

EXPECTED UTILITY AND RISK AVERSION

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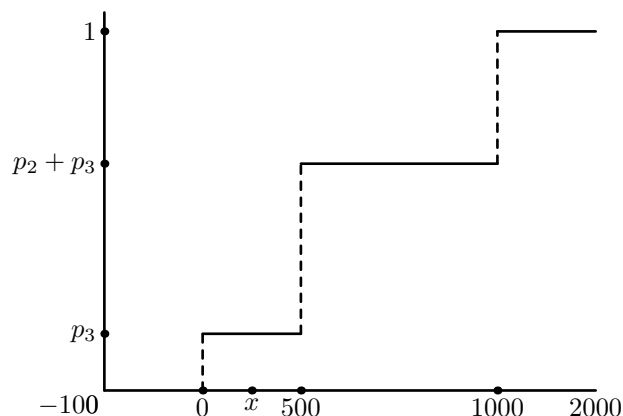
1. INTRODUCTION

This reading describes how people's aversion to risk affects the decisions they make about investment. Basically, the concepts used to do this analysis emerge naturally when people have expected utility preferences, but not otherwise. So, it illustrates one important way that expected utility is applied.

This analysis also makes it possible to illustrate how to do *comparative statics*. Comparative statics typically involve calculations designed to show the direction in which changes in the environment move people's optimal decisions. Convincing comparative static results are ones that hold even if you only impose weak restrictions on preferences. So, for instance, to explain how an increase in price affects a buyer's demand when he or she has Cobb-Douglas preferences is not very convincing because Cobb-Douglas preferences are a very special case. In fact, as we have already shown, an (uncompensated) increase in price will only reduce demand under very special conditions. That is why the method for doing comparative statics tend to be a little more sophisticated, and the questions asked tend not to be the most obvious ones.

1.1. Lotteries with a Continuum of Outcomes. In portfolio theory, outcomes are all monetary. We looked at monetary lotteries in some of the example above. Yet it doesn't make sense when thinking about stocks or bonds to restrict to only three or four outcomes, there are really an infinity of possible outcomes. The way to think about this is to think of the set of potential outcomes \mathcal{X} as an interval of \mathbb{R} . Of course, we can't assign a positive probability to each of these outcomes the way we have so far. So instead we describe a lottery as a *probability distribution function*, $F(x)$ that takes each possible value x in the set of *potential* outcomes and gives the probability that the *actual* outcome is less than or equal to x .

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We could actually describe the lotteries we have considered so far this way. For example, in the last chapter we thought about lotteries with three outcomes, in particular, the outcomes were $\{1000, 500, 0\}$, where each of these outcomes is supposed to be a monetary outcome. We assigned probability p_1 to the first outcomes, p_2 to the second and $p_3 = 1 - p_1 - p_2$ to the third.

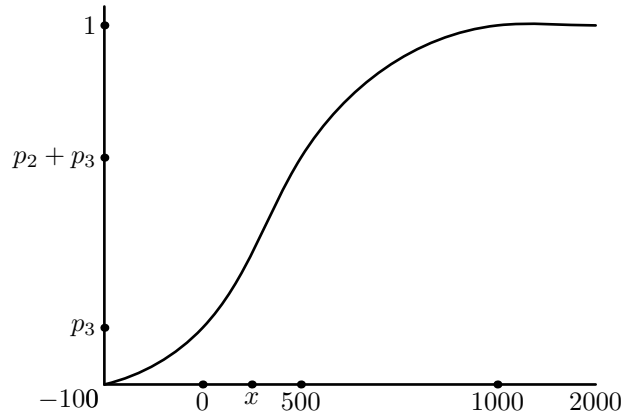
Suppose we think of $\mathcal{X} = [-100, +2000]$ as the set of possible outcomes. Then the following diagram shows you what the corresponding probability distribution function looks like for this lottery. The function $F(x)$ starts out at 0 when x is equal to -100 which is the lowest possible monetary payoff.

