# A NON-COOPERATIVE APPROACH TO HEDONIC EQUILIBRIUM

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Abstract. This paper studies Bayesian equilibrium in a worker firm matching problem in which workers choose their human capital investment and firms choose wages before the matching process occurs. Symmetric equilibrium exists, and supports assortative matching. When the number of traders is large, the highest types invest efficiently, while low types invest too much and receive wages that are too high in the sense that both the worker and firm could have been better off if they had agreed ex ante on a lower wage and investment. The environment is then an unusual one in the sense that large numbers of traders is not enough to support outcomes close to the competitive equilibrium.

## 1. Introduction

This paper considers a matching problem with two sided investments. This means that traders on both sides of the market choose characteristics that are costly to them in order to influence the quality of the partner with whom they ultimately match. There are a couple of recent papers that illustrate why investment in problems like this remains inefficient in equilibrium even when the number of traders on each side of the market is very large. Peters (2004) examines a model in which traders on the same side of the market are all identical, and in which utility isn't transferable post match. The paper shows that both sides of the market tend to over invest when there are many traders being matched. Felli and Roberts (2000) study an environment with transferable utility and show that inefficiency persists with large numbers of traders, though in their case the inefficiency is of the kind usually associated with the holdup problem, i.e., traders tend to under invest. This is in sharp contrast to competitive models of the two sided investment problem, where efficiency is guaranteed (Peters and Siow (2002) or Han (2002)).

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This paper involves a generalization of the argument in Peters (2004), which presents the basic over investment argument. Peters (2004) is restricted to an environment in which all traders on the same side of the market have the same preferences. As such, it permits only very limited distributions of investment, and in particular, makes it difficult to address the relationship between non-cooperative equilibrium and hedonic (fully competitive) equilibrium. The arguments in this paper borrow from that paper, but are designed to address this relationship. Investments are first determined by Bayesian equilibrium in a game in which traders' preferences are randomly drawn from known distributions. In this way, a rich variety of different investments occur in equilibrium. The 'hedonic' return to different levels of investment can then be calculated. Sequences of such equilibria are studied as the number of traders on each side of the market become large in order to characterize the way the hedonic return changes as traders' strategic impact on the outcome disappears.

The results are interesting in a number of ways. First, unlike most problems of this kind that have been studied in the literature, large numbers alone do not eliminate the inefficiencies associated with strategic play. In a matching problem, the strategic advantage that any player enjoys is limited by the existence of alternative partners who are very close substitutes. A worker in a labour market can't drop his investment in education because there is likely to be some other worker who does have education who can take his place. As the number of traders grows large, this other worker is likely to be a closer and closer substitute for him. It is tempting for this reason to jump immediately to a competitive solution. However, the analysis of the finite game illustrates that this is not appropriate. The reason is that the worst workers and firms retain their strategic advantage as the market becomes large. The logic is explained in Peters (2004). Some worker will end up with the lowest investment, no matter how large the market is. However, this worker will still be considerably better than the any of the workers who never expected to match in the first place. Assortative matching will mean that the firm with whom this worker is supposed to match in equilibrium won't be able to compete away any of the better workers in the market. So if this worst worker cuts investment, the worst firm really won't have any alternative but to hire this worst worker. Since better workers investments are driven up by the workers below them, this causes investments to fall for higher types, and the competitive solution must unravel.

The result illustrates one of the differences between the competitive approach to markets and the game theoretic approach. The competitive approach simply chooses off equilibrium payoffs to support the outcome that it wants to prevail. In a hedonic equilibrium (Rosen (1974)), the worst worker in the market thinks that if he cuts investment, the wage that he receives will fall because he will be forced to match with a lower wage firm. The non-cooperative analysis simply points out that there will not be such a firm. It tries instead to model the off equilibrium payoffs in an economically reasonable way. The pre-match investment problem is a great example of an environment in which this makes a difference.

Finally, the non-cooperative analysis of the pre-match investment problem provides much more than a critique of competitive theory. The results below illustrate how the competitive solution can be modified to account for this change in off equilibrium payoffs. The theorems below show how to characterize the limits of Bayesian equilibria when there are many traders by specifying a pair of strategy rules then checking to see whether the return that each trader type expects by making his best reply investment is equal to the best reply investment of his *hedonic* partner. The hedonic partner of a worker of type $y$ is the firm of type $\pi(y)$ such that the measure of firms with types better than $\pi(y)$ is equal to the measure of workers with types better than $y$. Strategy rules that satisfy this condition support allocations that resemble a *truncated hedonic equilibrium* as described in Peters (2006).

This limit equilibrium has a number of interesting properties. The allocation it supports looks very much like a competitive equilibrium for the best types in the market. They match assortatively, and choose investments that are bilaterally Pareto optimal. The market return to investment fully internalizes the externality that leads to the holdup problem in a bilateral relationship. However, the worst types in the market behave in a very non-competitive way. Though the Bayesian equilibrium assures that these low types are assortatively matched, in the limit, it will appear as if there is a lot of pooling at the lower end of the market. Many of the lowest types will make nearly the same investment and be paid nearly the same wage.

However, for most of these types, the investments that they make will be too large, in the sense that post match, the firm would have preferred to pay a lower wage to a worker with a lower level of investment and the worker would have been more than happy to accept such a compromise. This prediction about the low end of the market is one of the testable consequences of this theory. It predicts some thing that ought to look roughly like a mass point in the distribution of wages and investments that lies strictly above the minimum wage. For example, this is consistent with dual peaks in the distribution of human capital investment at high school and university level.

It is also possible to make a comparison between the investment and wage levels that prevail in the limit of Bayesian equilibrium outcomes, and those that prevail in a simple competitive equilibrium. All worker types invest more and are paid higher wages in the limits of Bayesian equilibrium than they are in competitive equilibrium.

## 2. Fundamentals

The market consists of $m$ firms and $n$ workers with $n > m$. Each firm has a privately known characteristic $x$. If is commonly believed that these are drawn from a distribution $F$ on a closed connected interval $X = [\underline{x}, \overline{x}] \subset \mathbb{R}^+$. This characteristic measures the value of worker investment to the firm. Firms with higher types have higher marginal value for worker human capital. Similarly, each worker has a type $y$ that affects his or her investment cost. Again it is assumed that these are independently drawn from a distribution $G$ on a closed connected interval $Y = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}^+$. The distributions $F$ and $G$ are both assumed to be differentiable, with both $F'$ and $G'$ uniformly bounded above.

Each firm has a single job that it wants to fill with one worker. Each worker wants to fill one job. In order to match, firm $i$ chooses a wage $w_i \in W \subset \mathbb{R}^+$. Each worker $j$ chooses a human capital investment $h_j \in H \subset \mathbb{R}^+$. Workers and firms are then matched assortatively, with the most skilled worker (the worker with the highest $h_j$) being hired by the firm with the highest wage, and similarly for lower wages and investments. Ties are resolved by flipping coins.[1]

Payoffs for firms and workers depend on their characteristic, their investment or wage, and on the investment or wage of the partner with whom they are eventually matched. The payoff of a firm who offers wage $w_i$ and is matched with a worker of type $h_j$ is

$$v(x_j) h_j - w_i \tag{2.1}$$

where $v(x_j)$is a monotonically increasing function of $x_j$ that is bounded away from 0 on $X$. The corresponding payoff for a worker whose investment is $h_j$ who finds a job at wage $w_i$ with a firm of type $x$ is

$$w_i - c(h_j)\tau(y_j) \tag{2.2}$$

where $c$ is a strictly convex increasing function of $h_j$ with bounded derivative and $\tau(\cdot)$ is a decreasing function. Furthermore, marginal

[1] Since attention is focused on symmetric equilibrium in which all traders use monotonic strategies, ties occur with zero probability. The tie breaking rule is then inconsequential.

costs are assumed to be uniformly bounded in the sense that $0 \leq \frac{\partial c(h_j)}{\partial h_j} < B$ for all $y_j \in Y$, all $h_j \in H$, where $B$ is some finite positive number.

