Midterm Fall 2018
Econ 600

Do each question, each has equal marks.

1. Let $\mathcal{X}$ be a set consisting of a finite number of elements, $\{B_i \subset \mathcal{X}\}$ a collection of choice sets consisting of all non empty subsets of $\mathcal{I}$ that contain at least 2 elements, and C a choice correspondence satisfying the weak axiom of revealed preference. Prove (using only the weak axiom, don't refer to utility functions) that (a) there is an outcome w in $\mathcal{X}$ such that there is no 2 element set B_i such that $w \in C(B_i)$, or (b) $w \in C(B_i)$ implies that $x \in C(B_i)$ for every $x \in B_i$. Give a counterexample for the case where $\{B_i\}$ does not contain all subsets of $\mathcal{X}$.
   Answer: A constructive proof. Pick any point $x_0 \in \mathcal{X}$. Now just search all the experiments in B. If we can't find a B_0 such that $x_0 \in C(B_0)$ then we have found the element we want and completed the proof. Otherwise, if there are some B where $x_0 \in C(B)$ then we can check them to see whether there is one, say B_0 such that $x_0 \in C(B_0)$ and B_0 contains a point $x_1 \notin C(B_0)$. If we can't find such a B_0 then again, we have found the element we want and have proven the result.
   We could call what we just did 'checking x_0' to see if it satisfies the condition. Now lets just 'check x_1'. If the check succeeds, again we have proven the theorem. As above, if it fails, the we will be able to find an $x_2 \neq x_1$ to check. The key point in this proof is that x_2 can't be equal to x_0. If it were, then x_0 and x_1 are in the intersection of B_0 and B_1. Since $x_0 \in C(B_0)$ then by the weak axiom $x_0 \in C(B_1)$, whereas by construction $x_2 \notin C(B_1)$. If we have come this far, then x_2 is the element we want. To see this just take the set $B' = \{x_0, x_1, x_2\}$. x_2 can't be in $C(B')$ because if it were, the weak axiom would force it to be in $C(B_1)$ which it isn't by construction. It can't be in $C(B_0)$ because if it was, the weak axiom would require that it be in $C(B_1)$, again a contradition.
   The contradiction to the theorem is the $\mathcal{X} = \{x_0, x_1, x_2\}$ with $C(\{x_0, x_1\}) = \{x_0\}$,$C(\{x_1, x_2\}) = \{x_1\}$, $C(\{x_0, x_2\}) = \{x_2\}$ which satisfies the weak axiom.

2. In the directed search problem we discussed in class with two workers and two firms, suppose the probability that each worker is a 'good' worker (what we called λ) is equal to $\frac{3}{8}$, the high wage firm offers wage $\frac{3}{4}$ while the low wage firm offers $\frac{1}{2}$. Describe the Bayesian Nash equilibrium. What is the highest value of λ for which there is an equilibrium where good workers only apply at wage $\frac{3}{4}$? What is the lowest value of at which 'bad' workers use a pure strategy? Assuming you see an outcome in which only one of the workers is employed at wage $\frac{3}{4}$ (while the other worker is unemployed), write down a formula for the probability with which the employed worker is a good worker (you can write this formula as a function of λ and π^* without explicitly solving for π^*)? Write down the corresponding formula for the probability that the worker employed at the high wage is a good worker if the other worker is employed at the low wage? Find these two

posterior probabilities explicitly for the case where $\lambda = \frac{1}{2}$
Answer: The good worker applies at wage $\frac{3}{4}$ for sure, the bad worker applies at $\frac{3}{4}$ with probability $\frac{25}{128}$ - just substitute into the formula in the text. When $\lambda = 2/3$ the good worker is just indifferent about applying at wage $\frac{3}{4}$ and wage $\frac{1}{2}$ when he or she expects the other to apply to the high wage for sure. If the value of λ goes any higher, the good workers start to mix. When $\lambda = \frac{1}{2}$ applying at the high wage firm is weakly dominated, they start to mix when λ falls below that.
On the Bayes rule question

$$\Pr\left(\text{employed worker is good}|\text{wage is}\frac{3}{4},\text{other worker is unemployed}\right) =$$

$$\frac{\Pr\left(w\text{is }\frac{3}{4},\text{other is unemployed}|\text{Good}\right)\Pr(\text{Good})}{\Pr\left(\text{wage is }\frac{3}{4},\text{other is unemployed}\right)}$$

$$\frac{\left(\frac{\lambda}{2}+(1-\lambda)\pi\right)\lambda}{\left(\frac{\lambda}{2}+(1-\lambda)\pi\right)\lambda+(1-\lambda)\left((1-\lambda)\frac{\pi}{2}\right)}$$

The formula for the other case is similar. When $\lambda = \frac{1}{2}$, $\pi^* = 0$ so the posterior probability the worker is a good worker is 1 in both cases.

3. In the Ellsberg experiment, a subject is shown a box containing 100 red and black balls but is not told how many of each color are in the box. Then she is asked to bet on drawing, say a red ball from the box. Subjected expected utility suggests that the subject will make decisions based on a prior belief p that she thinks represents the probablity that a ball drawn from that box will be red. There is no way to know what this subjective belief is before an experiment, but it has been suggested that an experimenter might discover it in the following way:

   - The experimenter creates a bet in which he promises to choose a number q randomly somewhere in the interval between 0 and 1. To do this, he'll use a distribution with a strictly positive density function $f(\cdot)$ which is continuous between 0 and 1. The subject is then asked to name a number $p' \in [0,1]$. If $q \geq p'$ the subject will play a lottery that pays \$1 with probability q (and \$0 otherwise). If $q < p'$ the subject is paid \$1 if a red ball is drawn from the box and \$0 otherwise. Assuming the subject is a subjective utility decision maker, will this method induce the subject to reveal her actual belief p when she plays? Prove or give a counter example.
     Answer: The lottery is subjective only if $q < p'$ so the subject will choose p to maximize

$$F(p')\,pu(R)+\int_{p'}^{1} qu(R)\,f(q)\,dq$$

The first order condition is

$$u\left(R\right)pf\left(p'\right)=p'u\left(R\right)f\left(p'\right)$$

which has unique solution $p'=p$.

- In the bet described in the previous bullet, what does the subject's choice of p' reveal if the subject is known to be uncertainty averse (in the sense that they have multiple priors and evaluate them by lowest expected utility).
  Answer: The lowest prior the dm believes is possible.
- A second alternative is proposed as simpler - the subject asked to give a number r between 0 and 1. The subject will then be paid $\$\left(1-(1-r)^2\right)$ if a red ball is drawn from the box, and $-\$r^2$ otherwise. If the subject truly is a subjective utility maximizer, will her choice reveal her subjective probability p? If so, you should be able to prove it. If not, explain why and give an example to illustrate. If your answer is no, are there any special conditions under which her answer will reveal her subjective probability?
  Answer: Now the subject maximizes

$$pu\left(1-(1-r)^2\right)+(1-p)\,u\left(-r^2\right)$$

which gives first order condition

$$pu'\left(1-(1-r)^2\right)2\left(1-r\right)=\left(1-p\right)u'\left(-r^2\right)2r$$

which gives that the optimal r depends on u. Simplifying

$$\frac{p2\left(1-r\right)}{\left(1-p\right)2r}=\frac{u'\left(-r^2\right)}{u'\left(1-(1-r)^2\right)}$$

which shows that $r=p$ if and only if

$$u'\left(-p^2\right)=u'\left(1-(1-p)^2\right)$$

which is true if the dm is risk neutral.