

# Problems For All Chapters

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## 1 Preferences

1. What is a *preference relation*? What is it defined over?
2. What are the two properties of a *rational* preference relation? For each of these properties, describe a realistic situation in which the property would fail.
3. What does it mean for a person to be ‘rational’ in a particular economic environment? What does it not mean?
4. Why is it “almost tautological” to “assume that a person chooses the alternative that he or she prefers most”? Given this, where does the predictive content of economic theory come from? (i.e. what is the testable hypothesis?)
5. Why is it “a waste of time to argue whether or not consumers are rational”? What is the real objection that one might make?
6. Explain why a *rational* preference relation implies that indifference curves can not have a point in common.
7. Consider a game of tennis. If you are standing to the right, I prefer to hit the ball to the left. But, if I hit the ball to the left, you prefer to go to the left. But, if you are standing to the left, I prefer to hit the ball to the right. Thus, a consequence of my preference to hitting the ball to the left is that I prefer to hit the ball to the right (and vice versa). Does this mean my preferences are intransitive?

8. What does it mean for a function,  $u$ , to *represent* a preference relation on  $X$ ? (i.e what properties must an arbitrary function possess in order for it to be called a 'utility' function?)
9. What are the benefits of using a utility function, as opposed to working with its underlying preference relation?
10. What does it mean for a preference relation to be 'monotonic'? Describe a scenario where the assumption of monotonicity would be reasonable, and one where it would not.
11. Show that a utility function can not exist if the preference relation is not transitive. *Hint:* Try a 'proof by contradiction.'
12. Pick an arbitrary consumption bundle  $(x, y) \in \mathbb{R}_+^2$  and draw the sets  $B$  and  $W$  (in  $\mathbb{R}_+^2$ ) for the following situations:
  - (a) I like consuming  $x$ , and the more the better. I am completely indifferent to the amount of  $y$  that I consume.
  - (b) As long as my consumption of  $x$  is less than  $\bar{x}$ , the more the better. After  $\bar{x}$  the less the better. I am completely indifferent to the amount of  $y$  that I consume.
  - (c) I like both  $x$  and  $y$ , but a unit of  $x$  must be paired with a unit of  $y$  in order for me to enjoy it (like left and right shoes).
  - (d) I prefer the bundle  $(\bar{x}, \bar{y})$  to all other bundles. If I have any other bundle, then I am indifferent between having this bundle and having nothing  $(0,0)$ .
13. This question is designed to take you through the steps of the proof of the Theorem presented in the text (the existence of a utility function) by using specific description of preferences.

Garrett likes owning shoes, but only gets use out of them if they are in a pair. Let  $L$  be the number of left shoes he owns, and  $R$  be the number of right shoes he owns. Now consider adding more shoes to the bundle  $(L, R)$ . In particular, if Garrett were to add  $\ell \geq 0$  left shoes and  $r \geq 0$  right shoes, then he would (strictly) prefer the new bundle,  $(L + \ell, R + r)$ , to the original bundle,  $(L, R)$ , if and only if *both*  $\ell$  and  $r$  are strictly positive. For example, this implies that he is indifferent to having more left shoes but not more right shoes.

14. Are these preferences rational (complete and transitive)? Are they monotonic?
- Draw (with  $L$  on the vertical axis, and  $R$  on the horizontal axis) the following bundles and sets.
    - $Z$ .
    - An arbitrary bundle,  $(L, R) \in \mathbb{R}_+^2$ .
    - $B$  and  $W$  (relative to the above bundle).
    - $P^+$  and  $P^-$  (again, relative to the above bundle).
    - The 'indifference bundle',  $z \in Z$ .
  - Construct the utility function as suggested in the proof (*hint*: Pythagoras' theorem might come in handy).
  - Can you think of a simpler utility function?
  - Check that both utility functions (the suggested one, and the simpler one) actually represent preferences.
15. Repeat Question 13 using the following description of preferences.
- Ken also like shoes, but unlike Garrett, he does not wear them. He makes 'shoe sculptures' out of them. As far as he is concerned, a left shoe is equally as good as a right shoe. That is, suppose that we were to add more shoes to Ken's initial bundle of  $(L, R)$ . In particular, if we were to add  $\ell \geq 0$  left shoes and  $r \geq 0$  right shoes, then he would (strictly) prefer the new bundle,  $(L + \ell, R + r)$ , to the original bundle,  $(L, R)$ , if and only if *either*  $\ell$  or  $r$  is strictly positive<sup>1</sup>
16. These questions are designed to highlight the role of the various assumptions used in the proof.
- What part of the proof would fail if we removed each of the following assumptions?
    - Continuity.
    - Monotonicity.
    - Transitivity.
    - Completeness.

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<sup>1</sup>This includes the case in which both are strictly positive.

Which of these are ‘crucial’ in the sense that a failure of the assumption does implies that a utility function will not exist?

- (b) The proof was only concerned with consumption bundles in  $\mathbb{R}_+^2$  (two-good consumption bundles). What problems, if any, would be introduced if we considered consumption bundles in  $\mathbb{R}_+^k$  ( $k$ -good consumption bundles), where  $k > 2$ ?
- (c) The proof was by ‘construction:’ it showed that a utility function existed (under the stated assumptions) by actually building such a function. Is this suggested function unique? Prove your claim.

17. Draw a graph showing the indifference curve associated with each of the following utility functions when the level of utility is equal to 1

- $u(x, y) = \max\{x, y\}$
- $u(x, y) = x + 2y$
- $u(x, y) = x - 2y$

18. Simone has an endowment of \$20 in period 1 and \$40 in period 2. Her utility is  $u(c_1, c_2) = c_1 + c_2$ . She can put money in a bank account in period 1, and get it back again in period 2 without interest. She can also borrow money from a loan shark. For each dollar she borrows in period 1, she must pay back \$1.50 in period 2. She can also buy a chair in period 1 for \$10 that she knows that she will be able to resell in period 2 for \$15.

- Draw her budget line assuming that she does not buy the chair (labeling you diagram carefully).
- Draw her budget line if she does buy the chair.
- Should she buy the chair, or not? What consumption bundle should she pick in each case?
- Predict her consumption, and whether or not she will buy the chair if her utility function is instead given by  $u(x, y) = \min\{x, y\}$ .

19. Solve the problem

$$\max 2x^{\frac{1}{2}} + 2y^{\frac{1}{2}}$$

subject to the constraints  $3x + y \leq 10$ ;  $x \geq 0$ ;  $y \geq 0$  using the method of Lagrangian multipliers. Write down the Lagrangian function, and each of the first order conditions before solving the problem.

20. Solve these problems:

- (a) Solve for the optimal value for  $x$  and  $y$  in the problem  $\max u(xy) = \frac{xy}{x+y}$  subject to  $px + y \leq W$ ,  $x \geq 0$ ,  $y \geq 0$  using the method of Lagrangians. Hint: the derivative of  $u(x, y)$  with respect to  $x$  is  $\frac{y^2}{(x+y)^2}$ .
- (b) Find the demand function associated with utility function  $u(x, y) = x + xy$  (Hint: there are corner solutions to this problem - the Lagrangian method is too slow for this problem).
- (c) Solve the problem maximize  $u(x, y) = x$  subject to  $2x + y \leq 10$ ,  $x \geq 0$ , and  $y \geq 0$  using the method of Lagrangians (i.e., find the optimal value for  $x$  and  $y$  and all the multipliers).

21. A consumer has downward-sloping indifference curves for goods  $x$  and  $y$ . Below the  $45^\circ$  line, the indifference curves are straight lines with slope  $-\frac{1}{2}$ . When the curves hit the  $45^\circ$  line, they become steeper. Above the  $45^\circ$  line they are also straight lines but their slope is equal to  $-2$ . Can you use the method we used to prove the existence of a utility function to provide a utility function that represents these preferences?