For every value of x between -100 and 0 the probability that the outcome of the lottery is less than or equal to these values is constantly zero. However, the probability that the outcome of the lottery is less than or equal to 0 is exactly equal to the probability that 0 occurs, in other words, p_3 , the probability that we have been assigning to the worst outcome. As the monetary payoff travels between 0 and 500 , say a value like x in the diagram, the probability that the outcome is less than or equal to x is then constant at p_3 . Suddenly it jumps up to $p_2 + p_3$ when x is 500 , and so on, until it reaches 1 and stays there after x is 1000 .

The previous chapter showed how the independence axiom makes it possible to compare different lotteries by computing the *expected utility* associated with those lotteries. The expected utility associated with the lottery $\{p_1, p_2, p_3\}$ for example, is given by

$$p_1 u(1000) + p_2 u(500) + p_3 u(0)$$

The lottery $\{p_1, p_2, p_3\}$ is the same as the probability distribution described in the Figure above.



Now suppose that we want to describe a lottery with an infinite number of outcomes, for example, a share that has a return that could be anything between -100 and 2000 . We can't describe such a lottery by assigning probabilities to each possible outcome in this case. There are just too many of them. We could, however, still describe the lottery by giving the probability distribution function associated with it. An infinite lottery might have a probability distribution that looked like the one in the next figure.

This lottery (i.e., the probability distribution function associated with this lottery) has the same values at 0 , 500 , and 1000 as the previous lottery - p_3 , $p_2 + p_3$, and 1 . However, at a point like x , the probability that the outcome in this lottery is less than or equal to x is strictly higher than p_3 which was its value in the previous lottery.

The expected utility theorem suggests that if the independence axiom holds for compound lotteries described as distribution functions, then we should be able to assign a utility value to *all* the outcomes between -100 and 2000 . Then if we could somehow multiply these utility values by something like probabilities associated with each outcome we would have a single utility value for the lottery. The way to do this is to *integrate* the utility values multiplied by the amount by which the probability distribution function increases at each possible outcome. This expression looks like

$$\int_{-100}^{2000} u(x) dF(x)$$

where $dF(x)$ means the *change* in the probability distribution function. For a lottery like the first of the two above, dF is equal to zero at every point except for the three special outcomes, 0 , 500 , and 1000 . Then, dF takes the values p_3 , p_2 and p_1 respectively. Then if we add up by

integrating as above we just get the sum

$$p_1u(1000) + p_2u(500) + p_3u(0)$$

In a case like the second lottery, dF is the *derivative* of the probability distribution (sometimes called its density). In that case you would write dF as $F'(x)dx$ where $F'(x) = \frac{dF(x)}{dx}$. That is the formulation you have probably seen before.

As an example, one special probability distribution function is the *uniform* distribution function. The idea behind it is that every possible outcome is equally likely. So if you take a point midway through the set of outcomes, i.e. 1050, then the probability the outcome is less than or equal to 1050 should be equal to $\frac{1}{2}$. The probability distribution that does this is

$$F(x) = \frac{x - (-100)}{2000 - (-100)}$$

The expected utility associated with this lottery is

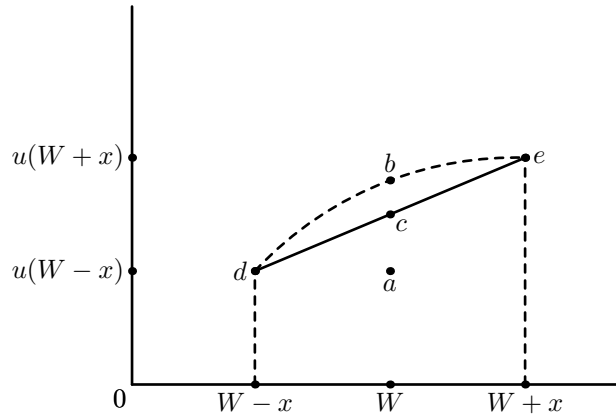
$$\frac{1}{2100} \int u(x)dx$$

because of the fact that $F'(x) = \frac{1}{2100}$. Then using your high school algebra, the expected utility associated with the lottery whose probability distribution function is uniform is just the ratio of the area under the utility function between -100 and 2000 , to the area in the rectangle whose sides have length 1 and 2100 .