**Assumption 1.** *The sets $H$ and $W$ are bounded intervals and $H \times W$ contains all pairs $(h, w)$ such that $v(x_i)\, h - y \geq 0$ and $w - c(h)\, \tau(y) \geq 0$ for some pair $(x, y)$.*

The matching game is a simultaneous move game among all workers and firms. Each firm picks a non-negative wage, each worker pick a non-negative investment. Then workers and firms are matched assortatively. For simplicity, we assume that firms always hire the best available worker ex post, even if this involves a loss.

## 3. Bayesian Equilibrium

We begin with an analysis of the Bayesian game.[2] In what follows, $h$ and $w$ refer to strategy rules, $\tilde{h}$ and $\tilde{w}$ refer to vectors of size $n$ and $m$ respectively of realized investments and wages. Similarly $\tilde{y}$ and $\tilde{x}$ are vectors of realized types for workers and firms respectively, while $y_j$ and $x_i$ refer to specific types. For any vector $x \in \mathbb{R}_n$, $x_{k:n}$ means the $k^{th}$ lowest element of the vector.

Attention is restricted to symmetric equilibrium in which individuals on the same side of the market all use the same strategy rule. For the moment assume that these strategy rules are monotonic, so that ties occur with probability 0. Then the outcome a trader enjoys depends on his or her rank in the distribution of realized types. Suppose this rank is $k$, so that $k-1$ of the others have lower types. Then $k-1$ of the others should have lower investments if all workers are using the same monotonic strategy.

If $k \leq n-m$, then there are at least $m$ workers with higher rank, and they will take up all the positions offered by the $m$ firms. Otherwise, the worker will be matched with the firm whose rank is $k-(n-m)$,

Let $h : Y \rightarrow H$ be the investment strategy used by workers. A Bayesian equilibrium for the workers' investment game is a strategy

[2] An alternative and very sensible model would involve workers choosing human capital in the first stage, while firms compete for realized human capital levels ex post. The main difference would be that in the latter case workers should realize that their own investments would affect the equilibrium of the subgame among firms. Since we are mostly interested in large games where this effect is small, it seems sensible to stick the simpler simultaneous game form.

rule $h$ such that $h\left(\underline{y}\right)=0$ and for almost every $y_j, y' \in Y$,

$$\sum_{t=n-m}^{n} \binom{n-1}{t} G^t\left(y_j\right)\left(1-G\left(y_j\right)\right)^{n-1-t} \mathbb{E}\tilde{w}_{t+1-(n-m):m} - c\left(h\left(y_j\right)\right)\tau\left(y_j\right) \geq$$

$$\sum_{t=n-m}^{n} \binom{n-1}{t} G^t\left(y'\right)\left(1-G\left(y'\right)\right)^{n-1-t} \mathbb{E}\tilde{w}_{t+1-(n-m):m} - c\left(h\left(y'\right)\right)\tau\left(y_j\right).$$

The initial condition follows from the fact that if $h\left(\underline{y}\right)>0$, then the worst worker type could cut investment without affecting his rank in for any array of types of the other workers.

The first order condition for maximization yields the differential equation

$$h'\left(y_i\right)c'\left(h\left(y_i\right)\right)\tau\left(y_j\right) = \sum_{t=n-m}^{n-1} \gamma_t'\left(y_i\right)\mathbb{E}\tilde{w}_{t-(n-m)+1:m} \tag{3.1}$$

where

$$\gamma_t\left(y_i\right) = \binom{n-1}{t} G^t\left(y_i\right)\left(1-G\left(y_i\right)\right)^{n-1-t}.$$

Under our assumptions, the Lipshitz condition holds, and the differential equation has a unique solution. Since each of the $\gamma'$ functions integrates to $\gamma$, we can integrate both sides to get

$$\int_{\underline{y}}^{y} h'\left(y_i\right)c'\left(h\left(y_i\right)\right)\tau\left(y_i\right)dy_i = \sum_{t=n-m}^{n-1} \gamma_t\left(y_i\right)\mathbb{E}\tilde{w}_{t-(n-m)+1:m}. \tag{3.2}$$

Integrating by parts and using the boundary condition, gives

$$\tau\left(y_i\right)c\left(h\left(y_i\right)\right)h\left(y_i\right) - \int_{\underline{y}}^{y_i} c\left(h\left(\tilde{y}\right)\right)h\left(\tilde{y}\right)\tau'\left(\tilde{y}\right)d\tilde{y} = \overline{w}\left(y_i\right),$$

where $\overline{w}\left(y_i\right)$ is the right hand side of (3.2).

A similar argument applies for firms. The equilibrium condition is

$$v\left(x_i\right)\sum_{t=0}^{m-1} \binom{m-1}{t} F^t\left(x_i\right)\left(1-F\left(x_i\right)\right)^{m-1-t} \mathbb{E}\tilde{h}_{t+(n-m+1):n} - w\left(x_i\right) \geq$$

$$v\left(x_i\right)\sum_{t=0}^{m-1} \binom{m-1}{t} F^t\left(x_i'\right)\left(1-F\left(x_i'\right)\right)^{m-1-t} \mathbb{E}\tilde{h}_{t+(n-m+1):n} - w\left(x_i'\right) \tag{3.3}$$

The first order conditions for this problem yield the differential equation

$$w'\left(x_i\right) = v\left(x_i\right)\sum_{t=0}^{m-1} \phi_t'\left(x_i\right)\mathbb{E}\tilde{h}_{t+(n-m+1):n} \tag{3.4}$$

where

$$\phi_t(x_i) = \begin{pmatrix} m-1 \\ t \end{pmatrix} F^t(x_i')(1-F(x_i'))^{m-1-t}.$$

Integrating by parts and using the boundary condition $w(\underline{x}) = 0$ gives the formula

$$w(x_i) = v(x_i)\overline{h}(x_i) - v(\underline{x})\overline{h}(\underline{x}) - \int_{\underline{x}}^{x_i} v'(\tilde{x})\overline{h}(\tilde{x})\,d\tilde{x} \tag{3.5}$$

where

$$\overline{h}(\tilde{x}) = \sum_{t=0}^{m-1}\phi_t(\tilde{x})\,\mathbb{E}\tilde{h}_{t+(n-m+1):n}$$

is mean worker investment.

Since the equilibrium strategies are increasing, the worst firm will offer the lowest wage with probability 1. It will then match with the worker who has the $n-m+1^{st}$ lowest investment among workers. So $\overline{h}(\underline{x}) = \mathbb{E}\tilde{h}_{(n-m+1):n}$, which gives the equilibrium wage function as

$$w(x_i) = v(x_i)\overline{h}(x_i) - v(\underline{x})\,\mathbb{E}\tilde{h}_{(n-m+1):n} - \int_{\underline{x}}^{x} v'(\tilde{x})\overline{h}(\tilde{x})\,d\tilde{x}.$$

Observe that the functions $\overline{w}(y_i)$ and $\overline{h}(x_j)$ are respectively the expected wage paid to a worker of type $y_i$ when he plays his eqilibrium strategy, and the expected quality of the worker with whom a firm of type $x_j$ matches when it plays its equilibrium strategy. Simple rearrangement gives the equilibrium payoff of the same worker and firm as $-\int_{\underline{y}}^{y_i} c(h(\tilde{y}))\,h(\tilde{y})\,\tau'(\tilde{y})\,d\tilde{y}$ and $v(\underline{x})\,\mathbb{E}\tilde{h}_{(n-m+1):n} + \int_{\underline{x}}^{x} v'(\tilde{x})\overline{h}(\tilde{x})\,d\tilde{x}$ respectively.

