal to* $\max_w v(x')\,\hat{h}_\infty(w) - w$*;*
(iii) $\hat{h}_\infty(w_\infty(x')) = h_\infty(\pi^{-1}(x'))$ *for each* $x' \geq \underline{x}$*.*

One final Lemma is important in the description of equilibrium.

Lemma 8. $h_0 > h^*(y_0)$ *and* $w_0 > w^*(\underline{x})$*.*

3.1. **The lowest wage and investment.** The characterization so far does not provide any information on the investment h_0 made by the marginal worker, or the wage w_0 paid by the lowest firm type beyond the fact that this wage is positive and the investment h_0 is larger than the privately optimal investment of the marginal worker. This section aims to provide more information on these values. Ultimately, the bound established in this section will be used to show that Bayesian equilibrium need not converge to hedonic equilibrium.

Consider the limits $\hat{w}_\infty(h')$ and $\hat{h}(w')$ defined on the intervals $[h^*(y_0), h_0]$ and $[0, w_0]$ respectively. It is difficult to pin down the relationship between these two functions because the set of worker and firm types who make these investments in equilibrium vanishes in the limit. However, from Lemma 7, $\hat{h}(w')$ is a linear function which coincides with the graph of an iso-profit curve (in (h, w) space) for the worst firm through (h_0, w_0). Specifically, it is the iso-profit curve that passes through the point (h_0, w_0). Similarly the function $\hat{w}_\infty$ as defined on this region coincides with the graph of an indifference curve of the marginal worker passing through the point $(h^*(y_0), 0)$ (and through the point (h_0, w_0)). Let $\omega : H \to W$ be the graph of this indifference curve.

The payoff function $\hat{w}_\infty$ can be decomposed in a useful way. For each m, Bayesian equilibrium strategies are monotonic functions of worker type with $h_m(\underline{y}) = 0$. For large enough m, $h_m(y') \geq h_0$ for each $y' > y_0$. As a result, for each $h' \in (h^*(y_0), h_0)$ and each m, there is some worker type who invests h' in equilibrium. Workers with higher types invest more, workers with lower types invest less. As a result, when m is large, there is some chance that the investment h' will not result in a match at all. Let $Q_m(h')$ be the probability that an investment h' results in a match in a game with m firms. The function $Q_m(h')$ gives the probability that $n - m$ of the other $n - 1$ workers invest less than h' in equilibrium. Let $W_m(h')$ be the wage the worker expects to be paid *conditional on matching* in the same game. Then by definition $\hat{w}_m(h') = Q_m(h') W_m(h')$. Apart from the fact that $W_m(h') \geq \hat{w}_m(h')$, it is difficult to say much about the wage function $W_m(h')$. However, when m is large, a matched worker with investment $h' < h_0$ is very unlikely to match with a firm offering a wage above w_0 since the partners of such firms have investments that converge in probability to something that is at least as large as h_0. This fact provides a useful bound.

Lemma 9. *The expected investment of the partner of a firm who offers wage* 0 *is bounded above as follows*

$$\hat{h}_\infty(0) \leq \int_{h^*(y_0)}^{h_0} h' \frac{d\omega(h')}{w_0}.$$

To understand this bound, consider a worker who makes an investment h' that lies between $h^*(y_0)$ and $h(y_0)$. This investment will result in a match if it is larger than the $n-m^{th}$ order statistic of the other workers' investments. So $Q_m(h') = \Pr\left\{\tilde{h}_{n-m:n-1} < h'\right\}$. If the worker does get a job and m is very large, the probability that she matches with a firm whose type is larger than $\underline{x}$ must go to zero, since every such firm must eventually match with his or her hedonic partner with high probability. This ensures that the wage she receives conditional on matching is less than w_0 with arbitrarily high probability when m is large.

Thus for m large enough, it must be that

$$\hat{w}_m(h') \leq \Pr\left\{\tilde{h}_{n-m:n-1} < h'\right\} w(\underline{x}).$$

Then taking limits, exploiting the idea that the marginal worker has to be indifferent to all investments between $h^*(y_0)$ and $h(y_0)$, and that the marginal worker matches with her hedonic partner if she makes her equilibrium investment, gives

$$\frac{\lim_{m\to\infty} \hat{w}_m(h')}{w(\underline{x})} = \frac{\omega(h')}{\omega(h(y_0))} < \lim_{m\to\infty} \Pr\left\{\tilde{h}_{n-m:n-1} < h'\right\}.$$

The reason this inequality is useful is that the expected quality of the worst firm's partner when it offers the bilateral Nash wage w^* is the expected value of the $n-m+1^{st}$ order statistic of workers investment. The inequality gives a stochastic bound on the distribution of this order statistic, which bounds the expectation.