1.2. Risk Aversion. As described above, the independence axiom implies that there is a *utility for wealth* function that people use to evaluate lotteries. One reasonable question to ask is whether we could guess something about the shape of this function from other properties of behavior. For example, the St. Petersburg bet that we discussed in the last chapter has an infinite expected value, but no one would pay an infinite amount to engage in the bet.

An even simpler example might be the following - I propose a new grading scheme for the exam. You can choose either of the following options - either take the mark you get, or I will flip a coin - heads I raise your mark by 10 points, tails I lower your mark by 10 points. Your expected grade is the same under either scheme. You probably wouldn't want the second scheme because it is risky. An A grade could turn into a B, a pass could turn into a fail (or conversely).

The second grading scheme involves what is called a *fair* gamble in the sense that the expected gain to you of taking the bet is exactly zero. It seems plausible that most people simply wouldn't be interested in



taking a fair bet.¹² This kind of behavior does have some implications for the shape of the utility function. You can see what the implications are in the next figure.

Suppose our consumer has some initial level of wealth W . A fair bet is one whose expected return is zero. One example would be a bet that pays x with probability $\frac{1}{2}$, but which costs you x with the same probability. Doing nothing leaves you with W for sure. The independence axiom implies that there is a utility level $u(W)$ associated with this outcome. The outcomes $W-x$ and $W+x$ also have associated

¹However, it should be added that the bets that people take in a casino are not fair. For example, if you fed coins into a slot machine for the rest of your life you would lose a lot of money. This doesn't seem to stop people from going to casinos.

²The lotteries that you buy into at the corner store (like the Lotto 6-49) represent a strange variant on what happens in casinos. The way the lotteries work is roughly as follows - the lottery sells tickets. From the total revenue they earn by ticket sales, they take out a big chunk to cover operating costs and to give themselves some profit. The rest is put in a pool. A number is then drawn randomly by dropping balls from an urn (or some other method that is independent of the number of tickets). If one or more people has the winning number they split all the money in the pool. The interesting event occurs when no one has the ticket. Then, when the lottery is run the next period, a portion of ticket sales is added to the pool which already contains the pool from the last lottery. If no one wins for a long time, the pool can become so large that the lottery ticket will be a more than fair bet. For example, as this is written the chance of winning the 6-49 is about 1 in 18 million, but the current pool of money to be won is about 40 million. So the lottery ticket appears to have an expected value of a little over \$2. It only costs \$2 to buy a ticket. So it appears that everyone should buy a ticket. Many people do, leading to enormous revenues and profits for the lottery corporation. This is a bit misleading because the odds of winning are independent of the number of bidders. That means that if many people bid, more than one person is likely to have a winning ticket. The average payout to a winner will be considerably smaller than \$40 million for this reason, which makes the expected value of the ticket smaller than \$2.

utility values $u(W-x)$ and $u(W+x)$. These are marked in the diagram. The expected utility of the lottery we just described is then

$$\frac{1}{2}u(W-x) + \frac{1}{2}u(W+x)$$

In the diagram, this utility level is the distance from the point W on the horizontal axis up to the point c in the diagram.³ The assertion that our consumer would rather not engage in this bet means that the utility value of W (for sure) must be above c at a point like b . Since this preference not to take fair bets is likely to be true no matter what x is and no matter what W is, means that the utility for wealth function must lie everywhere above the line segment from d to e . In other words, the utility for wealth function must be concave.

1.3. Measuring Aversion to Risk. Stocks and bonds are more complex than lotteries because the monetary outcomes usually aren't fair. Most traded stocks for example have a positive expected return. The thing that makes stock trading interesting is that it involves considerable risk. Whether or not an investor will want an initial public offering of some stock depends partly on the risk characteristics of the stock, and partly on the investors own attitudes toward risk. The economics literature has suggested some interesting ways of separating these two things.

