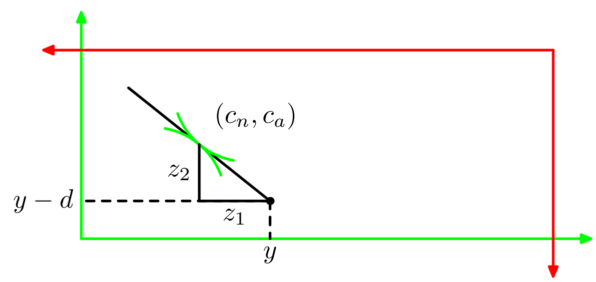
FINANCE

MICHAEL PETERS

The edgeworth box provides a nice way to think about trade in markets. The kind of market for which that theory probably best applies is financial markets. This note explains how to use the edgeworth box to understand asset pricing.

The approach begins with two traders and two goods. The first step to understanding all this is to think about income in different states as representing two distinct goods. Insurance is an example in which this is true. An insurance policy only pays you money if you have an accident. Money means a lot more to you when you have an accident than when you don't, which is why you might be willing to give up money when you don't have an accident by paying an insurance premium in order to get it when you do have an accident.

Lets start with a simple Walrasian equilibrium in a two good exchange economy similar described with an Edgeworth box.



The figure describes a Walrasian equilibrium for an economy with two goods. Trader 1 starts at the origin, and has quantity y of good 1 and $y-d$ of good 2. He makes a trade with trader to in which he exchanges z_1 units of good 1 for z_2 units of good 2. This takes both of them to the point where the indifference curves are tangent so that the allocation is pareto optimal.

In finance, we want to interpret the goods by imagining that there is an as yet unrealized event, say a recession. Trader 1 has income y if there is no recession, but has much lower income $y-d$ if there is.

Before it is known whether or not the recession occurs, there is a market in which two 'securities' can be traded. The first security, a, pays 1 dollar if and only if there is no recession. Security b pays 1 dollar if and only if there *is* a recession. We imagine that the way the event contingent transfer occurs is that trader 1 sells z_1 unit of security a, for which he receives total revenue qz_1. He uses the revenue to buy $z_2 = qz_1$ units of security b from individual 2.

The, once the event is realized, when a recession occurs, trader 2, since she has sold z_2 units of security b, is obliged to pay 1 z_2 dollars. If the recession doesn't occur, then 1 is on the hook since he is the one who sold off $z_1$units of security a.

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There is a second way to look at this tradeoff. From 1's perspective, what he does is to plan how much income he would like to have in each of the two events. His plan is to arrange it so that he has income c_n if there is no recession and c_a when there is a recession. To accomplish this, he realizes that he has to purchase and sell assets that will force him to pay z_1 if there is no recession, but will leave him with a payment of z_2 when a recession occurs.

The way he might do this is to take the matrix of asset returns, given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and post multiply it by his portfolio (z_1, z_2) viewed as a column vector to get the payments that he wants. In other words, he would solve the equation

$$\begin{bmatrix} c_n - y \\ c_a - y + d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

which obviously has the solution described above.

As we are working here with a Walrasian equilibrium, trader 2 is happy with the trade she makes, which is to receive $c_n - y$ when there is no recession, but to pay $c_a - y + d$ when there is.

At this point, we could ask what the prices of the securities have to be so that the market for them clears. To see this, we just consider the Walrasian equilibrium first and imagine that the price of good 1 is q while the price of good 2 is 1. The bundle (c_n, c_a) has cost $qc_n + c_a$which is equal to $qy + y - d$ since it is a Walrasian equilibrium. Then of course, $q(c_n - y) + (c_a - y + d) = 0$, from which it is obvious that the price of asset a has to be q to get this to work.

We could express this as a problem is computing a security price ρ for asset 1 such that

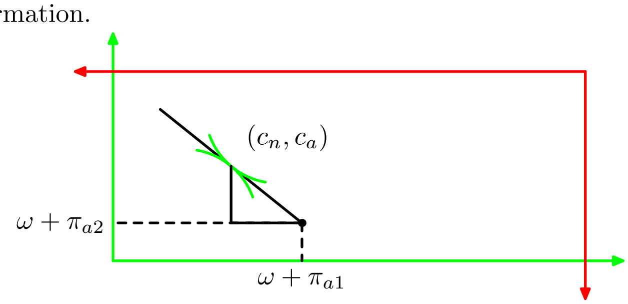
$$\rho z_1 + z_2 = \rho(c_n - y) + (c_a - y + d) = 0$$

or

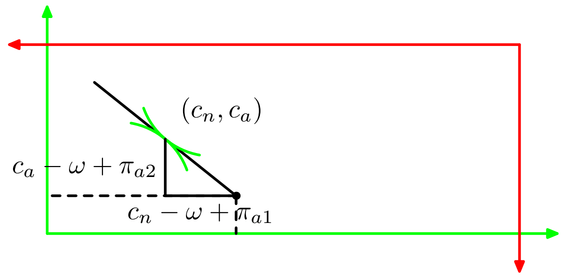
$$\rho = -\frac{c_a - y + d}{c_n - y} = q.$$

So far this is pretty obvious, so lets enrich the model to make it look more like a stock market. Suppose that our two traders are entrepreneurs. Each of them has an inheritance ω that they have no matter what. But each has a start up venture they are running. The start up of trader 1 gives profits $\pi_{a1} > 0$ if there is no recession and $\pi_{a2} < 0$ if there is. Trader 2 has another start up that does well in a recession. Her company earns $\pi_{b1} < 0$ if there is no recession, but make a profit π_{b2} if there is a recession.