**Theorem 2.** *A Bayesian equilibrium $(h, w)$ exists for which both $h$ and $w$ are monotonically increasing differentiable strategy rules.*

The proof follows the method in Peters (2004) so we simply outline the argument. Begin with arbitrary vectors $(h^e, w^e)$ of expected order statistics for investment and wages and use these to compute solutions to (3.1) and (3.4). The Lipschitz conditions guarantee that these solutions exist. These solutions vary (sup norm) continuously as the parameters $(h^e, w^e)$ change. Recompute the vectors of expected order statistics using the solutions to (3.1) and (3.4). The mapping from vectors of expected order statistics to expected order statistics defined this way is continuous in the usual sense. The boundedness of the set of feasible investments and wages ensures that this is a continuous mapping from a compact set into itself. The existence of a fixed point is then ensured by the Brouwer fixed point theorem.

We are interested in the hedonic relationships $\hat{w}(h') = \{\overline{w}(y') : h(y') = h'\}$ and $\hat{h}(w') = \left\{\overline{h}(x') : w(x') = w'\right\}$ that are traced out by the Bayesian equilibrium of the matching investment game. The basic idea is to check whether these resemble the hedonic relationship associated with competitive equilibrium. Of course, this is only likely to be true when the number of workers and firms is very large. For this reason we consider the 'limits' of the various functionals as the number of workers and firms becomes large.

Index the game by the number of firms, $m$, and suppose that the number of workers $n > m$. We want to consider sequences of game like this chosen such that the limit of the ratio of workers to firms converges to $k > 1$ as $m$ goes to infinity. By Theorem 2, there is for each $m$ a pair of monotonic and differentiable Bayesian equilibrium strategies $w_m$ and $h_m$. The equilibria define hedonic return functions $\hat{w}_m(\cdot)$ and $\hat{h}_m(\cdot)$ as defined above. They are both differentiable with derivatives bounded between 0 and the slope of the steepest worker or firm indifference curve on the set $H \times W$. This follows from the fact that the hedonic return function is tangent to some worker or firm indifference curve at every point in the support of equilibrium distributions. The slope of the workers' indifference curves are all equal to the marginal investment costs $\frac{\partial c(h)\tau(y_j)}{\partial h}$ which we have assumed to be bounded on $H \times W$. The implication is that the family of functions $\hat{w}_m$ is equi-continuous. Then each sub-sequence of function $\hat{w}_m$ that has a pointwise limit has a continuous pointwise limit. The same argument applies to the sequence $\hat{h}_m$. We are interested in the degree to which the pointwise limits of these function resemble standard hedonic return functions.

To emphasize how the limits of these return functions are unusual, it is useful to define them with reference to the weak limits of the Bayesian equilibrium strategy rules. In finite matching games, these strategy rules are continuous, differentiable, strictly increasing and bounded above for every $m$ from (3.4) and (3.1). By Helly's compactness theorem, there are then sub-sequences of these strategy rules that converge weakly to right continuous non-decreasing functions. We will proceed with a theorem that characterizes the strategy rules $h$ and $w$ that are candidates to be weak limits of sequences of equilibrium strategy rules.

## 4. Truncated Hedonic Equilibrium

We now provide a device for characterizing the limits of Bayesian equilibrium strategy rules. To understand this result, define $y_0$ to be the solution to $k(1 - G(y_0)) = 1$. With assortative matching, the worker with type $y_0$ would be the lowest worker type to be employed if there were a continuum of workers of mass $k$ distributed as $G$, and a continuum of firms of mass 1. We refer to the worker of type $y_0$ as the marginal worker. Let $\omega(h)$ be the graph of the marginal worker's indifference curve through the origin in $(h, w)$ space. This is the indifference curve the marginal worker would attain if he made no investment and was not offered a job.

**Definition 3.** A *truncated hedonic equilibrium* is an investment level $h_0 > 0$, a strictly increasing function $\hat{w} : [h_0, \infty] \to W$, and a non-negative constant $\beta \leq 1$ such that

(1) for each $h' \geq h$,

$$G\left(\left\{y' : \arg\max_{h_i \geq h_0} \hat{w}(h_i) - c(h_i)\tau(y') \leq h'; \max_{h_i \geq h_0} \hat{w}(h_i) - c(h_i)\tau(y') \geq 0\right\}\right)$$

is equal to

$$F\left(\left\{x' : \arg\max_{h_i \geq h_0} v(x') h_i - \hat{w}(h_i) \leq h'\right\}\right);$$

and

(2) the wage $\hat{w}(h_0)$ satisfies

$$v(\underline{x}) h_0 - \hat{w}(h_0) = \beta v(\underline{x}) \int_0^{h_0} h' \frac{d\omega(h')}{\omega(\underline{x})}. \tag{4.1}$$

For each $y \geq y_0$, let $\pi(y) = \{x : k(1 - G(y) = 1 - F(x))\}$. The motivation for defining truncated hedonic equilibrium is the following theorem:

**Theorem 4.** *Let $\{h_m, w_m\}$ be a sequence of Bayesian equilibrium investment strategies that converge weakly to a pair of non-decreasing functions $\{h, w\}$. Then there is a truncated hedonic equilibrium $(\beta, h_0, \hat{w})$ such that $h_0 = h(y_0)$ and $\hat{w}(h(y)) = w(\pi(y))$ for each $y \geq y_0$.*

The first condition in the definition of truncated hedonic equilibrium looks exactly like a standard hedonic equilibrium as it might be defined in (Rosen 1974), (Peters and Siow 2002) or (**?**). We discuss the relationship with hedonic equilibrium in more detail below. The main difference between the two solution concepts is the bound given in condition 2. To understand this bound, consider a worker who makes an investment $h'$ that lies between $h^*(y_0)$ and $h(y_0)$ (it is shown in the appendix that these two must be distinct). This will result in a match if it is larger than the $n - m^{th}$ order statistic of the other workers' investments. Let $\Pr\left\{\tilde{h}_{n-m:n-1} < h'\right\}$ be the probability that this $n - m^{th}$ order statistic is less than or equal to $h'$. If the worker does get a job and $m$ is very large , she will end up matching with one of the very low type firms since every worker whose type is larger than $y_0$ eventually investments more than $h'$. The exact wage she gets will depend on exactly what the order statistic of her investment is, and what the realized types of the firms on the other side of the market happen to be. It is impossible to say much about the wage she will receive when she matched, even when the number of workers and firms is very large. However, this wage is naturally bounded above by the wage $w\left(\underline{x}\right)$ since the wage she receives cannot exceed the wage offers of any positive fraction of firms in the limit.

Thus we have

$$\hat{w}_m\left(h'\right) \leq \Pr\left\{\tilde{h}_{n-m:n-1} < h'\right\} w\left(\underline{x}\right).$$

Then taking limits, exploiting the idea that the marginal worker has to be indifferent to all investments between $h^*\left(y_0\right)$ and $h\left(y_0\right)$, and that the marginal worker matches with her hedonic partner if she makes her equilibrium investment, gives

$$\frac{\lim_{m\to\infty}\hat{w}_m\left(h'\right)}{w\left(\underline{x}\right)} = \frac{\omega\left(h'\right)}{\omega\left(h\left(y_0\right)\right)} < \lim_{m\to\infty}\Pr\left\{\tilde{h}_{n-m:n-1} < h'\right\}.$$

The reason this inequality is useful is that the expected quality of the worst firms partner when it offers the bilateral Nash wage $w^*$ is the expected value of the $n-m+1^{st}$ order statistic of workers investment. The inequality gives a bound on the distribution of this order statistic, which bounds the expectation. The constant $\beta$ in the definition of equilibrium provides the appropriate scale factor to account for the possibility that this expectation doesn't attain this bound.

## 5. Truncated Hedonic Equilibrium and Hedonic Equilibrium

An hedonic equilibrium is simply a competitive allocation for a continuum of different goods. It then has two characteristics, markets clear, and the resulting allocaiton is pareto optimal. Condition 1 in the definition of truncated hedonic equilibrium is a market clearing condition relative to a competitive return function $\hat{w}\left(h\right)$, except for the restriction to upward deviations. A hedonic equilibrium is defined by using this same market clearing condition, removing the restriction the workers and firms choice, and forcing the allocation to be pareto optimal.