Let F be the probability distribution over outcomes that is associated with some lottery. The expected payoff associated with the lottery is

$$\mathbb{E}x = \int x dF(x)$$

where $dF(x)$ depends on the nature of the lottery as discussed above. Lets assume for now that the probability distribution function F has a density so that we can write the expected payoff (and all the expected

³To see this observe that the vertical distance between the points d and e is $u(W+x) - u(W-x)$. By similar triangles, the ratio of the vertical distance between a and c (call it $|c-a|$) to $u(W+x) - u(W-x)$ is the same as the ratio of the horizontal distance between a and d to $W+x - (W-x)$. This latter ratio is equal to $\frac{1}{2}$. Therefore

$$|c-a| = \frac{1}{2}(u(W+x) - u(W-x))$$

Now adding $u(W-x)$ to both sides gives

$$|c-d| + u(W-x) = \frac{1}{2}u(W-x) + \frac{1}{2}u(W+x)$$

utility calculations) using the better known manner

$$\mathbb{E}x = \int xF'(x)dx$$

Let W be any level of wealth. Lets try to calculate the maximum amount our consumer would be willing to pay to play this lottery F . Using the independence axiom we would find this price p by solving

$$u(W) = \int u(W - p + x) F'(x) dx$$

It is going to be tough to solve this exactly without more precise information about u so lets solve it 'approximately'.

First take a second order Taylor approximation to the utility function inside the integral sign on the right hand side of the equation. A first order Taylor approximation won't work very well here because it only gives a good approximation when x is small. However, there will typically be some risk that x could be quite large, so some correction for the curvature in u will help. The second order approximation is given by

$$\int \left\{ u(W) + u'(W)(x - p) + u''(W) \frac{(x - p)^2}{2} \right\} F'(x) dx$$

If this is a good approximation, then it is approximately equal to $U(W)$ when we choose the right p , so simplify the equation

$$\int \left\{ u(W) + u'(W)(x - p) + u''(W) \frac{(x - p)^2}{2} \right\} F'(x) dx = u(W)$$

to get

$$u'(W) \int (x - p)F'(x)dx + u''(W) \int \frac{(x - p)^2}{2} F'(x) dx = 0$$

or

$$p = \int xF'(x)dx + \frac{u''(w)}{u'(w)} \int \frac{(x - p)^2}{2} F'(x)dx$$

The second term in this expression has a $u''(w)$ which is negative if the utility for wealth function is concave. So the expression says that the maximum amount the consumer will be willing to pay for the lottery is equal to its mean return less a term consisting of two parts. One part is the integral which is related to the variance of the lottery or the degree to which it is spread out and unpredictable. The other part depends only on the consumer's utility for wealth function. The larger in absolute value is the ratio $\frac{u''(w)}{u'(w)}$, the less the consumer will be willing to pay.

This particular decomposition has proved to be very useful in thinking about portfolio theory. The term

$$-\frac{u''(W)}{u'(W)}$$

is referred to as the Arrow-Pratt measure of absolute risk aversion.

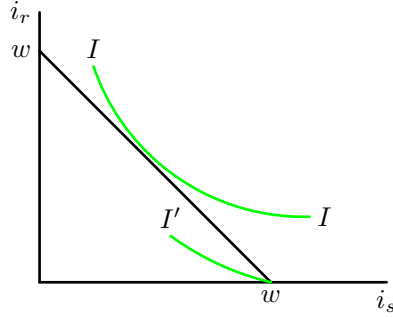
We will shortly see how this measure can be used to get some insight into the way that investment decisions work. For the moment, the main thing to note is that the Arrow-Pratt measure is a conceptual device that emerges with the help of the expected utility theorem. One natural assumption would seem to be that the risk premium that a consumer would be willing to pay to avoid risk would be lower if the consumer were more wealthy. The formulation so far shows exactly how to formalize this idea - the function $-\frac{u''(W)}{u'(W)}$ should be a decreasing function. We will come back to this idea momentarily.

1.4. The Portfolio Problem. A fairly simple version of the portfolio problem can be studied by assuming that there are exactly two different securities that an investor can buy. A *security* is a lottery. The consequences of the lottery are possible rates of return on investment. If the consequence of the lottery is a rate of return s , then the security pays $1 + s$ dollars tomorrow for each dollar that is invested in the lottery today. In the problem that we are going to analyze, one of the two securities is safe in the sense that the lottery that produces the rate of return gives a rate of return of zero for sure. So, each dollar invested today gives back exactly one dollar tomorrow, no more and no less. There is also a risky security where the probability distribution function for the random rate of return is F . Sometimes this rate of return will be positive and the security will pay back more than one dollar for each dollar invested. Other times the rate of return will actually be negative, and a dollar invested will return less than one dollar tomorrow.