Now lets take the first figure above, and just relabel it so that it coincides with this new information.



Here is the same diagram with the trades labeled.



Now we can imitate what we did before. We'll let $q = \frac{c_a - \omega - \pi_{a2}}{c_n - \omega - \pi_{a1}}$ be the price ratio that seems to support the Walrasian equilibrium.

To support this outcome, trader 1 has to buy a portfolio (z_1, z_2). As the assets are now profits in start up companies, the portfolio has to satisfy the following matrix equation:

$$\begin{bmatrix} c_n - \omega - \pi_{a1} \\ c_a - \omega - \pi_{a2} \end{bmatrix} = \begin{bmatrix} \pi_{a_1} & \pi_{b1} \\ \pi_{a2} & \pi_{b_2} \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

It is straightforward how we should do this, pre multiply both sides of the equation by the inverse matrix

$$\begin{bmatrix} \pi_{a_1} & \pi_{b1} \\ \pi_{a2} & \pi_{b_2} \end{bmatrix}^{-1} \begin{bmatrix} c_n - \omega - \pi_{a1} \\ c_a - \omega - \pi_{a2} \end{bmatrix} =$$

$$\frac{1}{\pi_{a1}\pi_{b2} - \pi_{b1}\pi_{a_2}} \begin{bmatrix} \pi_{b2} & -\pi_{b1} \\ -\pi_{a2} & \pi_{a_1} \end{bmatrix} \begin{bmatrix} c_n - \omega - \pi_{a1} \\ c_a - \omega - \pi_{a2} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

So we can compute the portfolio rather easily. The interpretation is just as before. Trader 1 is going to sell off z_1 of the shares in his venture and use the proceeds to purchase shares in the venture being run by trader 2.

All we have left to do at this point is to try to figure out how the shares of the two ventures should be priced. Lets use the convention that the shares in venture 2 should have a price 1, so that all we need to figure out is what the corresponding relative price ρ should be of the shares in venture 1.

Now it is just computation. We have to have $\rho z_1 + z_2 = 0$, which means

$$\frac{1}{\pi_{a1}\pi_{b2} - \pi_{b1}\pi_{a_2}} \begin{bmatrix} \rho & 1 \end{bmatrix} \begin{bmatrix} \pi_{b2} & -\pi_{b1} \\ -\pi_{a2} & \pi_{a_1} \end{bmatrix} \begin{bmatrix} c_n - \omega - \pi_{a1} \\ c_a - \omega - \pi_{a2} \end{bmatrix} = 0$$

If we want to solve this, we can obviously forget about the constant $\frac{1}{\pi_{a1}\pi_{b2} - \pi_{b1}\pi_{a_2}}$ and solve

$$\begin{bmatrix} \rho & 1 \end{bmatrix} \begin{bmatrix} \pi_{b2} & -\pi_{b1} \\ -\pi_{a2} & \pi_{a_1} \end{bmatrix} \begin{bmatrix} c_n - \omega - \pi_{a1} \\ c_a - \omega - \pi_{a2} \end{bmatrix} = 0$$

instead. This gives first

$$\begin{bmatrix} \rho\pi_{b2} - \pi_{a2} & \pi_{a1} - \rho\pi_{b1} \end{bmatrix} \begin{bmatrix} c_n - \omega - \pi_{a1} \\ c_a - \omega - \pi_{a2} \end{bmatrix} = 0$$

or

$$(\rho\pi_{b2} - \pi_{a2})(c_n - \omega - \pi_{a1}) + (\pi_{a1} - \rho\pi_{b1})(c_a - \omega - \pi_{a2}) = 0.$$

From the Walrasian equilibrium, this is

$$(\rho\pi_{b2} - \pi_{a2}) = -q(\pi_{a1} - \rho\pi_{b1})$$

or

$$\rho = -\frac{q\pi_{a1} - \pi_{a_2}}{\pi_{b2} - q\pi_{b1}}.$$

So this formula describes the basics of asset pricing. To compute the value of an asset, you need to know a couple of things. The first of which is the returns on other assets - even though we are trying to find the relative price of asset a we have to know π_{b1} and π_{b2} to do that. Second, we need to know the state price q, which we can compute by solving for a Walrasian equilibrium.