Graphs

- a *graph* G is a finite collection of vertices $\mathcal{V}$ and a collection of edges $\mathcal{E}$ that connect vertices. The *degree* of a vertex $d(v)$ is the number of edges connected to that vertex
- the degree of a graph $\triangle(G)$ is the maximum degree of all the vertices.
- edges are adjacent when they connect to the same vertex.

coloring

- a *coloring* of a graph G is a mapping from from the set of edges $\mathcal{E}$ into a finite set of colors C. The coloring is called a *proper coloring* if any two adjacent edges are assigned different colors. The *chromatic index* of a graph, $\chi(G)$ is the minimum number of colors required to provide a proper coloring for G
- Vizing Theorem - full version $\triangle(G) \leq \chi(G) \leq \triangle(G) + 1$
- A graph for which $\triangle(G) = \chi(G)$ is called a type 1 graph, others are referred to as type 2 graphs

Bipartite graphs

- A graph is *bipartite* when the set of vertices $\mathcal{V}$ can be divided into two disjoint subsets U and V in such a way that u is a vertex in U and (u, v) is an edge, then $v \in V$.
- Every many to many matching can be represented as a bipartite graph.
- **Theorem: Every bipartite graph is of type 1:** every bipartite graph of degree $\triangle$ can be colored in such a way that no two adjacent edges have the same color, and such that the range of the coloring contains only $\triangle$ distinct colors.

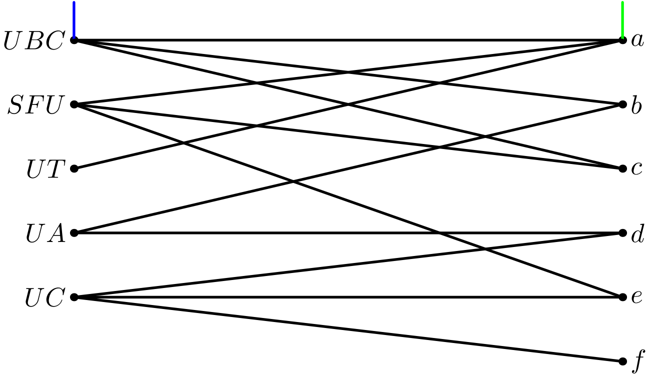
- in the diagrams above, there are at most three edges emanating from any vertex, i.e., three interviews. So label the interview slots $\{1, 2, 3\}$ and match this with the set of colors $\{$Red, Green,Blue$\}$. The theorem says that edges can be colored either red green or blue in such a way that no two adjacent edges have the same color,
- To see the analogy, just think of each vertex as either a college or an applicant, and each edge a match between a college and applicant. The colors are the interview slots. So the theorem says that we can assign the k interview slots to college applicant pairs in such a way that for every college, no pair of applicants that it is matched with is given the same interview time, and for each applicant, the applicant never has interviews scheduled with two colleges at the same time.

Proof (bipartite graph)

- Select an arbitrary subset E of the edges such that $|E| < \triangle(G)$. Then trivially each edge can be assigned a different color (slot) using fewer than $\triangle(G)$ colors which would constitute a proper coloring. So lets proceed inductively and suppose that for any subset E containing t edges, we can do a proper coloring using no more than $\triangle(G)$ colors.
- the next step is to add another edge (u, v) to E then recolor so that $E \cup (u, v)$ is properly colored using no more than $\triangle(G)$ colors.
- since (u, v) is part of G which has degree $\triangle(G)$, u and v both have a free color. Suppose u has color blue free. If v also has color blue free we just color the edge between them blue, and we are finished.

proof - contd

- if v does not have blue free, then v has some other color say green free. Then starting with v we want to create the maximal length chain possible starting at v whose colors alternate between blue then green. That means follow the blue edge from v to some u' and look for an edge attached to u' which is colored green and keep going until you can't find an edge of the corresponding color. When you find this vertex, it means that the vertex has blue free if you followed a green edge to get to it, or green free if you followed a blue edge to get to it.
- Notice this chain can't come back to u because there are only two colors, and the color that leads from a $v' \in V$ to a $u' \in U$ is always colored blue, but u has blue free
- Now recolor interchanging blue and green back through the chain.



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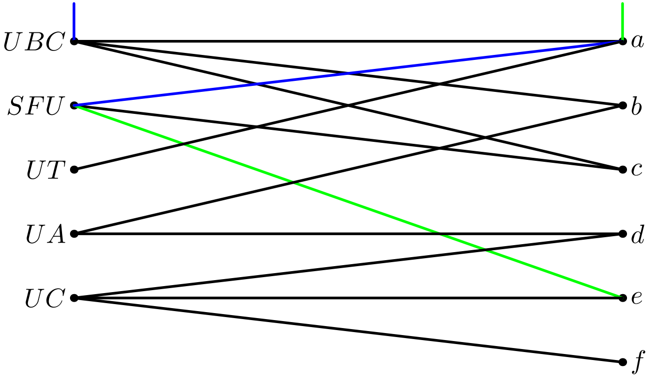

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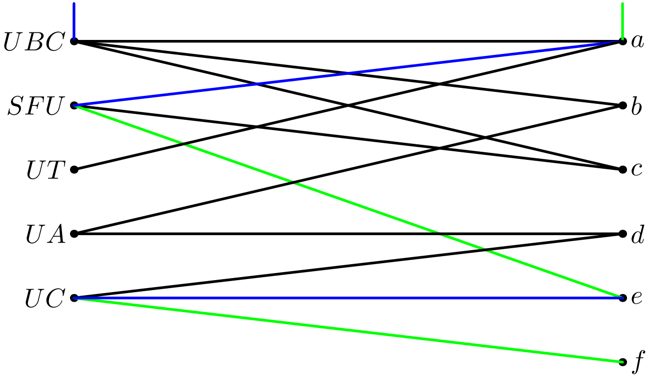

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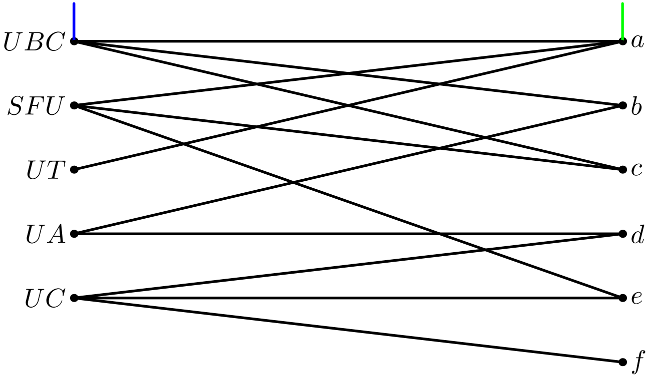

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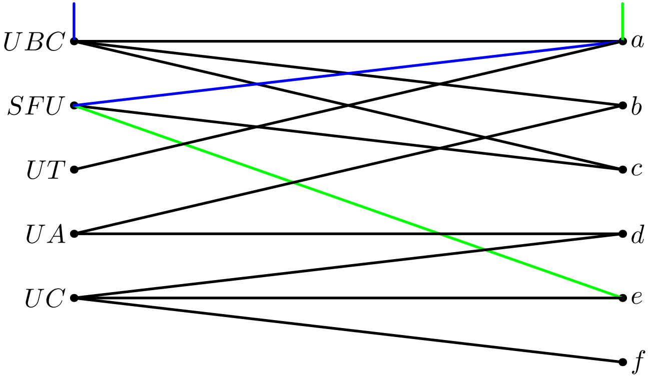

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