Intro

- let $\mathcal{X}$ be a set of alternatives, $\mathcal{X} \times \mathcal{X}$ is the Cartesian product of $\mathcal{X}$ with itself. A binary relation on $\mathcal{X}$ is a subset $\mathcal{P} \subset \mathcal{X} \times \mathcal{X}$
- orderings of alternatives can be thought of as binary relations - e.g. if x and y both elements of $\mathcal{X}$ and $(x, y) \in \mathcal{P}$ then one might say that x is at least as good as y, or $x \succeq y$

Examples

1. a set of alternative consumption bundles
2. a set of alternative policies with $x \succeq y$ meaning that x is 'socially preferred' to y
3. a set of alternative policies with $x \succeq y$ meaning that x would defeat y in a referendum between the two
4. a set probability distributions with $x \succeq y$ meaning that the distribution x first order stochastically dominates y
5. a set of strategy rules in a game with $x \succeq y$ meaning the rule x weakly dominates the rule y
6. a set of numbers with $x \succeq y$ meaning that x is bigger than y (an example where a binary relation is an ordering)

- some binary relations have strange properties - for example

	TC	BB	IS
C	1	2	3
F	2	3	1
M	3	1	2

  rows are parties, numbers represent their preferences over policies TC,BB and IS. Every policy is defeated in a majority vote against some alternative (Condorcet paradox).

- a *preference* relation is a special binary relation intended to represent an individual choice process - one imagines that it has two properties
  1. Completeness for any pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ either $x \succeq y$ or $y \succeq x$ or both.
  2. Transitivity for any $x, y, z \in \mathcal{X}$ $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$
- a preference satisfying these two properties is sometimes called a *rational preference relation*

Utility Functions

- a function $u : \mathcal{X} \to \mathbb{R}$ is called a *utility function* representing preference relation $\succeq$ if for all $x, y \in \mathcal{X}$

$$x \succeq y \iff u(x) \geq u(y)$$

- intransitive preference relations typically can't be represented by utility functions - if a binary relation $\succeq$ is intransitive, then there are three options x, y,and z such that $x \succeq y$; $y \succeq z$ but not $x \succeq z$. Now suppose there is a utility function representing this relation. Then $x \succeq y \Rightarrow u(x) \geq u(y)$ while $y \succeq z \Rightarrow u(y) \geq u(z)$ so that $u(x) \geq u(z)$ which by definition means that $x \succeq z$. Since we know this is false, the assertion that there is a utility function must also be false. This is an example of a proof *by contradiction.*

- a critical question is whether there is some way to infer the existence of a preference relation from something that you can observe.
- Let $\mathcal{B}$ be a family of subsets of $\mathcal{X}$ and $\mathcal{P}(\mathcal{X})$ the collection of all subsets of $\mathcal{X}$ (the power set of $\mathcal{X}$) - a correspondence $C : \mathcal{B} \to \mathcal{P}(\mathcal{X})$ is called a *choice correspondence* if $C(B) \neq \emptyset$ and $C(B) \subset B$ for all $B \in \mathcal{B}$
- the set B corresponds to the set of experiments or outcomes.
- the choice correspondence C satisfies the *weak axiom of revealed preference* if for any pair of sets B and B' and points $x \in B \cap B'$ and $y \in B \cap B'$, $x \in C(B)$ and $y \in C(B') \Rightarrow x \in C(B')$.

- example:
  - $\mathcal{X} = \{x, y, z\}$,
  - $\mathcal{B} = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$
  - and
  - $C(\{x, y\}) = \{x\}; C(\{x, y, z\}) = \{x, y\}$
  - fails the weak axiom because y is chosen given choice set $\{x, y, z\}$ and x is also in $\{x, y, z\}$. x is chosen in $\{x, y\}$ but y isn't

- every rational preference relation supports a choice correspondence in the obvious way

$$C_{\succeq}(B) = \{x \in B : x \succeq y \forall y \in B\}$$

provided that this set is always non-empty
- Theorem: every choice correspondence supported by a rational preference relation satisfies the weak axiom

Proof: Suppose not. Then there are sets $B, B'$and points $x \in B \cap B'$ and $y \in B \cap B'$ such that

- (i)$x \in C_{\succeq}(B)$; (ii) $y \in C_{\succeq}(B')$ and (iii) $x \notin C_{\succeq}(B')$.
- Since $C_{\succeq}$ is supported by a preference relation $x \succeq y$ by (i).
- By (iii) there is a point z in B' such that $z \succeq x$ but not $x \succeq z$ ($z \succ x$).
- By (ii) $y \succeq z \succ x$. Then $x \succeq y \succeq z$ but not $x \succeq z$, so the preference relation isn't transitive.