4. Truncated Hedonic Equilibrium

We now provide a device for characterizing the limits of Bayesian equilibrium strategy rules.

Definition 10. A *truncated hedonic equilibrium* is an investment level $h_0 > 0$, a strictly increasing function $\hat{w} : [h_0, \infty] \to W$, and a non-negative constant $\beta \leq 1$ such that

(1) for each $h' \geq h_0$,

$$G\left(\left\{y' : \arg\max_{h_i \geq h_0} \hat{w}(h_i) - c(h_i)\tau(y') \geq h'\right\}\right)$$

is equal to

$$F\left(\left\{x' : \arg\max_{h_i \geq h_0} v(x')h_i - \hat{w}(h_i) \geq h'\right\}\right);$$

and

(2) the wage $\hat{w}(h_0)$ satisfies

$$v(\underline{x})h_0 - \hat{w}(h_0) = \beta v(\underline{x}) \int_0^{h_0} h' \frac{d\omega(h')}{\omega(\underline{x})} \tag{4.1}$$

and

$$\hat{w}(h_0) - c(h_0)\tau(y_0) = -c(h^*(y_0))\tau(y_0)$$

The motivation for defining truncated hedonic equilibrium is the following theorem:

Theorem 11. *Let $\{h_m, w_m\}$ be a sequence of Bayesian equilibrium investment strategies that converge weakly to a pair of non-decreasing functions $\{h, w\}$. Then there is a truncated hedonic equilibrium $(\beta, h_0, \hat{w})$ such that $h_0 = h(y_0)$ and $\hat{w}(h(y)) = w(\pi(y))$ for each $y \geq y_0$.*

Condition 1 in the Theorem follows from Lemma 6 and its counterpart in Lemma 7. Condition 2 follows from Lemmas 9 and 4. The constant β given in the second condition reflects the inequality in Lemma 9. We turn now to the relationship between truncated hedonic equilibrium and hedonic equilibrium as a means of characterizing the limits of Bayesian Nash equilibrium.

5. Truncated Hedonic Equilibrium and Hedonic Equilibrium

An hedonic equilibrium is simply a competitive allocation for a continuum of different goods. It has two characteristics, markets clear, and the resulting allocation is pareto optimal. Condition 1 in Definition 10 is the market clearing condition. Pareto optimality requires a bit more.

Let (h^{**}, w^{**}) be the investment wage pair that maximizes

$$v(h', \underline{x}) - w'$$

subject to the constraint that

$$w' - c(h', y_0) \geq -c(h^*(y_0))\,\tau(y_0).$$

This is the wage investment pair that is bilaterally efficient for the worst firm type and the marginal worker type.

Definition 12. A *hedonic equilibrium* is a return function $\hat{w}_c(h)$ that satisfies the market clearing condition 1 along with the boundary condition $\hat{w}_c(h^{**}) = w^{**}$.

An immediate implication of Theorem 11 is the following:

Theorem 13. *A necessary condition for a sequence $\{h_m, w_m\}$ of Bayesian equilibrium strategies to converge to a hedonic equilibrium is*

$$v(\underline{x}) \int_{h^*(y_0)}^{h^{**}} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{\omega(h^{**})} - w^*(\underline{x}) \geq v(\underline{x})\, h^{**} - w^{**}.$$

This follows immediately from the fact that Bayesian equilibrium strategies converge to truncated hedonic equilibrium, and the fact that Condition 2 in Definition 10 can only be satisfied for some β less than 1 if the condition of the theorem is satisfied.

For the purposes of this paper, it is useful to focus on environments in which this necessary condition fails. In that case we can provide an unambiguous relationship between hedonic and truncated hedonic equilibrium To make the statements of some of the theorems a little less awkward, we refer to environments in which the necessary condition given in Theorem 13 fails as *regular* environments.

Regular environments are no more or less plausible than irregular ones - regularity refers to the fact that all truncated hedonic equilibrium have the same properties in regular environments. Whether or not an environment is regular depends on the shapes of workers' indifference curves (iso profit curves are all linear) and on the distributions of worker and firm types. The results we establish below surely apply in some irregular environments, however the methods provided here cannot be used to characterize them.

Proposition 14. *In every regular environment, and in every truncated hedonic equilibrium in that environment, every firm pays a wage that is higher than it would pay in a hedonic equilibrium. A positive measure of workers invest h_0 and are matched with firms who pay $\hat{w}(h_0)$. For each of the workers and firms in this pool, investments and wages are too high in the sense that any pair in the pool would both be strictly* *better off at a lower wage and investment. All other workers invest the same amount in both the hedonic and truncated hedonic equilibrium.*

Proof. We begin by showing that in the regular case, for any $\beta \leq 1$, there is a unique investment $h_0 > h^{**}$ consistent with condition 2 in the definition of Truncated Hedonic Equilibrium.