Let $\left(h^{**}, w^{**}\right)$ be the investment wage pair that maximizes

$$v\left(h', \underline{x}\right) - w'$$

subject to the constraint that

$$w' - c(h', y_0) \geq -c(h^*(y_0))\,\tau(y_0).$$

This is the wage investment pair that is bilaterally efficient for the worst firm type and the marginal worker type.

**Definition 5.** A *hedonic equilibrium* is a return function $\hat{w}_c(h)$ that satisfies condition 1 along with the boundary condition $\hat{w}_c(h^{**}) = w^{**}$.

An immediate implication of Theorem 4 is the following:

**Theorem 6.** *A necessary condition for a sequence $\{h_m, w_m\}$ of Bayesian equilibrium strategies to converge to a hedonic equilibrium is*

$$v(\underline{x}) \int_{h^*(y_0)}^{h^{**}} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{\omega(h^{**})} - w^*(\underline{x}) \geq v(\underline{x})\, h^{**} - w^{**}.$$

This follows immediately from the fact that Bayesian equilibrium strategies converge to truncated hedonic equilibrium, and the fact that Condition 2 in the definition can only be satisfied for some $\beta$ less than 1 if the condition of the theorem is satisfied.

It is difficult to relate the limits of Bayesian equilibrium to hedonic equilibrium when this necessary condition is satisfied. The reason is that truncated hedonic equilibrium imposes relatively weak conditions on outcomes. Different values of $\beta$ support different return functions, for example, and these relate to hedonic equilibrium in different ways. When the necessary condition fails, all truncated hedonic equilibria relate to the hedonic equilibrium in the same way. As a result, we will focus on environments where the necessary condition described above fails. To make the statements of some of the theorems a little less awkward, we refer to these as *regular* environments.

Regular environments are no more or less plausible than irregular ones - regularity refers to the fact that all truncated hedonic equilibrium have the same properties in regular environments. Whether or not an environment is regular depends on the shapes of workers' indifference curves (iso profit curves are all linear) and on the distributions of worker and firm types. The results we establish below surely apply in some irregular environments, however the methods provided here cannot be used to characterize them.

**Proposition 7.** *In every regular environment, and in every truncated hedonic equilibrium in that environment, every firm pays a wage that is higher than it would pay in a hedonic equilibrium. A positive measure of workers invest $h_0$ and are matched with firms who pay $\hat{w}(h_0)$. For* *each of the workers and firms in this pool, investments and wages are too high in the sense that any pair in the pool would both be strictly better off at a lower wage and investment. All other workers invest the same amount in both the hedonic and truncated hedonic equilibrium.*

*Proof.* We begin by showing that in the regular case, for any $\beta \leq 1$, there is a unique investment $h_0 > h^{**}$ consistent with condition 2 in the definition of Truncated Hedonic Equilibrium.

Define

$$\overline{h}(h') \equiv \beta \int_{h^*(y_0)}^{h'} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{\omega(h')}. \tag{5.1}$$

For any $h' \geq h^{**}$ define

$$\alpha(h') = \max\left\{\tilde{h} : v(\underline{x})\,\tilde{h} = v(\underline{x})\,h' - \omega(h')\right\} \tag{5.2}$$

By condition 2, $h_0$ must satisfy $\alpha\left(\overline{h}(h_0)\right) = h_0$. In the regular case, $\alpha\left(\overline{h}(h^{**})\right) > h^{**}$. Since $\overline{h}$ is increasing, and $\alpha$ is decreasing for $h' \leq \overline{h}(h^{**})$, $\alpha\left(\overline{h}(h')\right) > h'$ for each $h' \leq h^{**}$. Since there is evidently an $h' > \alpha\left(\overline{h}(h^{**})\right)$ such that $\alpha(h') = h^{**}$, there must be a unique $h_0 > h^{**}$ such that $\alpha\left(\overline{h}(h_0)\right) = h_0$.

Since $(h^{**}, w^{**})$ maximizes the worst firm's payoff conditional on the marginal worker receiving his outside option, and $h_0 > h^{**}$, it follows by continuity that there is a pair $(h', w')$ with $h' < h^{**}$and $w' < w^{**}$ such that both the marginal worker and the worst firm are strictly better off with this new wage investment pair.

Finally, for each worker $y' \geq y_0$, $h(y') = \arg\max_{h_i \geq h_0}\{\hat{w}(h_i) - c(h_i)\,\tau(y')\}$, so that in any truncated hedonic equilibrium $v(\pi(y_i)) \leq \hat{w}'(h(y')) \leq c'(h(y'))\,\tau(y')$ where $h(y')$ is the equilibrium strategy of a worker of type $y' > y_0$. From the result above, $h_0 > h^{**}$ in a regular environment, so $v(\underline{x}) < \hat{w}'(h(y_0)) < c'(h(y_0))\,\tau(y_0)$. By continuity, there is a non-degenerate interval of types $[y_0, y_1]$ for which this is true.

Consider $h' > h_0$. If some type $y'$ makes the investment $h'$ in the truncated hedonic equilibrium, then

$$v(\pi(y')) = \hat{w}'(h') = c'(h')\,\tau(y').$$

A similar condition holds with respect to the hedonic return function $\hat{w}_c$. Since the function on the right hand side of this last expression is strictly decreasing in $y'$ while the function on the left hand side is strictly increasing, there is at most one value for $y'$such that $v(\pi(y')) = c'(h')\,\tau(y')$. As a consequence if $h' > h_0$ is an investment that is made by some worker type in both the hedonic and truncated hedonic equilibrium, then $h'$ is made by the same type in each case, and $\hat{w}'(h') = \hat{w}'_c(h')$. Since $\hat{w}(h_0) > w^{**}$ and $\hat{w}'(h') = \hat{w}'_c(h')$ for each $h' > h_0$, it follows that $\hat{w}(h') > \hat{w}_c(h')$ for each investment $h' > h_0$ that is made along the equilibrium path in both equilibrium. □

This proposition provides a comparison between hedonic equilibrium and truncated hedonic equilibrium in regular environments. The truncation caused by the competition at the lower end of the wage and investment distribution drives up wages and generates inefficient outcomes for many worker and firm types.

An important property of this theorem is the apparent 'pool' in the distribution of investments and wages that this predicts. This isn't pooling in the sense in which this term in normally used. For every $m$ no matter how large, all workers and firms use monotonic strategies. So in each finite game there is complete separation. However, when the number of workers and firms is large, the equilibrium investment rises very quickly with type at first, then rises quite slowly for workers whose types are close to that of the marginal worker. As a consequence a lot of workers and firms appear to use the same wage and investment in the limit.

5.1. **How to use Truncated Hedonic Equilibrium.** In the regular case, truncated hedonic equilibrium provides a relatively simple way of characterizing the limits of Bayesian Nash equilibria of the investment game. The boundary condition provides a testable retriction on wage and investment distributions since it suggests there should be something that looks like an atom at the bottom of the distribution. One problem with using truncated hedonic equilibrium in this way is that truncated hedonic equilibrium supports multiple outcomes. In the regular case, these outcomes are all similar with respect to the appearance of the wage and investment distributions, and the efficiency of worker investments. The point of this section is to show that these similarities are robust enough to do simple comparative static exercises.

To begin, focus on the constant $\beta$ in the definition. This constant is required because the limit theorem only provides a bound on the distribution of the $n-m+1^{st}$ order statistic that determines the worst firm's payoff when it offers 0 wage. Since the limit theorem doesn't provide an exact bound, truncated hedonic equilibrium is defined for all possible values of $\beta < 1$. However, since $\beta$ uniquely determines the minimum investment $h_0$ of workers, the following result can be used to provide some additional structure.

**Theorem 8.** *Let* $(h_0, \hat{w}, 1)$ *and* $\left(h_0^\beta, \hat{w}_\beta, \beta\right)$ *be a pair of truncated hedonic equilibia for which* $\beta < 1$. *Then* $h_0 < h_0^\beta$ *and* $\hat{w}_\beta(h) > \hat{w}(h)$ *for each* $h > h_0^\beta$.