22. Consider the following situation:

Geraldine prefers sports cars to hybrid cars because sports cars are faster and more powerful. She prefers SUVs to sports cars because SUVs are roomier. She prefers hybrid cars to SUVs, however, because they are more fuel-efficient and environmentally sustainable. *Honest Abe's Used Cars* charges \$1000 to trade in Geraldine's vehicle each time to "help her make up her mind."

Why is it that people "who exhibit intransitive preferences [...] quickly change their behavior when this is pointed out to them"?

## 2 Demand Theory

1. In what sense is the optimal consumption problem a special case of the more abstract choice problem introduced in the previous chapter? In particular, what is the special name given to the choice set  $\mathcal{X}$ ? What is a typical element  $x \in \mathcal{X}$ ?
2. Explain how the ‘existence of a utility function’ theorem allows us to make progress with understanding consumer demand.
3. In order to make sharper predictions about economic behaviour, we will often need to add assumptions (e.g. on preferences) to the more fundamental axioms. The collection of such assumptions are known as a *model*. What is the point of analysing a model, when a model is simply a set of assumptions?
4. What are the two ‘big’ assumptions made by Classical consumer theory? Think of a scenario in which each would be inappropriate.
5. Do the classical assumptions require that a consumer have an unchanging preference relation over  $\mathbb{B}$ ? That is, suppose that we observed a consumer’s choice of consumption bundle at two points in time,  $x_t^*$  and  $x_{t'}^*$  (where both of these are in  $\mathbb{R}^n$ ). Further, suppose that prices and incomes were identical in the two time periods. What predictions could we make about  $x_t^*$  and  $x_{t'}^*$  if we allowed for the possibility that the consumer’s preference relation over  $\mathbb{B}$  changed over time? Could we ever find evidence that our model is wrong?
6. There are two goods,  $x$  and  $y$ . Their prices are  $p_x$  and  $p_y$ , and income is  $W$ . Draw  $\mathbb{B}$ , the budget set. Show (graphically) the effect of
  - (a) An increase in  $W$ .
  - (b) An increase in  $p_x$ .
  - (c) All prices changing by a factor of  $\alpha$ .
  - (d) All prices and income changing by a factor of  $\alpha$ .

What is the implication of the last of these for the classical consumer theory assumptions?

7. A consumer is growing tomatoes and has  $W$ kg available for consumption in period 1. Whatever she doesn't eat in a period will grow by a factor of  $R$  by the next period. Let  $x_t$  represent the consumption of tomatoes in period  $t$ .
- Draw the budget set for the case when there are only two periods.
  - Describe this budget set in formal notation.
  - Describe this budget set in formal notation for the case in which there are  $T$  periods.
8. Derive the demand function when preferences are given by:
- $u(x, y) = x^2 + y^2$ . *Hint:* This highlights the fact that the FOCs are necessary but **not** sufficient - draw the indifference curves.
  - $u(x, y) = x + y$ .
9. Suppose that a consumer's preferences are described by a utility function,  $u(x)$ . Prove that it is also true that his preferences are represented by a new function,  $h(x)$ , which is defined as  $h(x) \equiv g(u(x))$ , where  $g$  is an increasing function (e.g. an exponential or logarithmic function).
10. Use the previous question to help you derive the demand function when preferences are given by

$$u(x, y) = Z^{f(x, y)}, \quad (1)$$

where  $Z > 1$  and

$$f(x, y) \equiv A - \frac{1}{\alpha \ln(x) + (1 - \alpha) \ln(y)}. \quad (2)$$

11. A quick inspection of the properties of the utility function often makes it much easier to derive the demand functions since the effects of the constraints are more transparent. For each of the utility functions below, identify the conditions under which positive quantities of both goods will be consumed at the optimum (so that the non-negativity constraints can be ignored when such conditions are met):

- (a)  $u(x, y) = x^\alpha y^\beta$ , where  $\alpha, \beta > 0$ .
  - (b)  $u(x, y) = x + \ln(y)$ .
  - (c)  $u(x, y) = x^\alpha + y^\alpha$ , where  $\alpha \in (0, 1)$ .
  - (d)  $u(x, y) = x^\alpha + y^\alpha$ , where  $\alpha \in [1, \infty)$ .
  - (e)  $u(x, y) = (x + y)(1 - x - y)$ .
  - (f)  $u(x, y) = (x^\alpha + y^\alpha)(1 - x^\alpha - y^\alpha)$ , where  $\alpha \in (0, 1)$ .
12. For each of the functional forms in the previous question, identify the conditions under which the income constraint will be binding.
  13. Suppose we want to test the basic assumption that a consumer's preferences are independent of the budget set. Describe the test proposed in the text. What is the extra assumption on preferences that was required? Can you think of another approach to testing this basic assumption?
  14. It is often thought that econometricians assist theorists by collecting and analysing data in order to determine whether a model is a plausible description of the world. This question shows that the reverse is sometimes true too - theorists may assist econometricians in terms of how they construct their econometric model.

Elliot the eager econometrician knows that consumer theorists talk about demand functions, and are especially interested in the effects of income and prices. Elliot thinks that theorists are full of hot wind and wants to test certain aspects of demand theory. To do this, each day he carefully records the prices of the  $n$  goods that he consumes. Along with this, he also records daily his income and his chosen consumption bundle (a quantity for each of the  $n$  goods). Thus, after a year or so he has a data set that contains observations on prices, incomes and demands. To facilitate his many probing tests, he needs to determine how prices and income influence his demand. To keep things simple, he decides to focus on his demand for pocket protectors. The most basic specification that comes to mind is the following econometric model:

$$x_{1t} = \beta_0 + \beta_w W_t + \alpha_1 p_{1t} + \sum_{i=2}^n \alpha_i p_{it} + \varepsilon_t$$



where  $x_{1t}$  is his demand for pocket protectors on day  $t$ ,  $W_t$  is his income on day  $t$  and  $p_{it}$  is the price of good  $i$  on day  $t$  (pocket protectors are good 1).

15. While waiting for his expensive statistical package to load, Elliot asks your opinion on his specification of the demand function. Do you see any problems? In particular think about:
  - (a) If demand theory is taken seriously, what *must*  $x_{1t}$  be when  $W_t = 0$ ? How does this restrict the possible values of  $\beta_0$  and  $\alpha_j$  ( $j = 1, \dots, n$ )? Does demand theory itself impose these latter restrictions on the partial effects of prices on demands?
  - (b) Again, if demand theory is taken seriously, then how must  $x_{1t}$  change when all prices and income are changed in the same proportion? How does this restrict what  $\beta_w$  and  $\alpha_j$  ( $j = 1, \dots, n$ ) can be? Does demand theory itself impose these latter restrictions on the partial effects of income and prices on demands?
16. The general mistake that Elliot has made was to forget that a demand function is *derived* from preferences. That is, he assumed a demand function (based on simplicity) rather than assuming a form on preferences. In other words, it may very well be the case that there are no preferences that would produce this particular demand function. Do you have any general advice to give Elliot before he decides on some other specification?
17. Use graphical methods to describe preferences (draw indifference curves) for goods  $x$  and  $y$  for the following scenarios.
  - (a) As income increases, the demand for good  $x$  falls.
  - (b) As the price of good  $x$  increases, so too does demand for  $x$ .
  - (c) As income changes, the ratio of the demand for  $x$  to the demand for  $y$  is constant.
  - (d) As the demand for  $y$  is unaffected by the price of good  $x$ .
18. Use the above question to answer the following.