Define

$$\overline{h}(h') \equiv \beta \int_{h^*(y_0)}^{h'} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{\omega(h')}. \tag{5.1}$$

For any $h' \geq h^{**}$ define

$$\alpha(h') = \max\left\{\tilde{h} : v(\underline{x})\,\tilde{h} = v(\underline{x})\,h' - \omega(h')\right\} \tag{5.2}$$

By condition 2, h_0 must satisfy $\alpha\left(\overline{h}(h_0)\right) = h_0$. In the regular case, $\alpha\left(\overline{h}(h^{**})\right) > h^{**}$. Since $\overline{h}$ is increasing, and α is decreasing for $h' \leq \overline{h}(h^{**})$, $\alpha\left(\overline{h}(h')\right) > h'$ for each $h' \leq h^{**}$. Since there is evidently an $h' > \alpha\left(\overline{h}(h^{**})\right)$ such that $\alpha(h') = h^{**}$, there must be a unique $h_0 > h^{**}$ such that $\alpha\left(\overline{h}(h_0)\right) = h_0$.

Since (h^{**}, w^{**}) maximizes the worst firm's payoff conditional on the marginal worker receiving his outside option, and $h_0 > h^{**}$, it follows by continuity that there is a pair (h', w') with $h' < h^{**}$and $w' < w^{**}$ such that both the marginal worker and the worst firm are strictly better off with this new wage investment pair.

Finally, for each worker $y' \geq y_0$,

$$h(y') = \arg\max_{h_i \geq h_0} \left\{\hat{w}(h_i) - c(h_i)\,\tau(y')\right\},$$

so that in any truncated hedonic equilibrium $v(\pi(y_i)) \leq \hat{w}'(h(y')) \leq c'(h(y'))\,\tau(y')$ where $h(y')$ is the equilibrium strategy of a worker of type $y' > y_0$. From the result above, $h_0 > h^{**}$ in a regular environment, so $v(\underline{x}) < \hat{w}'(h(y_0)) < c'(h(y_0))\,\tau(y_0)$. By continuity, there is a non-degenerate interval of types $[y_0, y_1]$ for which this is true.

Consider $h' > h_0$. If some type y' makes the investment h' in the truncated hedonic equilibrium, then

$$v(\pi(y')) = \hat{w}'(h') = c'(h')\,\tau(y').$$

A similar condition holds with respect to the hedonic return function $\hat{w}_c$. Since the function on the right hand side of this last expression is strictly decreasing in y' while the function on the left hand side is strictly increasing, there is at most one value for y'such that $v(\pi(y')) = c'(h')\,\tau(y')$. As a consequence if $h' > h_0$ is an investment that is made by some worker type in both the hedonic and truncated hedonic equilibrium, then h' is made by the same type in each case, and $\hat{w}'(h') = \hat{w}'_c(h')$. Since $\hat{w}(h_0) > w^{**}$ and $\hat{w}'(h') = \hat{w}'_c(h')$ for each $h' > h_0$, it follows that $\hat{w}(h') > \hat{w}_c(h')$ for each investment $h' > h_0$ that is made along the equilibrium path in both equilibrium. □

This proposition provides a comparison between hedonic equilibrium and truncated hedonic equilibrium in regular environments. The truncation caused by the competition at the lower end of the wage and investment distribution drives up wages and generates inefficient outcomes for many worker and firm types.

An important property of this theorem is the apparent 'pool' in the distribution of investments and wages that this predicts. This isn't pooling in the sense in which this term in normally used. For every m no matter how large, all workers and firms use monotonic strategies. So in each finite game there is complete separation. However, when the number of workers and firms is large, the equilibrium investment rises very quickly with type at first, then rises quite slowly for workers whose types are close to that of the marginal worker. As a consequence a lot of workers and firms appear to use the same wage and investment in the limit.

5.1. **How to use Truncated Hedonic Equilibrium.** In the regular case, truncated hedonic equilibrium provides a relatively simple way of characterizing the limits of Bayesian Nash equilibria of the investment game. The boundary condition provides a testable restriction on wage and investment distributions since it suggests there should be something that looks like an atom at the bottom of the distribution. One problem with using truncated hedonic equilibrium in this way is that truncated hedonic equilibrium supports multiple outcomes. In the regular case, these outcomes are all similar with respect to the appearance of the wage and investment distributions, and the efficiency of worker investments. The point of this section is to show that these similarities are robust enough to do simple comparative static exercises.

To begin, focus on the constant β in the definition. This constant is required because the limit theorem only provides a bound on the distribution of the $n - m + 1^{st}$ order statistic that determines the worst firm's payoff when it offers 0 wage. Since the limit theorem doesn't provide an exact bound, truncated hedonic equilibrium is defined for all possible values of $\beta < 1$. However, since β uniquely determines the minimum investment h_0 of workers, the following result can be used to provide some additional structure.