*Proof.* Borrowing from the proof of Proposition 7

$$\overline{h}_\beta(h') \equiv \beta \int_{h^*(y_0)}^{h'} \tilde{h} d \frac{\omega\left(\tilde{h}\right)}{\omega(h')}.$$

The bound $h_0$ satisfies $\alpha\left(\overline{h}_1(h_0)\right) = 0$. Now simply repeat the proof of Proposition 7 to show that there is a unique $h_0^\beta > h_0$ that satisfies $\alpha\left(\overline{h}_\beta\left(h_0^\beta\right)\right) = h_0^\beta$. As in Proposition 7, $\hat{w}'_\beta(h) = \hat{w}'(h)$ for $h > h_0^\beta$, from which the rest of the result follows. □

The implication of this theorem is that many of the interesting properties of the limits of Bayesian equilibrium for the investment game (at least in the regular case) can be discovered by characterizing truncated hedonic equilibrium in which $\beta = 1$. For example;

**Proposition 9.** *There exist non-degenerate distributions* $G$ *and* $F$ *of worker and firm types such that all workers make the same investment and all firms offer the same wage in the hedonic equilibrium in which* $\beta = 1$.

*Proof.* Since $h_0 > h^{**}$, $v(\underline{x}) < \hat{w}'(h_0) < c(h_0)\tau(y_0)$ in the hedonic equilibrium by condition 1 in the definition. Then there is an open interval $[y_0, y_1]$ for which $v(\pi(y')) < \hat{w}'(h_0) < c(h_0)\tau(y')$ for $y' \in [y_0, y_1)$. If the upper bounds of the supports of $G$ and $F$ are both less than $y_1$ and $\pi(y_1)$ respectively, the result follows. □

It is an immediate corollary of Theorem 8 that the same result is true for any $\beta < 1$, though, of course, the wage and investment that prevails in equilibrium will be higher the lower the value of $\beta$.

## 6. Conclusion

The paper shows the following things:

(1) The hedonic return functions $\hat{h}$ and $\hat{w}$ converge to continuous functions that 'agree' with one another for investments and wages that lie in the support of the equilibrium distributions. Agreement means that the wage that a worker expects to receive for a specific investment $h'$ coincides with the wage that firms think they have to offer to attract a partner of investment $h'$;

(2) The hedonic return functions below the support of the equilibrium distributions diverge;
(3) The lowest types of workers and firms pool together at low investment and wage levels. These investments are too high in the sense that a low type worker and firm who are matched together will joint prefer a lower investment wage combination than that which they are provided in equilibrium;
(4) The highest types of workers and firms match, make and investments and pay wages that are bilaterally efficient. In this sense they resemble the wages and investments that would prevail in a standard competitive equilibrium.

These properties are combined in a equilibrium concept for a continuum matching game in Peters (2006). There it is shown that limit investments all exceed those that prevail in a competitive equilibrium.

## 7. Appendix 1:The Convergence Theorem

7.1. **Characterization of Limits.** In this section, we characterize the functions $\hat{w}_\infty$, $\hat{h}_\infty$, $w_\infty$ and $h_\infty$ where $\hat{w}_\infty$ and $\hat{h}_\infty$ are the continuous limits of some subsequence of equilibrium payoff functions, while $w_\infty$ and $h_\infty$ are weak limits of some sub-sequence of equilibrium strategy rules. It is shown that $w_\infty$ and $h_\infty$ are truncated hedonic equilibria. Proof that are not explicitly included in the text are included in Appendix 2.

To begin, recall that $\tau(1-G(y_0)) = 1$. Define $h_0 \equiv h_\infty(y_0)$ and $w_0 \equiv w_\infty(\underline{x})$ and refer to these as the lowest investment and wage that are *realized in the limit*.[3] We begin with some intuitively straightforward results. The proofs are in the appendix.

**Lemma 10.** *$h_\infty(y') = h^*(y')$ for each $y' < y_0$.*

Only the highest $m$ worker types will match. The probability that a worker with type $y' < y_0$ will be one of the highest $m$ worker types goes to zero with $m$. Since the worker shouldn't expect to match, there is no point making an investment beyond that which is privately optimal.

The next Lemma is a consequence of the fact that the equilibrium return function must separate the types below and above $y_0$.

**Lemma 11.** *For each $h' \in [h^*, h_0]$, $\hat{w}_\infty(h') - c(h', y_0) = -c(h^*, y_0)$*

[3]This is an abuse of language in a number of ways. First, when there are finitely many players, equilibrium strategy rules are monotonic and continuous, so all wages and investments above $w^*$ and $h^*$ are in the support of the equilibrium strategies. Second, by weak convergence, the limit of type $y_0$'s equilibrium investment will be less than or equal to $h_0$.

7.2. **Hedonic Equilibrium.** Each of the payoff functions $\hat{w}_m$ and $\hat{h}_m$ is a hedonic return function. One obvious reason that they don't look like Rosen's hedonic return function is that there are two such functions instead of one. The next two theorems show how these two are reconciled in the limit. They show how a Rosen like hedonic return function can be constructed from limits of finite payoff functions. They also show how to use this limit hedonic function to check whether strategy rules could qualify as limits of sequences of equilibrium strategy rules.

**Lemma 12.** *For each* $y' \geq y_0$, $\lim_{m\to\infty} \hat{w}_m (h_m (y')) - c (h_m (y'), y')$ *is equal to* $\max_h \hat{w}_\infty (h) - c (h, y')$.

This Lemma says that if we knew the limit of the return function that the worker faced in the Bayesian equilibrium, then we could reconstruct the weak limit of his strategy rule by finding his best outcome on this return function.

For the next Lemma, define for each $y \geq y_0$, $\pi (y) = \{x : \tau (1 - G (y)) = F (x)\}$. We refer to a seller of type $\pi (y)$ as the *hedonic partner* of a worker of type $y$. The next Lemma shows how to construct the limit return function from the weak limits of the strategy rules and the hedonic partner function.

**Lemma 13.** $\hat{w}_\infty (h_\infty (y')) = w_\infty (\pi (y'))$ *for each* $y' \geq y_0$.

Parallel arguments establish exactly the same results for firms:

**Lemma 14.** *(i) for each* $w' \in [w^*, w_0]$, $v(\underline{x}) \hat{h}_\infty (w') - w' = v (\underline{x}) \hat{h}_\infty (w_0) - w_0$*;*
*(ii) for each* $x' \geq \underline{x}$, $\lim_{m\to\infty} v (x') \hat{h}_m (w_m (x')) - w_m (x')$ *is equal to* $\max_w v (x') \hat{h}_\infty (w) - w$*;*
*(iii)* $\hat{h}_\infty (w_\infty (x')) = h_\infty (\pi^{-1} (x'))$ *for each* $x' \geq \underline{x}$.

One final Lemma is important in the description of equilbrium.

**Lemma 15.** $h_0 > h^* (y_0)$ *and* $w_0 > w^* (\underline{x})$.

7.3. **A Bound on the payoff to the firm with the lowest wage.** This section provides the theorem that determines $h_\infty (y_0)$ and $w_\infty (\underline{x})$.

When the worst firm offers $w^*$ it will be the lowest wage firm with probability 1, since the equilibrium strategy $w_m$ is strictly increasing for every $m$. As a consequence, the worst firm will match with the worker who has the $n - m + 1^{st}$ lowest investment with probability 1.

Consider any investment $h'$ that lies between $h^*$ and $h_\infty (y_0)$. This investment will result in a match with *some* firm if and only if it exceeds the $n - m^{th}$ lowest order statistic of the investment of the other $n - 1$

workers. Let $\Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\}$ be the probability with which this order statistic is less than $h'$.