The investor has w dollars to invest in these two securities. His *ex post* income (i.e., his income tomorrow after the rate of return on the lottery is realized) when he invests i_s in the safe security and i_r in the risky security, is given by

$$i_s + i_r(1 + s)$$

A pair (i_s, i_r) is called a *portfolio*. Each portfolio generates a different lottery over monetary outcomes. Then relying on the independence axiom, we can invoke the expected utility theorem and conclude that there is a utility for wealth function u such that one portfolio (and its



associated lottery) (i_s, i_r) is preferred to another (i'_s, i'_r) if

$$\int u(i_s + i_r(1+s)) F'(s) ds \geq \int u(i'_s + i'_r(1+s)) F'(s) ds.$$

If that is true, then the investor should choose the portfolio that maximizes

$$\int u(i_s + i_r(1+s)) F'(s) ds$$

subject to the constraints that

$$i_s + i_r \leq W$$

$$i_r \geq 0$$

$$i_s \geq 0$$

We could solve this using Lagrangian methods, recalling that before you do so, you need to tease out some properties of the solution in order to guess which constraints are likely to be important. To see a way to do this, simply imagine that the integral above is just a particular function

$$U(i_s, i_r) \equiv \int u(i_s + i_r(1+s)) F'(s) ds$$

and that the first constraint above is just a budget constraint where the prices of both securities are just equal to 1. So, we should be able to find the optimal portfolio by finding the highest indifference curve that touches the budget constraint, as the indifference curve II does in Figure 1.

The slope of the indifference curve is given by the marginal utility of the good on the horizontal axis (i.e., i_s) divided by the marginal utility

associated with the good on the vertical axis (i.e., i_r). Differentiating gives

$$-\frac{\frac{\partial U(i_s, i_r)}{\partial i_s}}{\frac{\partial U(i_s, i_r)}{\partial i_r}} = -\frac{\int u'(i_s + i_r(1+s)) F'(s) ds}{\int u'(i_s + i_r(1+s)) (1+s) F'(s) ds}$$

Now, following our usual procedure, we can try to check the conditions under which we might find a solution at the corner where all wealth is invested in the riskless asset. This would occur if the indifference curve happened to be steeper than the budget line. So, let's evaluate the slope of the indifference curve at the bundle $(w, 0)$. Substituting into the formula above gives

$$-\frac{\int u'(w) F'(s) ds}{\int u'(w) (1+s) F'(s) ds} = -\frac{1}{1 + \int s F'(s) ds}$$

This means that whether or not the investor optimally invests in the risky asset depends entirely on $\int s F'(s) ds$. If this is positive, the (absolute value of the) slope of the indifference curve is smaller than 1 and the tangency has to occur somewhere up the budget line to the left of $(w, 0)$. On the other hand, if the mean value of s is less than zero, then the indifference curve will be steeper than the budget line, and the optimal solution will be at the corner of the budget set.

This is called the *diversification theorem*. If there is a risky asset whose expected return exceeds the expected return on the safe asset, then the optimal portfolio will always involve some investment in the risky asset.

1.4.1. Comparative Statics and Wealth. We might assume that wealthy people invest more in risky assets. This assertion seems that it must be true. Surprisingly, this is not always the case. It is difficult to see why this might be by using intuition alone. As I have often mentioned, this is when mathematics can be very useful. Intuition is rarely wrong: it just doesn't give the whole story. Math can often help you think out the parts that you tend to gloss over when you think intuitively. Often the biggest insights in economics come by understanding the complexities that intuition can't see.