Theorem 15. *Let* $(h_0, \hat{w}, 1)$ *and* $\left(h_0^\beta, \hat{w}_\beta, \beta\right)$ *be a pair of truncated hedonic equilibria for which* $\beta < 1$*. Then* $h_0 < h_0^\beta$ *and* $\hat{w}_\beta(h) > \hat{w}(h)$ *for each* $h > h_0^\beta$*.*

Proof. Borrowing from the proof of Proposition 14

$$\overline{h}_\beta(h') \equiv \beta \int_{h^*(y_0)}^{h'} \tilde{h} d\frac{\omega\left(\tilde{h}\right)}{\omega(h')}.$$

The bound h_0 satisfies $\alpha\left(\overline{h}_1(h_0)\right) = 0$. Now simply repeat the proof of Proposition 14 to show that there is a unique $h_0^\beta > h_0$ that satisfies $\alpha\left(\overline{h}_\beta\left(h_0^\beta\right)\right) = h_0^\beta$. As in Proposition 14, $\hat{w}'_\beta(h) = \hat{w}'(h)$ for $h > h_0^\beta$, from which the rest of the result follows. □

The implication of this theorem is that many of the interesting properties of the limits of Bayesian equilibrium for the investment game (at least in the regular case) can be discovered by characterizing truncated hedonic equilibrium in which $\beta = 1$. For example;

Proposition 16. *There exist non-degenerate distributions* G *and* F *of worker and firm types such that all workers make the same investment and all firms offer the same wage in the hedonic equilibrium in which* $\beta = 1$*.*

Proof. Since $h_0 > h^{**}$, $v(\underline{x}) < \hat{w}'(h_0) < c(h_0)\tau(y_0)$ in the hedonic equilibrium by condition 1 in the definition. Then there is an open interval $[y_0, y_1]$ for which $v(\pi(y')) < \hat{w}'(h_0) < c(h_0)\tau(y')$ for $y' \in [y_0, y_1)$. If the upper bounds of the supports of G and F are both less than y_1 and $\pi(y_1)$ respectively, the result follows. □

It is an immediate corollary of Theorem 15 that the same result is true for any $\beta < 1$, though, of course, the wage and investment that prevails in equilibrium will be higher the lower the value of β.

Finally, we illustrate a simple comparative static result by manipulating the truncated hedonic equilibrium directly. The exercise illustrates an interesting property of both hedonic and truncated hedonic equilibrium - what happens at the bottom of the wage and investment distribution can change everything else. It also illustrates an interesting difference between hedonic and truncated hedonic equilibrium. In particular, we impose a minimum wage. To make the result interesting, we imagine this minimum wage is below the lowest wage w^{**} that prevails in a pure hedonic equilibrium. The imposition of such a wage will not affect the hedonic equilibrium at all. Since the lowest wages in a truncated hedonic equilibrium are higher than w^{**}, this minimum wage will also be below the lowest wage that prevails in the truncated hedonic equilibrium. However, the truncated hedonic equilibrium will be affected by the minimum wage, as the following proposition shows.

Proposition 17. *In a truncated hedonic equilibrium in which $\beta = 1$, the imposition of a minimum wage will raise equilibrium investments of the lowest worker types who match, raise all wages, and increase the size of the pool of workers and firms at the bottom of the wage distribution.*

Proof. In the initial equilibrium with $\beta = 1$

$$v\left(\underline{x}\right) h_0 - w_0 = v\left(\underline{x}\right) \int_{h^*(y_0)}^{h_0} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{w_0}$$

and

$$w_0 - c\left(h_0\right) \tau\left(y_0\right) = -c\left(h^*\right) \tau\left(y_0\right).$$

Suppose the minimum wage is $\overline{W} > 0$. Then the worst firm must offer $\overline{W}$ in equilibrium. It is immediate that

$$v\left(\underline{x}\right) h_0 - w_0 > v\left(\underline{x}\right) \int_{h^*(y_0)}^{h_0} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{w_0} - \overline{W}.$$

It is straightforward to use the method in Theorem 14 to show that there is a unique pair $(h', w') > (h_0, w_0)$ satisfying

$$v\left(\underline{x}\right) h' - w' = v\left(\underline{x}\right) \int_{h^*(y_0)}^{h'} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{w'} - \overline{W}$$

and

$$w' - c\left(h'\right) \tau\left(y_0\right) = -c\left(h^*\right) \tau\left(y_0\right).$$