**Lemma 16.** $\Pr_m \left\{ \tilde{h}_{n-m+1:n} \leq h' \right\}$ *converges pointwise (hence weakly) to* $\Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\}$ *as* $m$ *goes to infinity.*

We now prove the main characterization result in the paper. It bounds the payoff any firm receives by offering the lowest wage $w^*$.

**Lemma 17.** *The expected investment of the partner of a firm who offers wage* $w^*$ *is bounded above as follows*

$$\hat{h}_\infty (w^*) \leq \int_{h^*}^{h_0} h' \frac{d\hat{w}_\infty (h')}{w_0}$$

*where as above* $h_0$ *and* $w_0$ *are the lowest investment and wage that are realized in the limit.*

*Proof.* The limit payoff associated with any investment $h' \in (h^*, h_0)$

$$\lim_{m \to \infty} \hat{w}_m (h') = \lim_{m \to \infty} \Pr_m \left\{ \tilde{h}_{n-m:n-1} \leq h' \right\} \overline{W}_m (h')$$

where $\overline{W}_m (h')$ is the expected wage the worker receives conditional on finding a job after investing $h'$. By Lemma 16, this is equal to

$$\lim_{m \to \infty} \Pr_m \left\{ \tilde{h}_{n-m+1:n} \leq h' \right\} \overline{W}_m (h')$$

The function $h_m (y)$ is continuous and monotonically increasing for each $m$. So for any investment $h' \in (h^*, h_0)$, there is a unique worker type $y_m$ such that $h_m (y_m) = h'$.

A worker of type $y_m$ finds a job if and only if $n - m$ or more of the other workers have types less than $y_m$, so

$$\overline{W}_m (h_m (y_m)) = \sum_{t=0}^{m-1} \frac{\gamma_{n-m+t} (y_m)}{\sum_{k=0}^{m-1} \gamma_{n-m+k} (y_m)} \mathbb{E}\tilde{w}_{t+1:m}$$

where, as above, $\gamma_k (y_m)$ is the probability that exactly $k$ of the other workers have types below $y_m$.

By standard properties of the binomial distribution (and the fact that $\mathbb{E}\tilde{w}_{t+1:m}$ is a non-decreasing sequence), $\overline{W} (h_m (y_m))$ is a non-decreasing function. Fix $y' > y_0$. Since $y_m$ invests less than $h_0$, while each worker $y' > y_0$ invests more than $h_0$ for $m$ large enough, it must be that for large enough $m$, $y_m < y'$, so, again for large enough $m$, $\overline{W} (h_m (y_m)) \leq \overline{W} (h_m (y'))$. By Lemma 18, a worker of type $y' > y_0$ finds a job with probability 1 in the limit. So the conditional and unconditional wage earned by a worker of type $y'$ must be the same in the limit, i.e.,

$$\lim_{m\to\infty} \overline{W}(h_m(y')) = \lim_{m\to\infty} \hat{w}_m(h_m(y')) = \hat{w}_\infty(h_\infty(y')).$$

So

$$\lim_{m\to\infty} \hat{w}_m(h') \leq \lim_{m\to\infty} \Pr_m \left\{\tilde{h}_{n-m+1:n} \leq h'\right\} \hat{w}_\infty(h_\infty(y'))$$

for every $y' > y_0$. By the right continuity of $h_\infty$ at $y_0$ $\lim_{y'\downarrow y_0} \hat{w}_\infty(h_\infty(y')) = h_0$ so

$$\hat{w}_\infty(h') \leq \lim_{m\to\infty} \Pr_m \left\{\tilde{h}_{n-m+1:n} \leq h'\right\} w_0.$$

Then

$$\lim_{m\to\infty} \int_{h^*}^{h_0} h' d \Pr_m \left\{\tilde{h}_{n-m+1:n} \leq h'\right\} \leq$$

$$\int_{h^*}^{h_0} h' d \frac{\hat{w}_\infty(h')}{w_0}$$

by stochastic dominance. □

## 8. Appendix 2: - Proofs

**Lemma 18.** *Let $\underline{y} < y_a < \overline{y}$ and $\underline{y} < y_b < \overline{y}$. The probability that the number of other workers whose types are at least $y_a$ exceeds $n(1 - G(y_b))$ converges to one if $y_b > y_a$ and converges to zero if $y_b < y_a$. Similarly, the probability that the number of firms whose types are at least $x_a$ exceeds $m(1 - F(x_b))$ converges to one if $x_b > x_a$ and converges to zero if $x_b < x_a$.*

*Proof.* The number of the $n-1$ workers whose type exceeds $y_a$ is a random variable with mean

$$(n-1)(1 - G(y_a))$$

and variance

$$(n-1) G(y_a)(1 - G(y_a)).$$

As $n$ grows large, this random variable becomes approximately normal in the sense that for any $x$, the probability that the number of workers whose type exceeds $y$ exceeds $x$ converges to the probability that a standard normal random variable exceeds

$$\frac{x - (n-1)(1 - G(y_a))}{\sqrt{(n-1) G(y_a)(1 - G(y_a))}}.$$

Evaluating this for $x = n(1 - G(y_b))$, and replacing $n-1$ by $n$ verifies that the probability that the number of workers whose type exceeds $y$

is greater than $m$ converges to the probability that a standard normal random variable exceeds

$$\frac{n\left(1-G\left(y_b\right)\right)-n\left(1-G\left(y_a\right)\right)}{\sqrt{nG\left(y_a\right)\left(1-G\left(y_a\right)\right)}}=\sqrt{n}\frac{G\left(y_a\right)-G\left(y_b\right)}{\sqrt{G\left(y_a\right)\left(1-G\left(y_a\right)\right)}}$$

which goes to minus infinity when $y_a < y_b$ and plus infinity when $y_a > y_b$. The probability then converges to one in the first case and zero in the second. The proof is identical for firms. □

8.1. **Proof of Lemma 10.**

*Proof.* The equilibrium strategies are monotonically increasing. So a worker of type $y' < y_0$ will find a job only if the number of other workers who have types above $y'$ is less than $m = n\left(1-G\left(y_0\right)\right)$. Let $y_a = y'$ and $y_b = y_0$ and apply Lemma 18 to conclude that the the probability that the number of other workers whose types exceed $y'$ is at least $m = n\left(1-G\left(y\right)\right)$ converges to one with $m$. It follows that a worker of type $y'$ matches with very low probability when $m$ is large, and thus can't profitably invest more than $h^*$ in the limit. □

8.2. **Proof of Lemma 11.**

*Proof.* If $\hat{w}_\infty\left(h_\infty\left(y_0\right)\right)-c\left(h_\infty\left(y_0\right),y_0\right) > -c\left(h^*,y_0\right)$, then for some $y' < y_0$ and $m$ large enough,

$$\hat{w}_m\left(h_\infty\left(y_0\right)\right)-c\left(h_\infty\left(y_0,y'\right)\right) > -c\left(h^*,y'\right)$$

The equilibrium payoff of a worker of type $y'$ is given by

$$\hat{w}_m\left(h_m\left(y'\right)\right)-c\left(h_m\left(y'\right),y'\right)$$

By Lemma 10 and the continuity of the utility function, this converges to $-c\left(h^*,y'\right)$. As a consequence, a profitable deviation must exist for worker $y'$ for large enough $m$, a contradiction.

