Hedonic Equilibrium

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- Goods have characteristics $Z \in \mathbb{R}^K$
- sellers $\longmapsto$characteristics $X \in \mathbb{R}^m$
- buyers $\longmapsto$ characteristics $Y \in \mathbb{R}^n$
- each seller produces one unit with some quality, each buyer wants to buy 1 unit of some quality

- $p(z)$ is the price of a good of quality z
- buyers utility $u(z, y) - p$
- seller's utility $p - c(z, x)$
- all buyers and sellers can earn zero payoff by not trading
- No trade is characterized by an outcome $\overline{z}$.

- the measure of the set of sellers is G, the measure of the set of buyers is F
- a feasible allocation consists of a pair of feasible outcome functions $d : Y \rightarrow Z \cup \{\overline{z}\}$ and $s : X \rightarrow Z \cup \{\overline{z}\}$ satisfying $F(\{y : d(y) \in B\}) = G(\{x : s(x) \in B\})$for each measurable subset B of Z; and a pair of transfer functions $t_b : Y \rightarrow \mathbb{R}$ and $t_s : X \rightarrow \mathbb{R}$ satisfying $\int t_b(t)\, dF(y) - \int t_s(x)\, dG(x) = 0$.
- an allocation (d, s, t_b, t_s) is pareto optimal if there does not exist an alternative feasible allocation (d', s', t_b', t_s') such that $u(d'(y), y) - t_b'(y) \geq u(d(y), y) - t_b(y)$ and $t_s'(x) - c(s'(x), x) \geq t_s(x) - c(s(x), x)$ for almost all $y \in Y$ and x with strict inequality holding on subsets of strictly positive measure.

- Proposition 1: An allocation is pareto optimal if and only if

$$\int u\left(d\left(y\right),y\right)dF\left(y\right)-\int c\left(s\left(x\right),x\right)dG\left(x\right)\geq$$

$$\int u\left(d'\left(y\right),y\right)dF\left(y\right)-\int c\left(s'\left(x\right)\right)dG\left(x\right) \tag{1}$$

for every feasible allocation (d', s').

- Proof: Suppose first that the allocation (d, s, t_b, t_s) satisfies (1) but isn't pareto optimal, then there is an alternative feasible allocation $\left(d', s', t_b', t_s^{'}\right)$ which is at least as good for everyone, and strictly better for someone. If so

$$\int \left\{u\left(d'\left(y\right), y\right) - t_b'\left(y\right)\right\} dF\left(y\right) +$$

$$\int \left\{t_s'\left(x\right) - c\left(s'\left(x\right), x\right)\right\} dG\left(x\right) >$$

$$\int \left\{u\left(d\left(y\right), y\right) - t_b\left(y\right)\right\} dF\left(y\right) +$$

$$\int \left\{t_s\left(x\right) - c\left(s\left(x\right)\right)\right\} dG\left(x\right).$$

- Since $\int t_b(y)\,dF(y) - \int t_s(x)\,dG(x) = 0 = \int t'_b(y)\,dF(y) - \int t'_s(x)\,dG(x)$ by feasiblity, this contradicts the presumption that the allocation (d, s, t_b, t_s) satisfies (1) . For the other direction, suppose (d, s, t_b, t_s) is pareto optimal, but that contrary to the assertion in the theorem, there is an alternative feasible allocation such that

$$\int u(d(y), y)\,dF(y) - \int c(s(x), x)\,dG(x) <$$

$$\int u\left(d'(y), y\right)dF(y) - \int c\left(s'(x)\right)dG(x)$$

Define $\rho_b(y)$ to be the transfer such that

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$$u\left(d'(y), y\right) - t_b'(y) - \rho_b(y) =$$
$$u(d(y), y) - t_b(y)$$
for each y. Similarly, let
$$\rho_s(x) + t_s'(x) - c\left(s'(x), x\right) =$$
$$t_s(x) - c(s(x), x)$$
for each x.