Suppose that in the initial equilibrium, workers with types between y_0 and y_1 and firms with types between $\underline{x}$ and x_1 pool at (h_0, w_0). The types y_1 and x_1 are chosen so that the worker indifference curve and firm iso profit curve are tangent at (h_0, w_0). As marginal costs are increasing, worker y_1's indifference curve is steeper than firm x_1's iso profit curve at (h', w'), so more worker and firm types must be pooled to support the return function for higher investments and wages. □

6. Conclusion

The paper shows a number of things. First, in regular environments, Bayesian Nash equilibrium of the matching investment game do not converge to hedonic equilibrium. Instead they converge to truncated hedonic equilibria. Truncated hedonic equilibrium involve bunching of the lowest types, both in terms of the wages that they pay (and receive), and in terms of the investments that workers make. The wages and investments used by the lowest types are too high in the sense that both workers and firms with these types would be made strictly better off if they could lower both wages and investments. Wages are higher in truncated hedonic equilibrium than they are in hedonic equilibrium. Since the most profitable firms match with the same workers, and since these same workers have the same investments as they do in the hedonic equilibrium (for the highest types only), firms profits are lower in a truncated hedonic equilibrium than they are in a simple hedonic equilibrium.

7. Appendix 2: - Proofs

Lemma 18. *Let $\underline{y} < y_a < \overline{y}$ and $\underline{y} < y_b < \overline{y}$. The probability that the number of other workers whose types are at least y_a exceeds $n(1 - G(y_b))$ converges to one if $y_b > y_a$ and converges to zero if $y_b < y_a$. Similarly, the probability that the number of firms whose types are at least x_a exceeds $m(1 - F(x_b))$ converges to one if $x_b > x_a$and converges to zero if $x_b < x_a$.*

Proof. The number of the $n - 1$ workers whose type exceeds y_a is a random variable with mean

$$(n - 1)(1 - G(y_a))$$

and variance

$$(n - 1) G(y_a)(1 - G(y_a)).$$

As n grows large, this random variable becomes approximately normal in the sense that for any x, the probability that the number of workers whose type exceeds y exceeds x converges to the probability that a standard normal random variable exceeds

$$\frac{x - (n - 1)(1 - G(y_a))}{\sqrt{(n - 1) G(y_a)(1 - G(y_a))}}.$$

Evaluating this for $x = n(1 - G(y_b))$, and replacing $n - 1$ by n verifies that the probability that the number of workers whose type exceeds y

is greater than m converges to the probability that a standard normal random variable exceeds

$$\frac{n\left(1-G\left(y_b\right)\right)-n\left(1-G\left(y_a\right)\right)}{\sqrt{nG\left(y_a\right)\left(1-G\left(y_a\right)\right)}}=\sqrt{n}\frac{G\left(y_a\right)-G\left(y_b\right)}{\sqrt{G\left(y_a\right)\left(1-G\left(y_a\right)\right)}}$$

which goes to minus infinity when $y_a < y_b$ and plus infinity when $y_a > y_b$. The probability then converges to one in the first case and zero in the second. The proof is identical for firms. □

7.1. **Proof of Lemma 3.**

Proof. The equilibrium strategies are monotonically increasing. So a worker of type $y' < y_0$ will find a job only if the number of other workers who have types above y' is less than $m = n\left(1-G\left(y_0\right)\right)$. Let $y_a = y'$ and $y_b = y_0$ and apply Lemma 18 to conclude that the the probability that the number of other workers whose types exceed y' is at least $m = n\left(1-G\left(y\right)\right)$ converges to one with m. It follows that a worker of type y' matches with very low probability when m is large, and thus can't profitably invest more than h^* in the limit. □

7.2. **Proof of Lemma 4.**

Proof. If $\hat{w}_\infty\left(h_\infty\left(y_0\right)\right)-c\left(h_\infty\left(y_0\right),y_0\right) > -c\left(h^*,y_0\right)$, then for some $y' < y_0$ and m large enough,

$$\hat{w}_m\left(h_\infty\left(y_0\right)\right)-c\left(h_\infty\left(y_0,y'\right)\right) > -c\left(h^*,y'\right)$$

The equilibrium payoff of a worker of type y' is given by

$$\hat{w}_m\left(h_m\left(y'\right)\right)-c\left(h_m\left(y'\right),y'\right)$$

By Lemma 3 and the continuity of the utility function, this converges to $-c\left(h^*,y'\right)$. As a consequence, a profitable deviation must exist for worker y' for large enough m, a contradiction.

