1 Theory of Auctions

1.1 Independent Private Value Auctions

- for the moment consider an environment in which there is a single seller who wants to sell one indivisible unit of output to one of n buyers whose valuations are private, a buyer whose valuation is θ who trades at price p gets surplus $\theta - p$ the seller gets surplus p in this case. Each buyer's valuation is independently drawn from a distribution F on $[0, 1]$. F is continously differentiable and strictly increasing.

- auctions are implemented using a variety of indirect mechanisms

  1. in a first price auction each buyer submits a bid and the high bidder pays his bid, securites auctions, treasury bills, procurement, timber auctions
  2. in a second price auction each buyer submits a bid, the high bidder wins and pays the bid of the second high bidder, ebay, clearing house

3. english auction - the auctioneer asks for a starting bid, then asks for a volunteer to raise the bid, he continues to do this until no one offers to raise the bid, the last bidder offered to raise the bid wins and pays that bid
4. english button auction - each bidder who wants to bid begins by pressing and holding down a button. The price is then raised continuously and each bidder releases the button when the price gets too high. When there is only one bidder left holding down a button, that bidder wins and pays the price where the last bidder dropped out
5. dutch auction - the price falls continuously until some bidder yells stop. The yelling bidder wins and pays the price at which the price stopped. Dutch flower auctions
6. all pay auction - all bidders submit bids, the high bidder wins but all bidders pay what they bid.
7. Cremer McLean auction - a second price auction in which all bidders who want to participate have to agree ex ante to pay a fee (that might depend on the bids of the other bidders)

- Sometimes (increasing virtual valuations, iid valuations) the second price auction will be optimal.

- Even when auctions aren't optimal for the seller, many of them are comparable in terms of revenue.

- The following theorem allows you to compare all of these auctions but the last one when valuations are iid.

1.2 The optimal selling mechanism

An outcome function for the seller is a specification of what the seller wants to happen for each profile of valuations $\theta = \{\theta_1, \ldots, \theta_n\}$. An outcome consists of three things, a price that each buyer pays if he gets the object for sale, a price he pays if he doesn't get it, and the probability with which he gets the object. These depend on the profiles of valuations, so lets write them as $q_i(\theta)$, $p_i(\theta)$ and $p_i'(\theta)$, where $p_i(\theta)$ is the price that buyer i pays when he gets the good, and $p_i'(\theta)$ is the price he pays if he doesn't get the good, while $q_i(\theta)$ is the probability that i is actually given the good.

For example, suppose the seller wants to run the following mechanism - each buyer pays a fee κ to participate in the mechanism, then one of the buyers is chosen at random and given the good in exchange for a fixed fee $\overline{p}$. Then the seller would like to choose the functions that are independent of θ as follows:

$$q_i(\theta) = \frac{1}{n}$$

$$p_i(\theta) = \overline{p}$$

and

$$p'(\theta) = \kappa.$$

Hopefully you can see that though this represents an outcome function, the seller can't really expect it to happen. First of all, no buyer whose value is below $\overline{p}$ will participate. Even if their values are above $\overline{p}$ they will only participate if

$$(\theta - \overline{p})\frac{1}{n} \geq \kappa.$$

The seller can do anything she likes at this point as long as

$$\sum_i q_i(\theta) \leq 1.$$

She can assign any profile of payments she likes, including negative payments in $p(\theta)$ and $p'(\theta)$.