- we want the other way around - if we run a series of experiments and find that some agents' choices obey the weak axiom, can we conclude that the trader will behave *as if* he has a rational preference ordering? can we discover this preference ordering?
- Not generally - Example - $\mathcal{X}$ as above with
- $\mathcal{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$ and
- (i) $C(\{x, y\}) = x$, (ii) $C(\{y, z\}) = y$ and (iii) $C(x, z) = z$.
- Note that this set of choices implies intransivity because if the rationalizing preference relation exists, then $x \succ y$ by (i), $y \succ z$ by (ii) and $z \succ x$ by (iii). The weak axiom holds because the sets in $\mathcal{B}$ simply don't give the decision maker an opportunity to violate the weak axiom.

- Theorem: let C be a choice correspondence satisfying the weak axiom. Suppose that for any three distinct points x, y, and z in $\mathcal{X}$ there exist sets B and B' in $\mathcal{B}$ such that $B = \{x, y\}$ and $B' = \{x, y, z\}$. Then there is a rational preference relation supporting C.
  - Proof: Define the binary relation $\succeq_C$ as follows
  - $x \succeq_C y$ iff $\exists B : x \in B; y \in B$ and $x \in C(B)$.
  - Since C is defined on all sets in $\mathcal{B}$ and $\mathcal{B}$ contains all two element sets, then for any pair of points $\{x, y\}$ either
  - $x \in C(\{x, y\})$ or $y \in C(\{x, y\})$ or both.
  - This is equivalent to $x \succeq_C y$ or $y \succeq_C x$ or both.

- Suppose now that $x \succeq_C y$ and $y \succeq_C z$.
- $C(\{x, y, z\})$ must contain at least one point.
- If that point is x then $x \succeq_C z$ by definition, and the relation is transitive.
- If the point is y then since $x \succeq_C y$ there is some set B'' such that $y \in B''$, and $x \in C(B'')$, so by the weak axiom $x \in C(\{x, y, z\})$ which gives $x \succeq_C z$.
- If the point is z, use the same reasoning to show that $y \in C(\{x, y, z\})$, from which the same logic gives $x \in C(x, y, z)$ or $x \succeq_C z$.
- This proves that $\succeq_C$ is transitive.

- So $\succeq_C$ is a rational preference relation (note how the assumptions were used in this argument - what would go wrong if $\mathcal{B}$ did not contain all sets of the form $\{x, y, z\}$?).
- if the set of alternatives $\mathcal{X}$ were finite, which would be easier to check, a preference relation $\succeq$ is complete and transitive, or a choice correspondence C satisfies the weak axiom?
- $\succeq_C$ supports a choice correspondence. Is it the same as C?
- If $\mathcal{X}$ is finite, $\mathcal{B}$ consists of all subsets of $\mathcal{X}$ and C satisfies the weak axiom, can you construct a utility function that represents the preference relation $\succeq_C$?

- $\mathcal{X} \subset \mathbb{R}_n$ and $\succeq$ is a binary relation on $\mathbb{R}_n \times \mathbb{R}_n$.
- $\succeq$ *is continuous* if whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$are (convering) sequences of consumption bundles satisfying $x_n \succeq y_n$ for all n, then $\lim_{n\to\infty} x_n \succeq \lim_{n\to\infty} y_n$
- Theorem: let $\succeq$ be a continuous rational preference ordering satisfying the property that $x \geq x'$ implies $x \succeq x'$ and $x \neq x'$ and $x \geq x'$ together imply $x \succ x'$ (monotonicity). Then there exists a utility function u that represents $\succeq$.

- Proof: Let $e \in \mathbb{R}_+^N$ be such that $e = [1, 1, \ldots 1]$. Let

$$Z = \left\{ x' \in \mathbb{R}_+^N : x = \alpha e \text{ for some } \alpha \geq 0 \right\}$$

- For any $x \in \mathbb{R}_{++}^N$ there is a $z \in Z$ such that $z \geq x$ (one such would be $z \equiv [\max_j x_j] \cdot e$), and so by monotonicity, $z \succeq x$.
- Similarly, there is a $z' \in Z$ (i.e. 0) such that $x \geq z'$, and therefore $x \succeq z'$.
- So the sets $P^+(x) = \{z \in Z : z \succeq x\}$ and $P^-(x) = \{z' \in Z : x \succeq z'\}$ are both non-empty.
- By completeness of preferences, $z \succeq x$ or $x \succeq z$ for all $z \in Z$, so $Z = P^+(x) \cup P^-(x)$.

- $P^{+}(x)$ and $P^{-}(x)$ must have a point in common (if they don't then $P^{-}(x)$ is the complement of $P^{+}(x)$ in Z which means at least one of them must be an open set violating continuity).
- Furthermore they can have only one point in common by monotonicity. Let $\alpha(x)e$ be this point. The claim is that $\alpha(x)$ is the desired utility function.
- To see it, suppose that $\alpha(x) \geq \alpha(y)$. Then by monotonicity $\alpha(x)e \succeq \alpha(y)(e)$. By transitivity, $x \sim \alpha(x)e \succeq \alpha(y)e \sim y$ implies $x \succeq y$. The reverse implication is proved in a similar way.