- Collecting these transfers from sellers and redistributing them to buyers provides each buyer and seller exactly the same payoff under the allocation (d', s', t_b', t_s') as they receive under the original allocation $\left(d, s, t^b, t^s\right)$. Total receipts from buyers less payments to sellers are

$$\int \rho_b(y)\, dF(y) - \int \rho_s(x)\, dG(x) =$$

$$\int \left\{u\left(d'(y), y\right) - t_b'(y) - u\left(d(y), y\right) + t_b(y)\right\} dF(y) -$$

$$\int \left\{t_s(x) - c\left(s(x), x\right) - t_s'(x) + c\left(s'(x), x\right)\right\} dG(x) =$$

$$\int u\left(d'(y), y\right) dF(y) - \int c\left(s'(x), x\right) dG(x) -$$

$$\int u\left(d(y), y\right) dF(y) - \int c\left(s(x), x\right) dG(x) > 0$$

- So total receipts strictly exceed total payments. The difference can be used to make some traders better off without, so the original allocation is not pareto optimal.
- the function $p : Z \to \mathbb{R}$ is a price function if $\int p(d(y))\, dF(y) = \int p(s(x))\, dF(x)$ for every feasible pair of outcome functions $d(\cdot)$ and $s(\cdot)$

- a hedonic equilibrium is a price function p and a pair of feasible outcome functions (d, s)satisfying

$$u\left(d\left(y\right), y\right) - p\left(\left(y\right)\right) = \max\left[u\left(\underline{z}, y\right), \arg\max_{z \in Z}\left\{u\left(z, y\right) - p\left(z\right)\right\}\right]$$

and

$$p\left(s\left(x\right)\right) - c\left(s\left(x\right), x\right) = \max\left[-c(\underline{z}, x), \arg\max_{z \in Z}\left(p\left(z\right) - c\left(z, x\right)\right)\right]$$

for almost all $x \in X$ and $y \in Y$.

- under weak conditions hedonic equilibrium exists, the set of equilibrium typically isn't unique. The set of equilibrium pricing functions is convex.
- Proposition 2. Every hedonic equilibrium is pareto optimal.

- Proof: By the pareto optimality theorem, if the hedonic equilibrium allocation isn't pareto optimal, then there is an alternative allocation (d', s', t_b', t_s') such that

$$\int u\left(d'\left(y\right), y\right) dF\left(y\right) - \int c\left(s'\left(x\right), x\right) dG\left(x\right) >$$

$$\int u\left(d\left(y\right), y\right) dF\left(y\right) - \int c\left(s\left(x\right), x\right) dG\left(x\right).$$

Since (d, s) are part of a hedonic equilibrium, it must be that

$$u\left(d'\left(y\right), y\right) - p\left(d'\left(y\right)\right) \leq u\left(d\left(y\right), y\right) - p\left(d\left(y\right)\right)$$

and

- $$p\left(s'(x)\right) - c\left(s'(x), x\right) \leq p\left(s(x)\right) - c\left(s(x), x\right)$$

  for each x and y. Integrating and using feasibility, and the fact that $\int p\left(d'(y)\right) dF(y) = \int p\left(s'(x)\right) dG(x)$,

  $$\int u\left(d'(y), y\right) dF(y) - \int c\left(s'(x), x\right) dG(x) \leq$$

  $$\int u\left(d(y), y\right) dF(y) - \int c\left(s(x), x\right) dG(x)$$

  a contradiction.

- Assortative Matching and hedonic equilibrium.
- Assortative Matching Theorem: suppose X, Y and Z are each subsets of $\mathbb{R}$. Suppose that u is increasing in z and that $u_{zy} \geq 0$, and $c_{zx} \leq 0$. Then in every hedonic equilibrium, $p(z)$ is non-decreasing at each $z \neq \overline{z}$ such that $d(y) = z$ for some y;$y' > y$ implies $d(y') \geq d(y)$; $x' \geq x$ implies $s(x') \geq s(x)$.