The payoff to buyer i from any outcome θ is then $(\theta_i - p_i(\theta)) q_i(\theta) - (1 - q_i(\theta)) p'_i(\theta)$ which allows us to compute the expected payoff associated with the mechanism for each of the buyers

$$\mathbb{E}\{(\theta_i - p_i(\theta)) q_i(\theta) - (1 - q_i(\theta)) p'_i(\theta)\}$$

$$\int \cdots \int (\theta_i - p_i(\theta_i, \theta_{-i})) q_i(\theta_i, \theta_{-i}) - (1 - q_i(\theta_i, \theta_{-i})) p'_i(\theta_i, \theta_{-i}) \prod_{i' \neq i} dF(\theta_{i'}). \tag{1}$$

Once the seller has defined the three functions, she can use this payoff function to determine whether or not the auction is worth participating in, and whether it is *incentive compatible.*

As for the outside option, we don't have to compute any complicated arg max in the auction, we can just use the fact that a seller who doesn't like the bids that some buyer submits can just refuse to trade with the buyer. So the outside option value for each buyer can be set to 0 for every buyer independent of type. Individual rationality simply means that (1) is non-negative for each buyer, and for each of the buyer's types.

Each mechanism like this has a corresponding reduce form representation as follows:

$$P\left(\theta_i\right) = \mathbb{E}\left\{\sum_{i=1}^{n} p_i\left(\theta\right) q_i\left(\theta\right) + \left(1 - q_i\left(\theta\right)\right) p_i'\left(\theta\right)\right\}. \tag{2}$$

So all we need to do to find the best way to sell is to maximize the expectation of (2) subject to the incentive compatibility and individual rationality constraints defined by (1).

Correspondingly, we can write

$$Q_i\left(\theta_i\right) = \mathbb{E} q_i\left(\theta_i, \theta_{-i}\right).$$

The collection $\left\{P_i\left(\theta_i\right) Q_i\left(\theta_i\right)\right\}_{i=1,n}$ is sometimes called the *reduced form mechanism.*

Buyer i's expected payoff from participating in the mechanism is

$$Q_i(\theta_i)\,\theta_i - P_i(\theta_i)\,.$$

If the auction mechanism is incentive compatible, then

$$Q_i(\theta_i)\,\theta_i - P_i(\theta_i) \geq Q_i(\theta_i')\,\theta_i - P_i(\theta_i')\,.$$

Lets assume that Q_i and P_i are differentiable. Then this requires

$$Q_i'(\theta_i)\,\theta_i = P_i'(\theta_i)\,. \tag{3}$$

This expression is an identity (i.e., it is true for all values of θ_i, so from basic calculus

$$P_i(\theta_i) = \int_0^{\theta_i} P_i'(t)\,dt = \int_0^{\theta_i} Q_i'(t)\,t dt =$$

$$Q_i(\theta_i)\,\theta_i - \int_0^{\theta_i} Q_i(t)\,dt$$

(which is just integration by parts). There is no constant term at the beginning of this expression because $P_i(0) > 0$ would mean the mechanism did not satisfy individual rationality for a buyer of type 0.

If two functions are identically equal, so are their derivatives, so

$$P''(\theta_i) = Q''(\theta_i)\,\theta_i + Q'(\theta_i)\,.$$

What is important about this is the second order necessary condition

$$Q''(\theta_i)\,\theta_i - P''(\theta_i) = -Q'(\theta_i)$$

will be satisfied if the mechanism $\{p_i(\theta), q_i(\theta)\}_{i=1.n}$ satisfies $Q'(\theta) > 0$ (that is, higher types trade with higher probability).

Assuming (just to make life simple) we treat all the buyers the same way so that the functions P_i and Q_i are all the same, we can rewrite the seller's revenue (2) as

$$n\int_0^1 P(\theta_i)\,f(\theta_i)\,d\theta_i =$$

$$n\int_0^1 \left\{Q(\theta_i)\,\theta_i - \int_0^{\theta_i} Q(t)\,dt\right\} f(\theta_i)\,d\theta_i =$$

$$n\int_0^1 Q(\theta_i)\,\theta_i dF(\theta_i) - n\int_0^1\int_0^{\theta_i} Q(t)\,dt f(\theta_i)\,d\theta_i =$$

$$n\int_0^1 Q(\theta_i)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) f(\theta_i)\,d\theta_i =$$

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^n \left\{ q_i(\theta_i,\ldots,\theta_n)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right\} f(\theta_1)\ldots f(\theta_n)\,d\theta_1\ldots d\theta_n. \tag{4}$$

In this series of inequalities, the fourth one follows by integrating the previous expression by parts.