- (a) Is it possible for the quantity demanded of  $x$  to decrease as income increases, yet increase when its price increases?
- (b) Is it possible for the quantity demanded of  $x$  to increase as its price increases, yet increase when income increases?
- (c) Is it possible for the quantity demanded of *both*  $x$  and  $y$  to fall as income increases?

### 3 Discontinuous Budget Sets

1. In the text, a non-linear pricing problem was analyzed in which the per unit price increased after some threshold demand was reached. In this problem, we do the same thing, but assume that the per unit price falls. Let the per-unit price of good  $y$  be constant and equal to 1. The per unit price of good  $x$  is assumed to be  $p$  for each unit up to the  $n^{th}$ . Then each additional unit costs  $p - dp$ . In other words, each additional unit of output purchased above and beyond the first  $n$  costs *less* than the first  $n$  unit. Assume that preferences are given by

$$u(x, y) = x^\alpha y^{1-\alpha}$$

Repeat the exercise given in text, and show how demand for good  $x$  varies with  $\alpha$ .

2. Tickets for the World Cup Quidditch match have been sold in bundles of three, and are now being resold by scalpers. The lowest priced scalper offers his three tickets at a unit price of \$ 100. So if a consumer wants  $x \leq 3$  tickets, the price she will pay is  $100x$ . The next lowest price scalper is offering her three tickets for \$ 110, the cheapest after that is \$ 120 and so on. A consumer has preferences given by  $u(x, y) = x^\alpha y^{1-\alpha}$  where  $x$  stands for the number of tickets she purchases, and  $y$  is the amount of money she has left over for other stuff. She has \$ 3000 to spend on tickets and other stuff. Write a computer program that takes as its input the value of  $\alpha$  and outputs the number of tickets that the consumer with that value of  $\alpha$  will buy. You can assume that tickets are infinitely divisible.

Hint: If our consumer decides to buy  $x$  tickets, then she will pay as follows: compute  $x/3$  and take its integer part and call it  $j(x)$  (e.g. the

integer part of  $22/3$  is 7, so  $j(22/3) = 7$ ; let  $p_0 = 90$ ; compute

$$\sum_{i=1}^{j(x)} 3(p_0 + i10) + (p_0 + j(x)10)(x - j(x)3)$$

This procedure gives you the slope and position of the budget line for different values of  $x$ . Now generalize the method in the text to solve the problem.

One way to do this would be to write a short computer program which takes as its input a value for  $\alpha$  and outputs a demand and expenditure on tickets. Scripting on computers is a useful skill to develop as an economist. It doesn't really matter which language you use to write scripts (c, java, perl, bash, php etc) since they all use similar principals which you can apply in a variety of contexts, for example, statistics and econometrics packages, or computer algebra programs.

3. In the text, the response of a consumer with quasi-linear preferences to the imposition of a fixed fee is analyzed. Preferences in this problem are given by  $u(x, y) = y + \log(x)$ . Consumer income is  $W$  and the firm's pricing scheme requires that the consumer pay a fixed fee  $K$ , which entitles her to  $n$  'free' units of good  $x$ . Each unit in addition to the first  $n$  costs the consumer  $p$ . Hold the price  $p$  and the initial number of units  $n$  constant, and suppose that the firm using this scheme wants choose a fee  $K$  to maximize the profit it receives from this consumer. What fee would it choose? If you were the competitor of this firm (i.e., you are the firm selling good  $y$  what might you do to counteract this strategy by the firm?

3. Suppose that a consumer has \$100 to spend during the week and likes to use some of this money to play tennis. She can buy time at the local court for \$1 per hour, but the local court will only let her play for a maximum of 20 hours. If she wants to play for more than 20 hours, she has to pay a fee of \$20 at the private tennis club. Then she can buy additional time for \$2 per hour. Suppose that her preferences are represented by a Cobb-Douglas utility function  $u(x, y) = x^\alpha y^{1-\alpha}$  where  $x$  is the number of hours of tennis she plays and  $y$  is the money she has left over for other stuff. Draw a picture to describe the budget line she faces. Label each of the important points in the diagram. Give a formula to show how much time she will spend playing tennis at the local court and how much time she will spend at the private court as a function of the number  $\alpha$ .

4. Our consumer has \$ 20 per month to spend on her cell phone and

comic books. Comic books cost \$ 1 each. The phone company has offered her two different deals - for \$ 6 per month she can have 100 'free' minutes on the phone. Each minute thereafter will cost her 10 cents. Or, if she wants, she can take the deluxe plan, \$ 10 per month which entitles her to 150 free minutes. As a bonus, extra minutes are then charged at the lower rate of 5 cents. Draw the consumer's budget set in this case. Suppose her preferences are given by  $u(x, y) = x^\alpha y^{1-\alpha}$ . Explain how her choice of plan and telephone use should depend on  $\alpha$ .

## 4 Best Reply Behavior

1. Alice and Bob both produce Caesar salad dressing in the same kitchen. Each of them has an endowment of one unit of labor that can be used as effort in producing the dressing, or can be used to play video games. Both Alice and Bob have identical preferences given by:

$$U(x, y) = \ln(x) + y, \quad (3)$$

where  $x$  is the hours of TV watched and  $y$  is the output of Caesar salad dressing.

If an amount of effort,  $z_i$ , is used in Caesar salad dressing production by person  $i$ , then the amount of dressing produced is given by

$$y_i = f_i(z_i) = S_i z_i, \quad (4)$$

for  $i \in \{\text{alice}, \text{bob}\}$ . Note that  $S_i$  is the marginal product of production effort for person  $i$ . An hour of video game playing is produced one-for-one from the labor endowment. That is (by substituting in the resource constraint),  $x_i = 1 - z_i$ .

The thing is that making the dressing is hard when the other person is not in the kitchen very much (because it is time-consuming to look for the spoons etc, and it gets very boring and lonely). In particular, it turns out that

$$S_a = cz_b, \quad \text{and} \quad S_b = cz_a, \quad (5)$$

where  $c > 2$ .

Suppose that Alice guesses that Bob will spend  $z_b^*$  hours in the kitchen. Write out Alice's optimization problem.

2. What is her best response number of hours in the kitchen,  $z_a(z_b^*)$ ? Graph this best response in  $(z_a, z_b)$  space.
  - (a) Do the same exercise for Bob, but place his best response on the same graph that you have just drawn. *Hint:* The two are identical, so the problem (and graph) should be very similar.
    - i. Graphically identify the Nash equilibrium levels of production effort,  $(z_a^*, z_b^*)$ . Explicitly solve for the solution when  $c = 3$ . *Hint:* You will need to use the quadratic formula.
3. Consider the following (extreme form of) consumption externality. Alice is going to Bob's house for dinner, and has agreed to bring something to drink. After arriving at the store, she realizes that she has no idea what Bob is planning to cook. Neither she or Bob cares much what they eat. If Bob is cooking pizza, she (and Bob) would prefer that she bring beer. If Bob is cooking pasta, then she (and Bob) would prefer that she bring wine. Bob has a similar problem - what he wants to cook will depend on what he thinks Alice will bring. Which outcomes (choices of a food and a drink) seem reasonable? Does it seem possible to predict exactly what kind of drink Alice will buy?

## 5 Expected Utility

1. What are the two objects that define a *lottery*? Define the lottery associated with rolling a fair die.
2. How is a *simple lottery* different from a *compound lottery*? What is a *reduced lottery*?
3. Write the reduced lottery associated with the following (compound) lotteries.
  - (a) A coin is tossed. If it comes up heads, you roll a fair die and get an amount of money equal to the number that turns up (e.g. if you roll a four, you get \$4). If it comes up tails, you toss the coin six more times and get an amount of money equal to the

number of tails that comes up in the six tosses (e.g. if four of the six were tails, you get \$4).