Similarly, if $\hat{w}_\infty\left(h_\infty\left(y_0\right)\right)-c\left(h_\infty\left(y_0\right),y_0\right) < -c\left(h^*,y_0\right)$, a worker of type $y' > y_0$ must find it profitable for large enough m to cut investment to h^*. □

7.3. **Proof of Lemma 5.**

Proof. If the latter term is larger, there is immediately a profitable deviation from $h_m\left(y'\right)$ when m is large enough. If the former term is larger, let $\overline{h} = \lim_{m\to\infty} h_m\left(y'\right)$. From the contrary hypothesis, there is some m_0 large enough so that

$$\hat{w}_{m'}\left(h_{m'}\left(y'\right)\right)-c\left(h_m\left(y'\right),y'\right) > \max_h \hat{w}_\infty\left(h\right)-c\left(h,y'\right)+\epsilon$$

for each $m' > m_0$. Since the family of functions $\hat{w}_m$ is equi-continuous, there is for every $\epsilon' > 0$ a $\delta > 0$ such that $|h - h'| < \delta$ implies that $|\hat{w}_m(h) - \hat{w}_m(h')| < \epsilon'$ for all m. Find δ such that

$$\left|\hat{w}_m(h') - c(h', y') - \left(\hat{w}_m\left(\overline{h}\right) - c\left(\overline{h}, y'\right)\right)\right| < \epsilon$$

for all m when $\left|h' - \overline{h}\right| < \delta$. Then choose m' for large enough so that $\left|h_m(y') - \overline{h}\right| < \delta$. Then for all $m' > m_0$.

$$\hat{w}_{m'}\left(\overline{h}\right) - c\left(\overline{h}, y'\right) > \max_h \hat{w}_\infty(h) - c(h, y')$$

Taking the pointwise limit gives a contradiction. □

The next theorem provides the 'hedonic' part of the limit outcome. Let $y' \geq y_0$ and $\pi(y') = \{x : 1 - F(\pi(y')) = \tau(1 - G(y'))\}$. The firm type $\pi(y')$ is the 'hedonic partner' of a worker of type y', i.e., it is the firm type such that the measure of the set of firms who have larger types than $\pi(y')$ is equal to the measure of the set of workers who have better types than y'. By definition, $\pi(y_0) = \underline{x}$.

7.4. **Proof of Lemma 6.**

Proof. In equilibrium, both workers and firms use monotonically increasing strategies. So the k^{th} lowest worker type will match with the $k - (n-m)^{th}$ lowest firm type. For any $y' \geq y_0$, let $x' > \pi(y')$. The probability with which a worker of type y matches with a firm of type x'or better is equal to the probability with which the number of firms whose type is at least x' exceed the number of workers whose type is at least y'. Let $\pi(y') < x'' < x'$. By Lemma 18, the probability that the number of firms with types above x' exceeds $m(1 - F(x''))$converges to zero with m. On the other hand the probability that the number of workers with types above y' exceeds $m(1 - F(x'')) = n(1 - G(\pi(y'')))$ converges to 1 by Lemma 18, where $\pi(y'') = x''$. Hence, for any $x' > \pi(y')$, the probability with which a worker of type y' matches with a firm whose type is x' or above converges to zero. A similar argument establishes that the probability that a worker matches with a firm whose type is less than $\pi(y')$ also converges to zero (or is zero if $y' = y_0$ because $\pi(y_0) = \underline{x}$).

The functions h_∞ and w_∞ are both non-decreasing. So they are both continuous except at countably many points. Then in any open interval to the right of y' and $x' > \pi(y')$, there are points y'' and $x'' > \pi(y'')$ such that h_∞ is continuous at y'' and w_∞ is continuous at x''. By weak convergence, $\lim_{m\to\infty} w_m(x'') = w_\infty(x'')$ and $\lim_{m\to\infty} h_m(y'') =$

$h_\infty(y'')$. As $x'' > \pi(y'')$, by the reasoning in the previous paragraph,

$$\lim_{m\to\infty} \hat{w}_m(h_m(y'')) - c(h_m(y''), y'') < \lim_{m\to\infty} w_m(x'') - c(h_m(y''), y'')$$
$$= w_\infty(x'') - c(h_\infty(y''), y'')$$

The limit on the left hand side is a continuous function of type. Since h_∞ and w_∞ are both right continuous, we have

$$\lim_{m\to\infty} \hat{w}_m(h_m(y')) - c(h_m(y'), y') \leq w_\infty(\pi(y')) - c(h_\infty(y'), y')$$