- Proof: If p is decreasing at some z for which $d(y) = z$ for some y, then y can strictly improve his payoff by increasing his choice of z. Now for $y' > y$,

$$u\left(d\left(y'\right), y'\right) - p\left(d\left(y'\right)\right) \geq u\left(d\left(y\right), y'\right) - p\left(d\left(y\right)\right)$$

and

$$u\left(d\left(y\right), y\right) - p\left(d\left(y\right)\right) \geq u\left(d\left(y'\right), y\right) - p\left(d\left(y'\right)\right)$$

which implies

$$u\left(d\left(y'\right), y'\right) - u\left(d\left(y'\right), y\right) \geq u\left(d\left(y\right), y'\right) - u\left(d\left(y\right), y\right)$$

which by the cross partial assumptions requires $d\left(y'\right) \geq d\left(y\right)$. The same argument applies to c.

- In this kind of equilibrium the highest types buy and sell the highest qualities.
- example $u(y,z) = yz$, $c(x,z) = (1-x)z^2$, x uniform on $[0,1]$, y uniform on $[0,2]$.
- Notice that this market satisfies the assumptions of the assortative matching theorem.
- so the price function is increasing. Buyers with types in $[1,2]$ will buy from sellers whose types are in $[0,1]$.

- a buyer with type 1 must be just indifferent between buying and selling the lowest quality (since the assortative matching theorem says all the other buyers will buy higher qualities). Then $z_0 - p(z_0) = 0$. Furthermore the seller with the highest cost (seller 0) will supply the quality z_0. So $p(z_0) - z_o^2 \geq 0$.
- Then using the assortative matching property, seller x will supply buyer $1 + x$ with some quality. Since a hedonic equilibrium must be pareto optimal, this quality must be bilaterally optimal for the pair consisting of buyer $(1 + x)$ and seller x. This occurs when $(1 + x) z - (1 - x) z^2$ is maximized, or $(1 + x) = 2(1 - x) z$ or $z = \frac{1+x}{2(1-x)}$. Setting $x = 0$ gives $z_0 = \frac{1}{2}$.

- This describes the complete allocation. To find the price, note that the slope of a buyer's indifference curve in (p, z) space is $\frac{dp}{dz} = y$, while the slope of a seller's iso-profit curve is

$$\frac{dp}{dz} = 2(1-x)z$$

- Using the allocation information, we can compute the price. Each buyer and seller will choose the quality at which the slope of the hedonic price function $p'(z)$ is equal to the slope of his or her indifference curve. In other words, at each $z > \frac{1}{2}$, $p(z)$ must have slope equal to the slope of the indifference curve of the seller who chooses to produce that z. This seller is the one for whom $z = \frac{1+x}{2(1-x)}$, or $\frac{2z-1}{(2z+1)} = x$, which means

$$p'(z) = 2\left(1 - \frac{2z-1}{(2z+1)}\right)z = \frac{4z}{2z+1}$$

- from the boundary condition (a buyer of type 1 is indifferent) $p\left(\frac{1}{2}\right) = \frac{1}{2}$ we get $p(z) = \frac{1}{2} + \int_{\frac{1}{2}}^{z} \frac{4\tilde{z}}{2\tilde{z}+1} d\tilde{z}$.

- Hedonics without quasi-linearity - the pre marital investment problem
- market is divided into two parts, men-women, workers-firms, etc
- firms have characteristics $x \in X$, workers $y \in Y$
- firms choose a costly characteristic $w \in W$, workers choose a costly characteristic $h \in K$

- an allocation is a pair of (measurable) mappings $d(y)$ and $s(x)$ such that $F(\{y : d(y) \in B\}) = G(\{x : s(x) \in B\})$for each measurable subset B of Z.
- payoff to a firm is $v(w, h, x)$ where h is the characteristic of the worker who they hire, firms $u(w, h, y)$ where w is the characteristic of the firm that hires them.
- an allocation (d, s) is pareto optimal if there does not exist an alternative feasible allocation (d', s') such that $u(d'(y), y) \geq u(d(y), y)$ for almost every y and $v(s'(x), x) \geq v(s(x), x)$ for almost every x with strict inequality holding on a measurable subset.