- (b) A coin is tossed. If it comes up heads, you roll a fair die and get an amount of money equal to the number that turns up (e.g. if you roll a four, you get \$4). If it comes up tails, you start the process over again.
- (c) A coin is tossed. If it comes up heads, you toss a second coin. If this second coin comes up heads, you get \$10 today. If the second coin comes up tails, you get \$10 in one year's time. If the first coin instead comes up tails, you get \$10 in two years time.

4. Consider the following compound lottery, with probability  $\frac{1}{4}$  you will play the lottery  $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ , with probability  $\frac{3}{8}$  you play the lottery  $\{0, 1, 0\}$ , while with probability  $\frac{3}{8}$  you play the lottery  $\{0, 0, 1\}$  where each element in the list is the probability with which you receive payoffs  $\{1000, 500, 0\}$  respectively. What is the reduced lottery associated with this compound lottery.
5. You tell your friend the Monty Hall problem (game show host revealing doors etc.). They are convinced that it is irrelevant whether or not you switch doors based on the following argument. "No matter what door I choose, I will end up in a situation in which I am choosing between two doors, one of which will have the prize behind it. Since the prize is equally likely to be behind all doors, the chance that it is behind the door I chose is  $1/2$ , which is the same as the chance of it being behind the other door. Therefore, it doesn't make a difference if I change doors". Explain where your friend is making an error in their reasoning.
6. Consider again the Monty Hall problem. At first glance, what did you think the probability of winning the prize was? What is the actual probability of winning the prize (assuming you switch doors as suggested)?
7. We know that it is always better to switch doors in the 3-door version of the Monty Hall problem. Will this remain true for  $n$  doors? If no, at what value of  $n$  does it stop being worthwhile to switch doors? If yes, does the benefit to switching increase or decrease as  $n$  increases?

8. Change the Monty Hall problem so that the prize is initially placed behind door A with probability  $\frac{1}{2}$  (instead of probability  $\frac{1}{3}$  as in the problem we discussed in class). The prize is placed behind doors B and C with equal probability  $\frac{1}{4}$ . Suppose you choose door A. Monty then opens one of the other doors and shows you there is nothing behind it. You are offered the chance to switch doors. What is the probability that you will find the prize if you keep door A? What is the probability if you choose to switch to the other door? Do you think you could increase your chances of finding the prize by choosing a door other than door A to start with?
9. Consider the St. Petersburg Paradox. When considering how much to pay for this lottery, it was argued that you could not lose by paying \$2 or less since you are guaranteed to win at least this amount. On the other hand, there is no limit to how much you could win. However, if you win after  $t$  rounds the amount that you win will be finite. That is, if you win in round  $t$ , you get  $1/2^t$  which is a finite number if  $t$  is finite. That is, the most you can win is strictly less than infinity. However, it was shown that the expected value of the lottery (and therefore how much a risk-neutral individual would be willing to pay) is infinite. How could it be that someone is willing to pay an amount for a lottery that is greater than the highest possible amount they could win?
10. Suggest two reasons why no individual would be willing to pay an infinite amount for the lottery described in the St. Petersburg Paradox (besides things like no-one has that much money).
11. A simplifying assumption used throughout the chapter was that all the lotteries in  $\mathcal{L}$  had the same set of possible outcomes,  $\mathcal{X}$ . In what way does this assumption simplify the analysis? Is the assumption restrictive? Why or why not?
12. State the independence axiom both formally and intuitively. Can you think of a scenario where this axiom is likely to fail?
13. The independence axiom is concerned with 'mixing' or 'combining' lotteries with a third lottery. Care must be taken in understanding exactly what it means to 'mix' two lotteries. The question is designed to highlight a common error.

Suppose there are two possible and equally-likely states of the world: I toss a coin, and it either comes up heads or comes up tails. Lottery 1 gives you \$2 if it is heads, and \$0 if it is tails. Lottery 2 is the reverse - you get \$2 if it is tails, and \$0 if it is heads.

14. These two lotteries are clearly different (by name if by nothing else). Are the differences meaningful in terms of the treatment of lotteries given in the text? The following questions help with this issue.
  - (a) What is the difference between the  $\mathcal{X}$  associated with these two lotteries?
  - (b) What is the difference between the  $p$  associated with these two lotteries?
  - (c) Given your answers to the above two questions, what is the difference (if any) between the lotteries?
15. what are the three assumptions on the preference relation over  $\mathcal{L}$ ? Explain what each of these means. for each assumption think of a realistic situation in which the assumption would be inappropriate.
16. In the text, the set of feasible lotteries over three possible outcomes is drawn as a triangle in two dimensions. What would the set of feasible lotteries look like if there were only two outcomes? Can you imagine how to draw the set of feasible outcomes when there are 4 outcomes? How many dimensions are needed in order to draw the set of feasible lotteries when there are  $k$  outcomes? Why?
17. Consider the following lotteries  $q = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}$ ,  $q' = \left\{ \frac{1}{10}, \frac{3}{10}, \frac{3}{5} \right\}$  and  $q'' = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right\}$ . Create a compound lottery over these three lotteries in which each of them is run with equal probability. What is the reduced lottery associated with this compound lottery?
18. A coin is flipped at most three times. If it comes up heads on the first try, the game ends and you win \$2. If it comes up tails, it is flipped again. If it comes up heads on the second try, you win \$2. If it comes up tails, it is flipped again. If it comes up heads on the third try, you lose \$2. If it comes up tails, the game is over and no one wins anything. Draw a tree depicting the compound lottery over



compound lotteries that is involved in this process. Compute the reduced lottery associated with it. What is the expected value of this lottery?

19. In the Monty Hall problem, use Bayes rule to compute the posterior probability that the prize is behind door  $A$  once you have chosen  $A$  and the show host opens door  $C$ . (Bayes rule -  $\Pr\{A|C\} = \Pr\{A \cap C\} / \Pr\{C\}$ . In words, the probability of event  $A$  conditional on event  $C$  is the probability that both  $A$  and  $C$  occur, divided by the probability that event  $C$  occurs. Draw yourself a picture to see if you can understand why this should be so).
20. Suppose that the set of lotteries is equal to all lotteries over three outcomes,  $x_1$ ,  $x_2$  and  $x_3$  as above, with  $x_1$  being the best outcome, and  $x_3$  the worst. What are the best and worst lotteries  $b$  and  $w$  used in the expected utility theorem in this case? What are the utility values assigned to these lotteries by the construction in the theorem?
21. Show that the utility function constructed in the expected utility theorem above actually represents preferences in the sense that if  $u(p) > u(p')$  then  $p \succ p'$  and conversely.
22. Draw the lotteries that Allais suggested in the last section above in a diagram and show that anyone who prefers  $q$  to  $q'$  must also prefer  $p$  to  $p'$  if they have expected utility preferences.

## 6 Risk Aversion

1. What is a probability distribution function?
2. Suppose that the lottery  $(p, \mathcal{X})$  can be represented by a probability distribution function,  $F$ .
  - (a) Does this impose any restriction on the set of objects in the set  $\mathcal{X}$ ? For example, could  $\mathcal{X} = \{\text{red ball, green ball, blue ball}\}$ ?
  - (b) Suppose in addition to representing  $(p, \mathcal{X})$ ,  $F$  happens to be such that it has a *density*, in other words, it has a derivative. What additional properties does  $\mathcal{X}$  have to possess for this to be true? For example, could  $\mathcal{X} = \{-2, -1, 0, 1, 2\}$ ?