The problem that is discussed in this section also provides an opportunity to see the way that comparative statics is often done in applications. We are interested in what the effect of a change in wealth w

will be on the optimal portfolio. One way to figure this out is to try to find out how a change in wealth will affect the position of the tangency between the indifference curve and budget line. To see how this line of argument works, start by substituting the fact that $i_s = w - i_r$ into the tangency condition to get

$$\frac{\int u'(w - i_r + i_r(1 + s)) F'(s) ds}{\int u'(w - i_r + i_r(1 + s)) (1 + s) F'(s) ds} = 1$$

Simplifying the arguments of the functions gives

$$\frac{\int u'(w + i_r s) F'(s) ds}{\int u'(w + i_r s) (1 + s) F'(s) ds} = 1$$

Multiplying both sides by the denominator on the left gives

$$\int u'(w + i_r s) F'(s) ds - \int u'(w + i_r s) (1 + s) F'(s) ds = 0$$

Canceling the common terms gives the equation

$$(1.1) \quad \int u'(w + i_r s) s F'(s) ds = 0$$

The trick at this point is to assume that i_r is actually a function of w that adjusts in such a way that the equation above is always satisfied. Then, in fact,

$$\int u'(w + i_r[w] s) s F'(s) ds \equiv 0$$

Since this holds uniformly under this definition, we can differentiate both sides of the expression with respect to w and the derivatives will also be equal. In other words

$$\int u''(w + i_r[w] s) s F'(s) ds + \int u''(w + i_r[w] s) \frac{di_r[w]}{dw} s^2 F'(s) ds = 0$$

Solving for the derivative gives

$$\frac{di_r[w]}{dw} = - \frac{\int u''(w + i_r[w] s) s F'(s) ds}{\int u''(w + i_r[w] s) s^2 F'(s) ds}$$

We want to know how an increase in w will change the amount invested in the risky asset. In other words, we want to know whether the derivative of the optimal value of i_r with respect to a change in wealth will be positive. This expression almost gives the answer. The denominator is an integral. Each term in the integrand is negative provided the investor is risk averse (which gives $u'' < 0$). There is a minus sign in front of the fraction, and minus times minus is positive. So, we could conclude that the derivative is positive if we could show that the numerator is positive. Unfortunately, this is not obvious if it is true.

The derivative u'' is certainly negative, but s can be either positive or negative. The sign of the integral will depend on how big u'' is when s is negative compared with how big it is when s is positive.

The lesson of the comparative statics has now been discovered. To conclude that increases in wealth raise investment, we need more information. Or, we need to restrict the set of preferences that we think are plausible. Fortunately, there is a fairly easy restriction that will do this trick. We have described the Arrow-Pratt measure of absolute risk aversion $\frac{u''(w)}{u'(w)}$. It is proportional to the size of the risk premium associated with fair gambles. Suppose we assumed that this measure of risk aversion is decreasing with wealth. That would mean that we would be assuming (or restricting attention to) investors whose risk premium falls as they become more wealthy. Surely these investors must raise investment as their wealth increases. So, let's check this out.

Write out the Arrow-Pratt measure as it appears in our comparative static equation. Then we would have

$$-\frac{u''(w + i_r s)}{u'(w + i_r s)} \leq -\frac{u''(w)}{u'(w)}$$

whenever $s > 0$ by the assumption that the Arrow-Pratt measure is decreasing. Well the whole problem arises from the fact that we don't know that s is positive. So, let's try a trick. If s is positive, it must also be true that

$$-\frac{u''(w + i_r s)}{u'(w + i_r s)} s \leq -\frac{u''(w)}{u'(w)} s$$

The nice thing about this expression is that if we change the sign of s to negative (which of course means that $-\frac{u''(w + i_r s)}{u'(w + i_r s)} \geq -\frac{u''(w)}{u'(w)}$) then it would still be true that

$$-\frac{u''(w + i_r s)}{u'(w + i_r s)} s \leq -\frac{u''(w)}{u'(w)} s$$

So, this last expression is actually correct no matter what the sign of s . So, let's multiply both sides of this inequality by $u'(w + i_r s)$ then integrate the result over s to get

$$-\int u''(w + i_r s) s F'(s) ds \leq -\frac{u''(w)}{u'(w)} \int u'(w + i_r s) s F'(s) ds$$

If you look back, you will see that the right hand side of this equation is proportional to the left hand side of (1.1) which is zero. So, we have

$$-\int u''(w + i_r s) s F'(s) ds \leq 0$$

which is just the result we wanted (since it shows that $\int u''(w + i_r s) s F'(s) ds \geq 0$).

After all this work, what we have discovered is that an investor will increase his investment in the risky asset as his wealth rises provided his Arrow-Pratt measure of absolute risk aversion is decreasing as his wealth increases.