Similarly, if $\hat{w}_\infty\left(h_\infty\left(y_0\right)\right)-c\left(h_\infty\left(y_0\right),y_0\right) < -c\left(h^*,y_0\right)$, a worker of type $y' > y_0$ must find it profitable for large enough $m$ to cut investment to $h^*$. □

8.3. **Proof of Lemma 12.**

*Proof.* If the latter term is larger, there is immediately a profitable deviation from $h_m\left(y'\right)$ when $m$ is large enough. If the former term is larger, let $\overline{h} = \lim_{m\to\infty} h_m\left(y'\right)$. From the contrary hypothesis, there is some $m_0$ large enough so that

$$\hat{w}_{m'}\left(h_{m'}\left(y'\right)\right)-c\left(h_m\left(y'\right),y'\right) > \max_h \hat{w}_\infty\left(h\right)-c\left(h,y'\right)+\epsilon$$

for each $m' > m_0$. Since the family of functions $\hat{w}_m$ is equi-continuous, there is for every $\epsilon' > 0$ a $\delta > 0$ such that $|h - h'| < \delta$ implies that $|\hat{w}_m(h) - \hat{w}_m(h')| < \epsilon'$ for all $m$. Find $\delta$ such that

$$\left|\hat{w}_m(h') - c(h', y') - \left(\hat{w}_m\left(\overline{h}\right) - c\left(\overline{h}, y'\right)\right)\right| < \epsilon$$

for all $m$ when $\left|h' - \overline{h}\right| < \delta$. Then choose $m'$ for large enough so that $\left|h_m(y') - \overline{h}\right| < \delta$. Then for all $m' > m_0$.

$$\hat{w}_{m'}\left(\overline{h}\right) - c\left(\overline{h}, y'\right) > \max_h \hat{w}_\infty(h) - c(h, y')$$

Taking the pointwise limit gives a contradiction. □

The next theorem provides the 'hedonic' part of the limit outcome. Let $y' \geq y_0$ and $\pi(y') = \{x : 1 - F(\pi(y')) = \tau(1 - G(y'))\}$. The firm type $\pi(y')$ is the 'hedonic partner' of a worker of type $y'$, i.e., it is the firm type such that the measure of the set of firms who have larger types than $\pi(y')$ is equal to the measure of the set of workers who have better types than $y'$. By definition, $\pi(y_0) = \underline{x}$.

8.4. **Proof of Lemma 13.**

*Proof.* In equilibrium, both workers and firms use monotonically increasing strategies. So the $k^{th}$ lowest worker type will match with the $k - (n-m)^{th}$ lowest firm type. For any $y' \geq y_0$, let $x' > \pi(y')$. The probability with which a worker of type $y$ matches with a firm of type $x'$or better is equal to the probability with which the number of firms whose type is at least $x'$ exceed the number of workers whose type is at least $y'$. Let $\pi(y') < x'' < x'$. By Lemma 18, the probability that the number of firms with types above $x'$ exceeds $m(1 - F(x''))$converges to zero with $m$. On the other hand the probability that the number of workers with types above $y'$ exceeds $m(1 - F(x'')) = n(1 - G(\pi(y'')))$ converges to 1 by Lemma 18, where $\pi(y'') = x''$. Hence, for any $x' > \pi(y')$, the probability with which a worker of type $y'$ matches with a firm whose type is $x'$ or above converges to zero. A similar argument establishes that the probability that a worker matches with a firm whose type is less than $\pi(y')$ also converges to zero (or is zero if $y' = y_0$ because $\pi(y_0) = \underline{x}$).

The functions $h_\infty$ and $w_\infty$ are both non-decreasing. So they are both continuous except at countably many points. Then in any open interval to the right of $y'$ and $x' > \pi(y')$, there are points $y''$ and $x'' > \pi(y'')$ such that $h_\infty$ is continuous at $y''$ and $w_\infty$ is continuous at $x''$. By weak convergence, $\lim_{m\to\infty} w_m(x'') = w_\infty(x'')$ and $\lim_{m\to\infty} h_m(y'') =$

$h_\infty(y'')$. As $x'' > \pi(y'')$, by the reasoning in the previous paragraph,

$$\lim_{m\to\infty} \hat{w}_m(h_m(y'')) - c(h_m(y''), y'') < \lim_{m\to\infty} w_m(x'') - c(h_m(y''), y'')$$
$$= w_\infty(x'') - c(h_\infty(y''), y'')$$

The limit on the left hand side is a continuous function of type. Since $h_\infty$ and $w_\infty$ are both right continuous, we have

$$\lim_{m\to\infty} \hat{w}_m(h_m(y')) - c(h_m(y'), y') \le w_\infty(\pi(y')) - c(h_\infty(y'), y')$$

Let $h'' > h_\infty(y')$. $h_\infty$ is a weak limit of a sequence of increasing continuous functions $h_m$. Then it is right continuous. As a result, there must exist an open interval $I^+_{h''}(y')$ such for every $y''$ in the interval, $y'' > y'$ and $h'' > h_m(y'')$ for infinitely many $m$. A worker who invests $h''$ will match as if his type were better than $y''$ . By the reasoning of the previous paragraph, this means that investment $h''$will lead to a match with a firm whose type is at least $\pi(y'') > \pi(y')$ with probability converging to one with $m$. Since $w_\infty$ is non-decreasing, it can have at most countably many points at which it is discontinuous. Hence for any $h''$ it is possible to choose $y''$ such that $w_\infty(\cdot)$ is continuous at the point $\pi(y'')$. Since $w_\infty$ is the weak limit of the sequence of functions $w_m$ and $w_\infty$ is continuous at $\pi(y'')$, $w_\infty(\pi(y''))$ is also the pointwise limit of $w_m(\pi(y''))$. Then the payoff to the investment $h''$ must be converging to something at least as large as $u(h'')(a - y') + w_\infty(\pi(y''))$. Since $I^+_{h'''}(y') \subset I^+_{h''}(y')$ when $h''' < h''$, we can choose a decreasing sequence $(h''_n, y''_n)$ converging to $(h_\infty(y'), y')$ for which this inequality holds, it follows by the right continuity of $w_\infty$ that

$$\lim_{m\to\infty} \hat{w}_m(h_m(y')) - c(h_m(y'), y') \ge u(h_\infty(y'))(a - y') + w_\infty(\pi(y')).$$

Putting these two arguments together gives

$$\lim_{m\to\infty} \hat{w}_m(h_m(y')) = w_\infty(\pi(y'))$$

From the fact that $h_\infty$ has only countably many discontinuities, there is for every $y'$ a $y'' > y'$ arbitrarily close to $y'$ such that $h_\infty(\cdot)$ is continuous at $y''$. Hence $\lim_{m\to\infty} h_m(y'') = h_\infty(y'')$. Now using equi-continuity of the family $\hat{w}_m$, for any $\epsilon$, there is a $\delta$ such that $\left|\hat{w}_m\left(\tilde{h}\right) - \hat{w}_m(h_\infty(y''))\right| < \epsilon$ for every $m$ provided $\left|\tilde{h} - h_\infty(y'')\right| < \delta$. Choose $m'$ such that

$$|h_m(y'') - h_\infty(y'')| < \delta.$$

It follows that from $m$ bigger than $m'$ $|\hat{w}_m(h_m(y'')) - \hat{w}_m(h_\infty(y''))| < \epsilon$. So $\lim_{m\to\infty} \hat{w}_m(h_m(y'')) = \hat{w}_\infty(h_\infty(y''))$. The result then follows from the fact that the limit on the left (equilibrium payoff) is a continuous function of type, $\hat{w}_\infty$ is continuous, and $h_\infty(y)$ is right continuous. $\square$

## 8.5. **Proof of Lemma 15.**

*Proof.* From Lemmas 13

$$\hat{w}_\infty(h_\infty(y_0)) - c(h_\infty(y_0), y_0) = w_0 - c(h_0, y_0).$$

From Lemma 11

$$w_0 - c(h_0, y_0) = -c(h^*(y_0), y_0).$$

Similarly from Lemma 14

$$v(\underline{x})\, h_0 - w_0 = v(\underline{x})\, \hat{h}(w') - w'$$

for each $w' \leq w_0$. By the definition of equilibrium, it must be that for any $y' > y_0$,

$$\hat{w}_\infty(h_\infty(y')) - c(h_\infty(y'), y_0) \leq w_0 - c(h_0, y_0)$$

and

$$v(\underline{x})\, \hat{h}_\infty\left(w_\infty\left(\pi^{-1}(y')\right)\right) - w_\infty\left(\pi^{-1}(y')\right) =$$
$$v(\underline{x})\, h_\infty(y') - \hat{w}_\infty(h_\infty(y')) < v(\underline{x})\, h_0 - w_0.$$