3. If  $F$  does have a density (derivative) and that  $\mathcal{X} = [0, 1]$ . What does  $\int_0^x F'(x')dx'$  equal?
4. Again, using the assumption that  $\mathcal{X} = [0, 1]$ , consider a degenerate lottery that gives \$z for sure. Write the  $F$  that characterizes this lottery.
5. Write the  $F$  that characterizes the following lottery. A coin is tossed, and you get \$1 if it comes up heads and -\$1 if it comes up tails.
6. Intuitively explain what a 'risk premium' is. Explain how to calculate it.
7. What is a 'fair bet'? A person is said to be *risk averse* if they have a positive risk premium when faced with a fair bet. Suppose that I tell you that a person associates a risk premium of -\$5 with some lottery  $\mathcal{L}$  (not necessarily a fair lottery). What can we say about the person's attitude toward risk?
8. Is it possible that an individual has a different risk premium associated with different lotteries? Explain.
9. Is it possible that two individuals with the exact same preferences over all lotteries have a different risk premium when faced with identical lotteries? If not, explain why not. If so, what must be different across individuals?
10. Daron is unsure how much he will get through a student loan this year. There is a 50% chance that he will get \$2000 and 50% chance that he will get \$3000. He already has \$1000 saved.
  - (a) Express the lottery that Daron faces,  $\mathcal{L}$ , in terms of  $(p, \mathcal{X})$ . Now write it in terms of  $(W, F)$ .
  - (b) What allows us to express his relative preference for the lottery  $\mathcal{L}$  in the form:

$$U(\mathcal{L}) = p_1 u(x_1) + p_2 u(x_2) \quad ? \quad (6)$$

- (c) Suggest an interpretation of the function  $u(x)$ . Using your answer to the first part, fill in what  $(p_1, p_2, x_1, x_2)$  are in this case.

- (d) Write an expression for the *risk premium* associated with  $\mathcal{L}$ . Is this positive or negative? Does this make sense in terms of your intuitive understanding of what a risk premium represents?
  - (e) Calculate the risk premium if we assumed that  $u(x) = \sqrt{x}$ .
11. Explain carefully how the expected utility theorem is useful in the process of calculating a risk premium.
  12. What is the Arrow-Pratt measure of risk aversion? What is the relationship between the risk premium and the Arrow-Pratt measure of risk aversion? What simplifications were made in deriving this relationship? Under what cases would the simplifications be inappropriate?
  13. What is the 'Portfolio Problem'? In what way does each choice of the  $(i_s, i_r)$  pair generate a lottery? Write this lottery in terms of a probability distribution function,  $F$ .
  14. What does the *Diversification Theorem* say? Does it only apply to those investors that are not too risk averse? What was the intuitive argument based on? Describe an aspect of the real world that might cause individuals to behave in a manner that is seemingly inconsistent with the theorem.
  15. The *Diversification Theorem* was derived without actually ever solving for the optimal portfolio. This question verifies the Theorem by solving for the optimal portfolio.  
We are going to (implicitly) find the optimal investment in the risky asset,  $i_r$ , under the assumption of a specific form of  $u(x)$ .
  16. Argue that the wealth constraint  $i_r + i_s \leq W$  can safely be assumed to hold with equality (given the existence of a risk-free asset).
    - (a) Use this to write the objective function in the form:

$$\int u(W + i_r s) F'(s) ds \tag{7}$$

- (b) Show that if the agent is risk-neutral (i.e.  $u(x) = x$ ), the objective function can be written in the form:

$$\alpha_1 + \alpha_2 \cdot i_r, \quad (8)$$

where  $\alpha_1$  and  $\alpha_2$  are constants that are specific to  $W$  and  $F$ .

- (c) Remaining in the risk-neutral case, what condition on  $F$  will ensure that  $i_r$  is positive? Does the level of  $W$  matter (apart from  $W > 0$ )? Is this general, or a feature of risk-neutrality?
- (d) Now suppose that there is some risk-aversion. In particular, suppose that  $u(x) = x^\alpha$ , where  $\alpha \in (0, 1)$ . Suppose that the optimal portfolio had the property that  $i_r = 0$ . Use the FOC to derive the property of the lottery that must have induced this. Does this make sense?
17. Comparative static techniques were used to show that if an investor's preferences are such that the Arrow-Pratt measure of (absolute) risk-aversion is decreasing in wealth, then the investor will invest more in the risky asset as their wealth increases.
- (a) The fact that an investor invests more as wealth increases might have nothing to do with risk aversion, but rather simply that they have more available wealth. Does the amount invested in the risky asset increase with wealth for investors that have an Arrow-Pratt measure of risk aversion that is *constant* over wealth levels?
- (b) What happens to the *proportion* of wealth allocated to the risky asset as wealth increases (for an investor with a decreasing Arrow-Pratt measure of risk aversion)? That is, calculate the comparative static  $\partial(i_r(w)/w)/\partial w$ .
18. Suppose it was observed that wealthier individuals tended to invest more in risky assets. Mike claims that this is evidence of investors having preferences that are adequately described by a utility function that exhibits a decreasing level of risk aversion (as measured by the Arrow-Pratt measure). Eric disagrees. He claims that wealthier people invest more because they are less affected by transactions costs.

- (a) For Eric's claim to make sense, what kind of transactions costs must he be thinking of? For example, would his argument apply if one had to pay \$t for every dollar invested? What about if it cost \$t per transaction (regardless of size)?
- (b) Describe how you go about determining who (Mike or Eric) had the better interpretation of the observation that wealthier individuals invest more in the risky asset.

19. Compute the Arrow Pratt Measure of Absolute Risk Aversion for the following utility functions

- $u(w) = \ln(w)$
- $u(w) = a + bw - cw^2$  (For what values of  $b$  and  $c$  is this concave? What happens to the Arrow Pratt Measure outside this region?)
- $u(w) = -w^{-\beta}; \beta > 0$
- $u(w) = -e^{-w}$

20. Evaluate

$$\int u'(w + i_r s) s F'(s) ds$$

and

$$\int u''(w + i_r s) s F'(s) ds$$

for each of the utility functions above assume that  $s$  is distributed uniformly on the interval  $[-1, 3]$  (which means that  $F(s) = \frac{s+1}{4}$ ).

21. A more interesting question than the one in the reading is to ask whether an individual invests a higher *proportion* of his or her wealth in the risky asset as their wealth rises. Use the method in the reading to prove that this is true provided the Arrow Pratt measure of *relative risk aversion* is decreasing in wealth where the Arrow Pratt measure of relative risk aversion is given by

$$-\frac{u''(w)}{u'(w)}w$$

## 7 Insurance Theory

1. Write an expression for the profits of the insurance company in terms of  $p$ ,  $q$  and  $b$ , and derive the *actuarially fair* premium,  $q^*$ , as a function of  $p$  and  $b$ .
  - (a) Draw the 'budget set' for the consumer when the firm charges  $q^*$  per unit of net payout,  $b$ .
  - (b) Show how this set would change when the firm charges a premium higher than  $q^*$ .
  - (c) How does this increase in the premium affect the consumer's optimal insurance policy choice?
2. Martha is considering holding an outdoor cooking show for which she will get ticket sales of  $\$y$  (Martha has no other wealth - due to recent kind donations to the Expensive Lawyers Foundation). However, there is some chance that it will rain, in which case she has to partially refund the tickets to the show. In total, she will have to refund  $\$d$ . Martha is risk averse and has preferences represented by  $u(\cdot)$ .

She is considering buying insurance against the possibility of rain. She discovers that all insurance firms agree that the chance of rain on the day of her show is  $p$ , and as such are willing to offer any contract that will pay a net amount of  $b$  in the event of rain, in exchange for a premium of  $q$  such that their profits are zero.