Let $h'' > h_\infty(y')$. h_∞ is a weak limit of a sequence of increasing continuous functions h_m. Then it is right continuous. As a result, there must exist an open interval $I^+_{h''}(y')$ such for every y'' in the interval, $y'' > y'$ and $h'' > h_m(y'')$ for infinitely many m. A worker who invests h'' will match as if his type were better than y'' . By the reasoning of the previous paragraph, this means that investment h''will lead to a match with a firm whose type is at least $\pi(y'') > \pi(y')$ with probability converging to one with m. Since w_∞ is non-decreasing, it can have at most countably many points at which it is discontinuous. Hence for any h'' it is possible to choose y'' such that $w_\infty(\cdot)$ is continuous at the point $\pi(y'')$. Since w_∞ is the weak limit of the sequence of functions w_m and w_∞ is continuous at $\pi(y'')$, $w_\infty(\pi(y''))$ is also the pointwise limit of $w_m(\pi(y''))$. Then the payoff to the investment h'' must be converging to something at least as large as $u(h'')(a - y') + w_\infty(\pi(y''))$. Since $I^+_{h'''}(y') \subset I^+_{h''}(y')$ when $h''' < h''$, we can choose a decreasing sequence (h''_n, y''_n) converging to $(h_\infty(y'), y')$ for which this inequality holds, it follows by the right continuity of w_∞ that

$$\lim_{m\to\infty} \hat{w}_m(h_m(y')) - c(h_m(y'), y') \geq u(h_\infty(y'))(a - y') + w_\infty(\pi(y')).$$

Putting these two arguments together gives

$$\lim_{m\to\infty} \hat{w}_m(h_m(y')) = w_\infty(\pi(y'))$$

From the fact that h_∞ has only countably many discontinuities, there is for every y' a $y'' > y'$ arbitrarily close to y' such that $h_\infty(\cdot)$ is continuous at y''. Hence $\lim_{m\to\infty} h_m(y'') = h_\infty(y'')$. Now using equi-continuity of the family $\hat{w}_m$, for any ϵ, there is a δ such that $\left|\hat{w}_m\left(\tilde{h}\right) - \hat{w}_m(h_\infty(y''))\right| < \epsilon$ for every m provided $\left|\tilde{h} - h_\infty(y'')\right| < \delta$. Choose m' such that

$$|h_m(y'') - h_\infty(y'')| < \delta.$$

It follows that from m bigger than m' $|\hat{w}_m(h_m(y'')) - \hat{w}_m(h_\infty(y''))| < \epsilon$. So $\lim_{m\to\infty} \hat{w}_m(h_m(y'')) = \hat{w}_\infty(h_\infty(y''))$. The result then follows from the fact that the limit on the left (equilibrium payoff) is a continuous function of type, $\hat{w}_\infty$ is continuous, and $h_\infty(y)$ is right continuous. $\square$

7.5. **Proof of Lemma 8.**

Proof. From Lemmas 6

$$\hat{w}_\infty(h_\infty(y_0)) - c(h_\infty(y_0), y_0) = w_0 - c(h_0, y_0).$$

From Lemma 4

$$w_0 - c(h_0, y_0) = -c(h^*(y_0), y_0).$$

Similarly from Lemma 7

$$v(\underline{x})\, h_0 - w_0 = v(\underline{x})\, \hat{h}(w') - w'$$

for each $w' \leq w_0$. By the definition of equilibrium, it must be that for any $y' > y_0$,

$$\hat{w}_\infty(h_\infty(y')) - c(h_\infty(y'), y_0) \leq w_0 - c(h_0, y_0)$$

and

$$v(\underline{x})\, \hat{h}_\infty\left(w_\infty\left(\pi^{-1}(y')\right)\right) - w_\infty\left(\pi^{-1}(y')\right) =$$
$$v(\underline{x})\, h_\infty(y') - \hat{w}_\infty(h_\infty(y')) < v(\underline{x})\, h_0 - w_0.$$

In words, this says that the market return function $\hat{w}(h)$ for $h > h_0$ must lie below the indifference curve of a worker of type y_0 through the point (h_0, w_0) and above the iso-profit curve of a firm of type $\underline{x}$ through the point (h_0, w_0). Since this property cannot be satisfied at the point $(h^*(y_0), w^*(\underline{x}))$, the result follows. $\square$

7.6. **Proof of Lemma 19.** The following Lemma is needed in the proof of the main theorem. By definition $Q_m(h') = \Pr_m\left\{\tilde{h}_{n-m+1:n} \leq h'\right\}$.

Lemma 19. $\Pr_m\left\{\tilde{h}_{n-m+1:n} \leq h'\right\}$ *converges pointwise (hence weakly) to* $\Pr_m\left\{\tilde{h}_{n-m:n-1} \leq h'\right\}$ *as* m *goes to infinity.*

Proof. Let y_m be the (unique) type such that $h_m(y_m) = h'$. By Lemma 3, $h_\infty(y) = y^*$ for every $y < y_0$. Then for any $h' > h^*$, there is m large enough such that $y_{m'} \geq y_0$ for $m' > m$.

$$\Pr_m\left\{\tilde{h}_{n-m+1:n} \leq h'\right\} =$$

$$= \int_{\underline{y}}^{y_m}\left\{1 - \left(\frac{1 - G(y_m)}{1 - G(z)}\right)^m\right\}\frac{(n-1)!}{(n-m-1)!\,(m-1)!}G(z)^{n-m-1}(1 - G(z))^{m-1}g(z)\,dz$$