- a hedonic equilibrium is a surface $\{z : g(z) = 0\}$ satisfying the restriction that for each measurable subset B of Z

$$G\left(\left\{x : \arg \max_{z:g(z)\geq 0} v(z,x) \in B\right\}\right) =$$

$$F\left(\left\{y : \arg \max_{z:g(z)\leq 0} u(z,y) \in B\right\}\right)$$

- **Proposition:**Every hedonic equilibrium is pareto optimal.

- Proof: Then $g(d'(y)) \geq 0$ and $g(s'(x)) \leq 0$ for all x and y with strict inequality holding on some subset of either Y or X of strictly positive measure. (the qualities allocated to every trader must be on the 'wrong' side of the budget line, otherwise, they would have chosen them in the hedonic equilibrium). Suppose that $g(d'(y)) > 0$ for some subset $A \subset Y$. Then by feasibility $\int_A d'(y)\, dF(y) = \int_{A'} s'(x)\, dG(x)$ for some subset $A' \subset X$. Since $g(d'(y)) > 0$ for each $y \in A$ by construction, then $g(s'(x)) > 0$ for each $x \in A'$, a contradiction.

- Example: firms pay wages w , workers make human capital investments h, y is uniform on $[0,2]$, x is uniform on $[1,2]$. Workers payoffs are $\ln(1+w) - h^2(2-y)$. Workers are risk averse, and have convex cost functions in production of human capital. The highest worker types have the lowest marginal costs of producing human capital.
- firms have payoffs $x\ln(1+h) - w$. Risk neutral, concave production function, highest types are most productive.
- lets assume the highest types match assortatively. Then a worker of type 1 matches with a firm of type 1 and

$$\ln(1+w_0) - h_0^2 = 0 \tag{2}$$

- the indifference curve for workers in (w, h) space has slope $2h(2-y)(1+w)$ while the indifference curve for firms has slope $\frac{x}{1+h}$
- By pareto optimality, a firm of type x should match with a worker of type $1+x$ at a wage investment pair that satisfies

$$\frac{x}{1+h} = 2h(2-x)(1+w)$$

- this is the set of (w, h) pairs at which a firm of type x and a worker of type $1+x$ find their indifference curves to be tangent.

- let $\alpha(h)$ be the firm type who chooses human capital investment h in the hedonic equilibrium. Then

$$\frac{\alpha(h)}{1+h} = 2h(2-\alpha(h))(1+w)$$

which gives the hedonic relationship

$$w = \frac{\alpha(h)}{(1+h)\,2h(2-\alpha(h))} - 1 \tag{3}$$

Now we have to choose the function α.

- It has to satisfy two properties First, when evaluated at h_0, (3) must evaluate to a wage that satisfies (2). Second, at each point h, the slope of the hedonic relationship must equal the slope of the indifference curve of a firm of type $\alpha(h)$. The slope of the indifference curve is $\frac{\alpha(h)}{1+h}$, while the slope of the hedonic line is found by differentiating (3) with respect to h. This yields a differential equation with boundary value, the solution determines α and the hedonic relationship. (the solution doesn't come in closed form here).
- a degenerate case worth looking at is to imagine that y has a point mass at 1 of size 2, while x has a point mass of size 1 at 1
- Then the solution can be computed from (2) and

$$w_0 = \frac{1}{(1+h_0)\,2h_0} - 1$$

- unraveling from the bottom
- a non-cooperative treatment is needed to determine what happens below the distribution