On a recent holiday to Jailstown, Martha met a very clever meteorologist that specialized in rain predictions. Martha phones this guy to get his inside<sup>2</sup> opinion on what the chance of rain is. He says that his extensive and top-secret analysis reveals that the chance of rain is  $p'$ .
3. Based on the insurance firms' belief that the probability of rain is  $p$ , write out the zero-profit relationship between  $q$ ,  $b$ , and  $p$ .
  - (a) Martha trusts the meteorologist, and believes that the probability of rain is not  $p$ , but in fact  $p'$ . Does Martha still buy full insurance? How does this depend on the advice,  $p'$ ?

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<sup>2</sup>The opinion is 'inside' in the sense that the insurance firms do not know it.

4. Let the utility for wealth function be given by:

$$u(x) = (\alpha + x)^\beta, \quad (9)$$

where  $\alpha, \beta \geq 0$  are parameters.

- (a) For what values of  $\beta$  is  $u$  a concave function? Does this depend on  $\alpha$ ?
- (b) Suppose that  $\beta < 1$ . For what values of  $\alpha$  is  $u'(x)|_{x=0} = \infty$ ?
- (c) Using the standard notation from the text, suppose that  $d = y$  (so that in the accident state the consumer has zero wealth). Use the first two parts to determine the parameter values such that a consumer with preferences that are given by  $u$  will buy some insurance *regardless of  $y$  and  $q/B$*  (as long as they are both positive). If you are not able to provide an intuitive answer, use Lagrangian techniques.

5. Ben loves hockey. He really enjoys it when his favorite team, the Calgary Flames, are playing well. He has an income of  $y$ , and gets a utility of  $u(y)$  when the Flames win. However, when the team loses, he gets upset and says that he would give up  $\$d$  dollars in order for the Flames to have played better and won (so that his utility when the team loses is  $u(y - d)$ ). Ben is **risk averse**, so that  $u''(\cdot) < 0$  (and  $u'(\cdot) > 0$ ).

A betting agency offers the following deal: you can pay  $\$q$ , and if the Flames lose you get a net payout of  $\$b$  (i.e.  $b$  is in addition to getting your  $\$q$  bet back). There is competition among betting agencies which implies that their profits are zero. Everyone (Ben, the betting agencies, and everyone else) evaluates the probability that the Flames will win at  $1 - p$ .

- 6. Write out the betting agency's profit function, and write the zero-profit relationship between  $q$  and  $b$ . How does  $q/b$  change as the Flames become more likely to win (i.e. as  $p$  decreases)?
  - (a) Does Ben find it worthwhile to take the bet? That is, solve Ben's optimal choice of  $b$  and  $q$  (subject to the zero-profit condition) and determine whether  $b^* > 0$ .

- (b) Does the answer to the last part surprise you? In particular, how would you reconcile the fact that Ben is betting on his favorite team losing? What about the fact that Ben is risk-averse?
  - (c) What would  $p$  have to be in order for Ben to not make any bets ( $b^* = 0$ )?
7. In what ways is a standard health insurance policy really insurance in the sense introduced in the text? In what ways is it not?
  8. Suppose that instead of there being just two states (accident or no accident), there are three states. There is a no-accident state ( $S_1$ ), an insurable-accident state  $S_2$ , and a non-insurable accident state,  $S_3$ . These states occur with probability  $p_1$ ,  $p_2$ , and  $1 - p_1 - p_2$  respectively. The consumer loses  $d_2$  in state 2, and loses  $d_3$  in state 3.

An insurance firm offers a policy which pays net benefits of  $b$  in the event that the state is  $S_2$ , and collects a premium of  $q$  in both of the other two states. Using Lagrangian methods, describe how the consumer's demand for insurance is affected by the introduction of the uninsurable-accident state.

## 8 First Welfare Theorem

1. The function that describes how a firm transforms  $x$  into  $y$  (and vice versa) was left quite general in the text. Name some sensible properties of  $f$ . For example, (in terms of the first figure) which quadrant(s) should the function never pass through? Why not?
2. Draw the  $f$  function associated with the following scenario. The two goods are bread ( $x$ ) and toast ( $y$ ). By exposing slices of bread to a fire, a firm is able to transform one slice of bread into one piece of toast. However, once turned to toast, the firm is not able to 'un-expose' the piece of toast so that it turns back into a slice of bread.
3. If there are  $\omega_x$  units of  $x$  before production, and  $z_x \leq \omega_x$  units after production, how many units of  $x$  were used as inputs in the firm's production process? How many units of  $y$  are produced using this level of input? Therefore, how many units of  $y$  are there after production (that is, what must  $z_y$  be)?



4. The previous question demonstrated that we can determine  $z_y$  if we know  $z_x$  (and, of course, we must know the endowment,  $(\omega_y, \omega_x)$ , and production function,  $f$ ). To be sure, explicitly write  $z_y$  as a function of  $z_x$ . What is this function commonly known as?
5. Must the endowment  $(\omega_x, \omega_y)$  lie on the production possibilities frontier? If not, what 'unusual' property must  $f$  possess at either  $z_x = \omega_x$  or at  $z_x : f(z_x - \omega_x) = 0$ ?
6. In reality, there are two main benefits to being a shareholder in a firm. First, you get a share of the firm's profits. Second, you get to vote on how the firm operates.

Under these maintained assumptions, we claimed that the second benefit is illusory because all shareholders would like the firm to maximize profits.

7. Explain the logic behind why all investors would vote to maximize profits.

- (a) What are the precise assumptions (there are at least three) that are creating this divergence between reality and the model?

*Hint:* Think of real-world motivations for individuals to hold voting rights in a firm, and then think about the assumptions used that would remove such motivations.

8. Carefully define a Walrasian equilibrium in a production economy. In particular, describe the objects (e.g. a consumption choice for each consumer) and the properties of these objects (e.g. the consumption plan is affordable). What is the main difference between a production economy and an exchange economy?
9. Briefly outline the argument explaining why equilibrium in an exchange economy is Pareto efficient (the first welfare theorem).

Since a production economy is basically an exchange economy in which these things called 'firms' effectively determine the 'endowments' that are to be exchanged, what must be true if the first welfare theorem still holds in a production economy?

10. Prove that the first welfare theorem (Walrasian equilibrium is Pareto efficient) is still true in a production economy.

*Hint:* Try a proof by contradiction: derive a contradiction that follows from the supposition that there exists alternative consumption plans such that these new plans make at least one consumer better off without making any other worse off.

11. Can a consumer in a production economy be made worse off than she would have been if she had refused to trade and instead consumed her endowment? Can a consumer be made worse off in a production economy relative to the welfare she experiences in the corresponding exchange economy?
12. Consider some equilibrium in an exchange economy. Now, suppose that a firm arises and has the technology to transform the goods into each other according to a production function  $f$ .
- (a) Is it possible that *all* consumers have a lower utility after the firm arises?
  - (b) Is it possible for *some* consumer to have a lower utility after the firm arises? Does this depend on whether the consumer owns shares in the firm?
  - (c) Suppose that some consumer is made worse off by the introduction of the firm, and that this consumer owns some shares in the firm. Surely, the consumer that is made worse off would not vote for the firm to pursue profit maximization (since this outcome makes him worse off by construction). In particular, he would surely rather try to convince the firm to produce something close to the exchange economy allocations (since he was better off in that equilibrium). What assumption makes this reasoning invalid?
  - (d) What do you find appealing about a Pareto efficient allocation? Why might a Pareto efficient allocation be undesirable?