The logic in this expression is based on the idea that the probability that $\tilde{h}_{n-m+1:n} \leq h'$ is equal to the expectation over all possible values for $\tilde{h}_{n-m:n-1}$ of the probability that at least one of the remaining m independent draws of $\tilde{h}$ lies in the interval $\left[\tilde{h}_{n-m:n-1}, h'\right]$. Substituting gives

$$\Pr_m\left\{\tilde{h}_{n-m+1:n} \leq h'\right\} = \Pr_m\left\{\tilde{h}_{n-m:n-1} \leq h'\right\} -$$

(7.1)

$$\int_{\underline{y}}^{y_m} \left(\frac{1-G(y_m)}{1-G(z)}\right)^m \frac{(n-1)!}{(n-m-1)!(m-1)!} G(z)^{n-m-1}(1-G(z))^{m-1} g(z)\,dz$$

Since the binomial term $\frac{(n-1)!}{(n-m-1)!(m-1)!} G(z)^{n-m-1}(1-G(z))^{m-1}$ is a probability, it is bounded above by 1, as is the function $\left(\frac{1-G(y_m)}{1-G(z)}\right)^m$. Define the function

$$\psi_m(z) = \begin{cases} \left(\frac{1-G(y_m)}{1-G(z)}\right)^m \frac{(n-1)!}{(n-m-1)!(m-1)!} G(z)^{n-m-1}(1-G(z))^{m-1} & z \leq y_m \\ 0 & \text{otherwise} \end{cases}$$

and evaluate the integral in (7.1) as

$$\int_{\underline{y}}^{\overline{y}} \psi_m(x)\, g(z)\,dz$$

As $\psi_m(z)$ is bounded above by the constant function 1 and converges almost everywhere to zero with m we conclude from the dominated convergence theorem that

$$\lim_{m\to\infty} \int_{\underline{y}}^{\overline{y}} \psi_m(z)\, g(z)\,dz = 0$$

Using this when taking the limit in (7.1) then gives the result. □

7.7. **Proof of Lemma 9.**

Proof. The limit payoff associated with any investment $h' \in (h^*, h_0)$ is given by

$$\lim_{m\to\infty} \hat{w}_m(h') = \lim_{m\to\infty} \Pr_m\left\{\tilde{h}_{n-m:n-1} \leq h'\right\} W_m(h').$$

Since the limit of a product is the product of the limits

$$\lim_{m\to\infty} \Pr_m\left\{\tilde{h}_{n-m:n-1} \leq h'\right\} W_m(h') =$$

$$\lim_{m\to\infty} \Pr_m\left\{\tilde{h}_{n-m:n-1} \leq h'\right\} \lim_{m\to\infty} W_m(h').$$

Since $h' < h_m(y')$ for any $y' > y_0$ and m large enough, this last expression is less than or equal to

$$\lim_{m\to\infty} \Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\} \lim_{m\to\infty} W_m\left(h_m\left(y'\right)\right).$$

Since a worker of type y' matches with probability 1 in the limit, $\lim_{m\to\infty} W_m\left(h_m\left(y'\right)\right) = \lim_{m\to\infty} \hat{w}_m\left(h_m\left(y\right)\right)$. By Lemma 6,

$$\lim_{m\to\infty} \hat{w}_m\left(h_m\left(y\right)\right) = \hat{w}_\infty\left(h_\infty\left(y'\right)\right).$$

Putting these things together gives

$$\lim_{m\to\infty} \Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\} W_m\left(h'\right) \leq \lim_{m\to\infty} \Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\} \hat{w}_\infty\left(h_\infty\left(y'\right)\right),$$

or, since this is true for all $y' > y_0$,

$$\lim_{m\to\infty} \hat{w}_m\left(h'\right) \leq \lim_{m\to\infty} \Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\} w_0 = \lim_{m\to\infty} \Pr_m \left\{ \tilde{h}_{n-m+1:n} \leq h' \right\} w_0$$

from Lemma 19. Finally, by Lemma 4, $\lim_{m\to\infty} \hat{w}_m\left(h'\right) = \omega\left(h'\right)$, so that

$$\frac{\omega\left(h'\right)}{w_0} \leq \lim_{m\to\infty} \Pr_m \left\{ \tilde{h}_{n-m+1:n} \leq h' \right\}.$$

In words, what this says is that the limit distribution of the $n-m+1^{st}$ order statistic of worker investments is stochastically dominated by the distribution function that is defined by the ratio $\frac{\omega(h')}{w_0}$. The result in the theorem then follows by taking expectations using this distribution. $\square$

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