In words, this says that the market return function $\hat{w}(h)$ for $h > h_0$ must lie below the indifference curve of a worker of type $y_0$ through the point $(h_0, w_0)$ and above the iso-profit curve of a firm of type $\underline{x}$ through the point $(h_0, w_0)$. Since this property cannot be satisfied at the point $(h^*(y_0), w^*(\underline{x}))$, the result follows. $\square$

## 8.6. **Proof of Lemma 16.**

*Proof.* Let $y_m$ be the (unique) type such that $h_m(y_m) = h'$. By Lemma 10, $h_\infty(y) = y^*$ for every $y < y_0$. Then for any $h' > h^*$, there is $m$ large enough such that $y_{m'} \geq y_0$ for $m' > m$.

$$\Pr_m\left\{\tilde{h}_{n-m+1:n} \leq h'\right\} =$$

$$= \int_{\underline{y}}^{y_m} \left\{1 - \left(\frac{1 - G(y_m)}{1 - G(z)}\right)^m\right\} \frac{(n-1)!}{(n-m-1)!\,(m-1)!} G(z)^{n-m-1} (1 - G(z))^{m-1} g(z)\, dz$$

The logic in this expression is based on the idea that the probability that $\tilde{h}_{n-m+1:n} \leq h'$ is equal to the expectation over all possible values for $\tilde{h}_{n-m:n-1}$ of the probability that at least one of the remaining $m$

independent draws of $\tilde{h}$ lies in the interval $\left[\tilde{h}_{n-m:n-1}, h'\right]$. Substituting gives

$$\Pr_m \left\{\tilde{h}_{n-m+1:n} \leq h'\right\} = \Pr_m \left\{\tilde{h}_{n-m:n-1} \leq h'\right\} -$$

(8.1)

$$\int_{\underline{y}}^{y_m} \left(\frac{1-G(y_m)}{1-G(z)}\right)^m \frac{(n-1)!}{(n-m-1)!(m-1)!} G(z)^{n-m-1} (1-G(z))^{m-1} g(z)\, dz$$

Since the binomial term $\frac{(n-1)!}{(n-m-1)!(m-1)!} G(z)^{n-m-1} (1-G(z))^{m-1}$ is a probability, it is bounded above by 1, as is the function $\left(\frac{1-G(y_m)}{1-G(z)}\right)^m$. Define the function

$$\psi_m(z) = \begin{cases} \left(\frac{1-G(y_m)}{1-G(z)}\right)^m \frac{(n-1)!}{(n-m-1)!(m-1)!} G(z)^{n-m-1} (1-G(z))^{m-1} & z \leq y_m \\ 0 & \text{otherwise} \end{cases}$$

and evaluate the integral in (8.1) as

$$\int_{\underline{y}}^{\overline{y}} \psi_m(x)\, g(z)\, dz$$

As $\psi_m(z)$ is bounded above by the constant function 1 and converges almost everywhere to zero with $m$ we conclude from the dominated convergence theorem that

$$\lim_{m\to\infty} \int_{\underline{y}}^{\overline{y}} \psi_m(z)\, g(z)\, dz = 0$$

Using this when taking the limit in (8.1) then gives the result. □

8.7. **Proof of Lemma 2.**

8.8. **Lemma 19.**

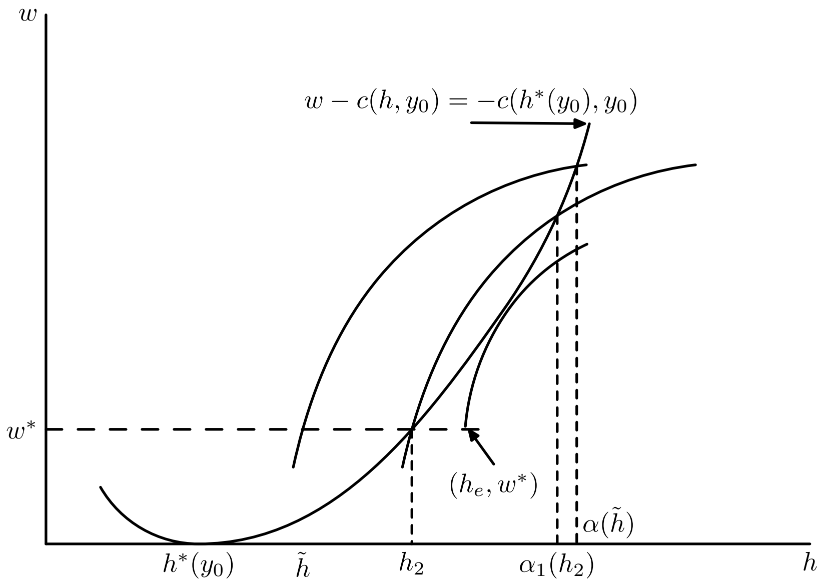
**Lemma 19.** *There is a unique $h'$ such that*

$$h' \in \alpha\left(\overline{h}(h')\right) \tag{8.2}$$

*where $\alpha$ and $\overline{h}$ are defined by (5.1) and (5.2) in the text.*

*Proof.* From (5.1), $\overline{h}$ is a continuous function. From (5.2), $\alpha(\cdot)$ is the collection of points at which the iso-profit curve through $\left(\overline{h}(h'), w^*\right)$ cuts the indifference curve through $(h^*(y_0), 0)$. Let $h_2$ be the investment such that

$$c(h_2, y_0) - w^*(\underline{x}) = -c(h^*(y_0), y_0).$$




There are two distinct cases to consider depending on the slope of iso-profit curve at $(w^*, h_2)$. If the iso-profit curve is flatter than the indifference curve at that point, then

$$h_2 > \overline{h}(h_2).$$

and $\alpha\left(\overline{h}(h_2)\right)$ contains a point that exceeds $h_2$. The function $\overline{h}$ is continuous, strictly increasing and unbounded. The correspondence $\alpha\left(\overline{h}(h')\right)$ is empty when $h'$ is large enough. Otherwise, the maximum element of $\alpha\left(\overline{h}(h')\right)$ is continuously decreasing as $h'$ increases. It follows from the mean value theorem that there is an $h' > h_2$ such that the maximum element of $\alpha\left(\overline{h}(h')\right)$ is equal to $h'$.

The second possibility is that the iso-profit curve is steeper than the indifference curve at $(h_2, w^*)$. This would be the case if the privately optimal wage of the lowest firm were 0. Then $\alpha\left(\overline{h}(h_2)\right)$ contains a point $h'' > h_2$ as illustrated in the diagram below. Then as $h'$ increases above $h_2$, the average $\overline{h}(h')$ rises continuously, while the maximum point in $\alpha\left(\overline{h}(h')\right)$ falls continuously and the minimum point $\alpha\left(\overline{h}(h')\right)$ rises continuously. Since $\overline{h}(\cdot)$ is increasing, each point in $\alpha\left(\overline{h}(h'')\right)$ is strictly less than $h''$. Then the existence of a solution to $h' \in \alpha\left(\overline{h}(h')\right)$ somewhere in the interval $[h_2, h'']$ follows from the mean value theorem. The uniqueness of this point follows from monotonicity.

□

# References


Felli, L., and K. Roberts (2000): "Does Competition Solve the Holdup Problem?," London School of Economics working paper.

Han, S. (2002): "Menu Theorems for Bilateral Contracting Problems," University of Toronto, working paper.

Peters, M. (2004): "The Pre-Marital Investment Game," University of British Columbia Working Paper.

——— (2006): "Truncated Hedonic Equilibrium," University of British Columbia Working Paper.

Peters, M., and A. Siow (2002): "Competing Premarital Investments," *Journal of Political Economy*, 110(3), 592–609, University of Toronto working paper UT-ECIPA-PETERS-00-01.

Rosen, S. (1974): "Hedonic Prices and Implicit Markets: Product Differentiation," *Journal of Political Economy*, 82(1), 34–55.