## 9 Public Goods

1. Define a *public good*, and give three examples. Is noise pollution a public good?
2. Do public goods tend to be over- or under-provided? Explain, and describe one possible solution.
3. Which assumption underlying the first welfare theorem is violated when studying an economy with a public good?
4. Consider the voluntary contribution game. Describe how agent 2's consumption of the private good affects agent 1's utility (i.e. write  $u_1(x_1, y)$  in the form  $u_1(x_1, x_2, \cdot)$ ). Is agent 1's utility increasing or decreasing in  $x_2$ ? Explain.
5. Carefully define a *Nash Equilibrium*. How is it different from a Walrasian equilibrium?
6. There are two agents with identical utility functions given by

$$u_i(x_i, y) = x_i^\beta + y, \quad (10)$$

where  $x_i$  is the level of consumption of the private good for agent  $i$ ,  $y$  is the consumption of the public good, and  $\beta \in (0, 1)$ . The public good is produced using the production function:

$$y = f\left(\underbrace{\overbrace{\omega_1 - x_1}^{\text{1's contribution}} + \overbrace{\omega_2 - x_2}^{\text{2's contribution}}}_{\text{Total Contribution}}\right) = (\omega_1 - x_1 + \omega_2 - x_2)^\beta, \quad (11)$$

where  $\omega_i$  is agent  $i$ 's endowment of the private good, and  $\beta \in (0, 1)$ .

- (a) Write agent 1's utility in terms of  $x_1$  and  $x_2$  (and  $\omega_1$  and  $\omega_2$ ). In  $(x_1, x_2)$  space, draw (roughly) a family of indifference curves for agent 1. What general shape do they have?
- (b) Suppose that agent 1 thinks that agent 2 is going to consume  $\bar{x}_2$  units of the private good. What is the optimal choice of  $x_1$  (in terms of  $\bar{x}_2$ )?

- i. How is this optimal choice affected by  $\bar{x}_2$ ? Intuitively, what is going on?
  - ii. How is this optimal choice affected by  $\omega_1$ ?  $\omega_2$ ?
  - iii. Draw a graph in  $x_1, x_2$  space that shows agent 1's optimal choice of  $x_1$  for any given level of  $\bar{x}_2$ .
- (c) Suppose that  $\omega_1 = \omega_2$  so that there is no quantitative difference between the two agents. Given this symmetry, what would you expect the relationship between the optimal  $x_1$  and the optimal  $x_2$  to be? Use this to calculate the Nash equilibrium private consumption pair  $(x_1, x_2)$ .
- (d) Suppose that a planner suspected that the Nash equilibrium was inefficient. The planner would like to set the private consumption levels so that it maximizes the sum of the two agents' utilities. Set up the planner's problem, and solve for the socially optimal private consumption levels  $(x_1^{**}, x_2^{**})$ .
- (e) Compare the socially optimal private consumption levels to the Nash equilibrium levels:
  - i. How does  $\beta$  affect the Nash equilibrium private consumption levels? How does it affect the socially optimal levels?
  - ii. Is the Nash equilibrium outcome inefficient (i.e. not Pareto efficient)? Is there too much or too little private consumption? Intuitively explain this result.
  - iii. How is the inefficiency affected by  $\beta$ ? That is, do the Nash equilibrium consumption levels get closer to the optimal levels when  $\beta$  approaches zero, or when  $\beta$  approaches one? Any intuition for why this is so?
- (f) Suppose that instead of assuming that each agent has a personal endowment,  $\omega_i$ , we instead assumed that the pair were given an endowment of  $\omega$  to 'share'. That is, each agent is free to consume from  $\omega$  but must take into account that whatever was left over (after both had consumed) was to be used in the production of the public good. Would this assumption change any of the above results? Why or why not?
- (g) Suppose that some outside body (like a government) wanted to help these agents out by donating  $z$  units of the private good

(as an endowment that can be allocated to the production of the public good as desired). Due to some kind of favoritism, the total of  $z$  is split as follows: agent 1 is to get  $\lambda z$  and agent 2 is to get  $(1 - \lambda)z$ , where  $\lambda \in [0, 1]$ .

- i. Suppose that there was no public good at all. Would agent 1 care about what  $\lambda$  was? Which would he most prefer?
- ii. Now introduce the public good in question. Now does agent 1 care? Why is this so?
- iii. Suggest a slight change to the model that would result in agent 1 caring about  $\lambda$  in the presence of a public good.

7. What happens to the Nash equilibrium of the voluntary contribution game as the number of agents increases? In particular, does efficiency get improved or worsened?
8. What is a *Lindahl price*? Describe how such prices are used to resolve the public goods problem.
9. How do you think agent 1's Lindahl price is affected by her endowment (keeping agent 2's endowment fixed)? What is your reasoning?

To derive this relationship in a particular economy, consider the following scenario. There are two agents, each with Cobb-Douglas preferences:

$$U(x_i, y_i) = x_i^\alpha y_i^{1-\alpha}, \quad (12)$$

for  $i = 1, 2$ . Each agent is endowed with  $\omega_i$  units of the private good which is then given to the firm in exchange for shares which gives the agent the rights to a proportion,  $\omega_i / (\omega_1 + \omega_2)$  of the firm's profits.

The firm transforms the private good into the public good with the simple linear technology (recall that  $y_1 = y_2 = y$ ):

$$y = \omega_1 + \omega_2 - x_1 - x_2. \quad (13)$$

Taking the price of the public good as the numeraire, the firm's profits are given by:

$$\pi = x_1 + x_2 + (p_1 + p_2)y. \quad (14)$$

10. Write the firm's constrained maximization problem.

- (a) Rather than using a Lagrangian approach, try substituting the constraint in directly. That is, write the firm's profit function in terms of endowments, prices, and  $(x_1, x_2)$ .
- (b) If an Walrasian equilibrium were to exist in this economy, one requirement is that the firm is choosing  $(x_1, x_2)$  to maximize profits. Another is that  $(x_1, x_2)$  must be feasible (i.e.  $x_1 + x_2 \in [0, \omega_1 + \omega_2]$ ). The latter implies the sensible restriction that the optimal production of the private goods can not be infinity or negative infinity.

Using this information, and the profit function derived in (10a), argue that the Lindahl prices must satisfy the following:

$$p_1 + p_2 = 1. \quad (15)$$

What does this imply that equilibrium profits will be?

- (c) Leaving the firm's production problem for now, let's look at the agents' consumption problem. Let  $M_i$  denote the income of agent  $i$ . Set up agent  $i$ 's constrained maximization problem, and derive the demand functions (for  $x_i$  and  $y_i$ ).
- (d) Now that we have the demands for  $y_1$  and  $y_2$  in terms of incomes and their (Lindahl) prices, we are able to derive another property of the Lindahl prices. What is the equilibrium relationship between  $y_1$  and  $y_2$ ? What does this relationship imply about how  $p_1$  and  $p_2$  are related to  $M_1$  and  $M_2$ ? Use the fact that  $M_i$  equals the proportion of firms profits that agent  $i$  has rights over to show that

$$\frac{p_1}{p_2} = \frac{\omega_1}{\omega_2}. \quad (16)$$

- (e) Use the two above restrictions on  $p_1$  and  $p_2$  to derive  $p_1$  as a function of  $\omega_1$  and  $\omega_2$ . How is  $p_1$  affected by an increase in  $\omega_1$ ? How does this compare with your initial reasoning?
11. How do you think the Lindahl price facing agent 1 is affected by agent 2's relative desire for the public good (relative to her desire for the private good)? Use the following modified structure provided in Question 9 to derive this.

- (a) Suppose that instead of  $\alpha$  being constant in the utility functions, that agent  $i$  has a preference parameter of  $\alpha_i$ . Re-derive the demand functions under this change.
- (b) Exploit the equilibrium relationship between  $y_1$  and  $y_2$  to derive a relationship between  $p_1/p_2$ , and  $(\alpha_1, \alpha_2, M_1, M_2)$ .
- (c) Use the fact that  $M_i$  equals the proportion of firms profits that agent  $i$  has rights over to show that

$$\frac{p_1}{p_2} = \frac{(1 - \alpha_1)\omega_1}{(1 - \alpha_2)\omega_2}. \quad (17)$$

- (d) Has anything changed on the production side? Re-iterate the arguments that can be used to show that

$$p_1 + p_2 = 1. \quad (18)$$

- (e) Use the above two conditions to derive  $p_1$  as a function of  $(\omega_1, \omega_2, \alpha_1, \alpha_2)$ . How is  $p_1$  affected by  $\alpha_2$ ? How does this compare to your initial reasoning?

12. We can verify that the Lindahl prices produce optimal choices. Consider the framework introduced in Question 9. For ease of calculation, suppose that  $\omega_1 = \omega_2$  (and retain the assumption that the preference parameter,  $\alpha$ , is the same across agents), so that the agents are completely symmetric.

- (a) Set up the planner's problem (do not solve yet). What does the planner maximize? Which variables does she choose? What are the constraints she faces?
- (b) Rather than setting up the Lagrangian, try substituting the public good production function in directly so that the objective function is completely in terms of  $x_1$  and  $x_2$  (and endowments of course).
- (c) Since the agent's are identical in every way (e.g. in preferences and endowments), what would you expect the relationship between the socially optimal  $x_1$  and  $x_2$ ? Use this relationship to simplify the objective function.

- (d) Solve for the optimal level of private consumptions,  $x_1$  and  $x_2$ .
  - (e) Compare this with the  $x_1$  and  $x_2$  induced by the Lindahl prices. Are they the same? Does this suggest an alternative method by which to calculate the Lindahl prices?
  - (f) Does the fact that the levels of private consumption are efficient imply that the level of the public good is efficient? Either make an argument for or against this, or manually verify whether it is true.
13. Informally speaking, Lindahl prices are a means by which to distort agents' incentives to contribute to the public good. The most obvious way to do this is to make the private good relatively more expensive. Consider again the framework in Question 9, and recall that  $p_1$  is the (Lindahl) price of agent 1's public good when the private good has price 1.
- (a) If the problem is that agents do not contribute enough to the public good, then given the above argument, would you expect  $p_1$  to be greater than or less than 1? What about  $p_2$ ? Was this the case?

If we let  $M_1$  be agent 1's income (his share of profits), then you should have calculated his demand for the private good to be

$$x_1^* = \alpha M_1. \quad (19)$$

This demand does not seem to be a function of  $p_1$  - a feature of the Cobb-Douglas specification.

- 14. How do you reconcile the fact that the Lindahl prices induced the efficient demand for the private good when the demand function depends only on income, and not prices directly?
- 15. Name two problems in using Lindahl prices as a solution to the public goods problem. In particular, think about problems that are likely to arise when trying to implement the mechanism. Can you think of a better solution to the public goods problem?



16. Consider an economy with two consumers. The total of the private good is  $W$ , and each consumer has  $\frac{W}{2}$  of this endowment. The public good can be produced from the private good using the simple linear technology  $y = x$ , i.e., each unit of the private good can be used to produce exactly one unit of the public good. Both consumers have preferences given by  $u(y, x) = y + \ln(x)$ . Find the equilibrium of the voluntary contribution game. Show that if  $W < 2$ , then no public good is produced in this equilibrium. Find the Lindahl equilibrium allocation. Show that if the aggregate endowment of the private good is less than 1, no public good will be produced in the Lindahl equilibrium. Interpret.
17. In the patent problem, assume that each dollar spent on a public good produces exactly one unit of the public good. There are two consumers. The aggregate endowment of money of both consumers is  $\omega$ , each consumer's individual endowment is  $\frac{\omega}{2}$ . Suppose both consumers have identical utility functions given by

$$\ln(x) + \ln(1 + y).$$

Draw the production possibilities frontier for this problem. Locate consumer 1's endowment in this diagram. Compute the equilibrium of the voluntary contribution game and label it in your diagram, showing consumption of good  $x$  by consumer 1 along with the total output of the public good.

18. Following the information given in problem 17, suppose that consumer 1 enjoys consumption  $x_1$  of the private good, while output of the public good is  $y_1$ . Write down the payoff that consumer 2 must receive in this outcome. Find the slope of consumer 2's indifference curve (in the space of all  $(x_1, y_1)$  pairs) by totally differentiating consumer 2's payoff as function of  $x_1$  and  $y_1$ . What is the slope of this indifference curve? Does it always slope the same way? Why or why not?
19. Once consumer 2 has been awarded a patent, suppose consumer 2 sets the price of the public good at  $p$ . Consumer 1 is now able to purchase whatever quantity of the public good that he likes. For each  $x_1 < \frac{\omega}{2}$ , find some quantity  $y_1$  such that the slope of consumer

1's indifference curve at the point  $(x_1, y_1)$  is equal to the slope of the line running from the point  $(x_1, y_1)$  to the point  $(\frac{\omega}{2}, 0)$ . Use the corresponding solution to draw a picture representing consumer 1's offer curve for this problem. Explain the sense in which this curve describes all the consumption bundles  $(x_1, y_1)$  that consumer 2 could force consumer 1 to take by setting an appropriate price  $p$  for the public good.

20. Find the point where this offer curve is tangent to consumer 2's indifference curve in  $(x_1, y_1)$  space, and compare this point to the point corresponding to the equilibrium of the voluntary contribution game.
21. In the environment described in Problem 17, suppose that consumer 1 has preferences given by

$$\ln(x) + 2 \ln(1 + y)$$

while consumer 2 has the same preferences as before. Would it be possible to make both consumers better off by taxing money from consumer 2 and giving it consumer 1?

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1. Consider an economy with 2 goods and two consumers. Good  $x$  is a private good. Each consumer owns exactly one unit of good  $x$ . The other good is a public good which is produced using good  $x$  as an input. There is no endowment of the public good, however each unit of good  $x$  can be used to produce exactly one unit of the public good. The complication is that only consumer 1 is able to produce the public good - consumer 2 can produce nothing on his own. The consumers have identical quasi linear utility functions given by  $U(x, y) = \ln(y) + x$ . Negative consumption of good  $x$  is okay (think of it as borrowing).
  - (a) What is the equilibrium of the voluntary contribution game.
  - (b) Starting from the equilibrium of the voluntary contribution game, suppose that consumer 2 (the non-producer) proposes to match

any additional contributions that consumer 1 makes to production of the public good one for one, provided that consumer 1 agrees to use these matching grants to produce the public good. Show that as long as consumer 2 can limit consumer 1's contributions to the public good, then this scheme can be used to make both consumers better off.

- (c) What is the Lindahl equilibrium price for the public good. How much public good is produced in the Lindahl equilibrium.
- (d) Interpret the Lindahl equilibrium as one in which a single price taking competitive firm is given the exclusive right to sell both public and private good. What happens if the firm instead acts as a monopolist? In particular, show that a monopoly firm (who maximizes its own profit) will produce less of the public good than the producer does in the voluntary contribution game.

## 11 Constrained Optimization

1. As described in class, let the utility function for good  $x$  and  $y$  be  $y + \ln(x)$  (natural logarithm). Write out the Lagrangian function and first order conditions. Find the demand function. Find the income and substitution effects (in derivative form) of an increase in the price of good  $x$ , and  $y$ .
2. Solve for the Cobb Douglas demand functions for utility function  $u(x, y) = x^\alpha y^\beta$  (in this case let  $\alpha + \beta > 1$ ). Also suppose that the consumer has an endowment of  $x_0$  units of good  $x$  and  $y_0$  units of good  $y$  (so the consumer maximizes utility subject to the constraint that the value of chosen consumption is less than or equal to  $px_0 + qy_0$ ). Find the income and substitution effects in derivative form.
3. For utility function  $y + bx$  find the demand function when  $p < b$  and  $q = 1$ . Can you find the Lagrangian multipliers for this case?