Now the constraint on q_i is that for every profile of types $\theta_1\ldots,\theta_n$, the sum $\sum_{i=1}^n q_i(\theta_i)$ should be less than or equal to one. This means that for every profile of types, the sum is a weighted average of the virtual valuations of each type. If for some profile of types, none of these virtual valuations are positive, then this expression suggests that setting all the q_i to zero is the best thing to do. While if one of more of the virtual valuations is positive, then the best thing to do is to set $q_i = 1$ for the largest such virtual valuation.

This tells us exactly how to maximize revenue. Choose and r such that $r = \frac{1-F(r)}{f(r)}$ so that the virtual valuation is exactly equal to 0 when it is evaluated at r. For each profile of values $(\theta_1, \ldots, \theta_n)$, if the highest value for θ_i is less than or equal to r, don't sell to anyone, otherwise sell for sure to the highest bidder. (these assertions follow from the monotonicity of the virtual valuation function). As we will show in the next section, this is exactly what happens when you run an auction with reserve price r. Any pricing rule you use to resolve the auction will work provided it gives

$$P(\theta_i) = Q(\theta_i)\,\theta_i - \int_0^{\theta_i} Q(t)\,dt$$

for each θ_i.

1.3 Revenue Equivalence Theorem

- *Revenue Equivalence Theorem*: Suppose that buyer valuations are identically and *independently* distributed according to some known distribution F whose support is an interval in $\mathbf{R}$ and for which $\theta -$

$\frac{1-F(\theta)}{f(\theta)}$ is an increasing function. Suppose further that the indirect mechanism that guides trade has an equilibrium in which the buyer with the highest valuation trades if and only if his valuation is at least r, and that a buyer with valuation r gets an expected payoff equal to zero. Then the seller's expected revenue from this indirect mechanism is

$$n \int_r^1 F^{n-1}(\theta) \left[\theta - \frac{1-F(\theta)}{F'(\theta)}\right] F'(\theta)\, d\theta$$

Furthermore, each buyer's expected payment is given by

$$P(\theta) = F^{n-1}(\theta)\,\theta - \int_r^\theta F^{n-1}(x)\, dx$$

- Proof: This follows the mechanism design argument in the previous lecture, but I give it here for completeness
- each buyers' expected payoff is given by

$$Q_i(\theta)\,\theta - P_i(\theta)$$

where Q_i is the probability with which the buyer trades and P_i is the expected payment the buyer makes to the seller.

- by assumption the indirect mechanism has an equilibrium in which the buyer with the highest valuation trades, so this trading probability is the same for every one and equal to $F^{n-1}(\theta)$ for buyers whose valuations at at least r, and it is equal to zero otherwise

- since it cannot pay for a buyer to behave as if his type were different from his true type in any equilibrium, it must be that

$$F^{n-1}(\theta)'\,\theta = P_i'(\theta)$$

for every buyer whose valuation is at least r

- Integrating by parts gives

$$P_i(\theta) = \int_0^\theta F^{n-1}(s)'\, s ds = F^{n-1}(\theta)\,\theta - \int_r^\theta F^{n-1}(s)\, ds \qquad (5)$$

- this gives the result for buyers' expected payoff. The seller's expected revenue is the sum of the expected revenue for each buyer, or

$$n\int_0^1 P_i(\theta)\,dF(\theta) = n\int_r^1 \left\{F^{n-1}(\theta)\,\theta - \int_r^\theta F^{n-1}(s)\,ds\right\} dF(\theta)$$

- because $P_i(t) = 0$ if $t < r$.
- to integrate this by parts write it first as

$$n\left\{\int_r^1 \left\{F^{n-1}(\theta)\,\theta\right\} dF(\theta) - \int_r^1 \int_r^\theta F^{n-1}(s)\,ds dF(\theta)\right\}.$$

In the second term in the brackets, think of

$$u(\theta) = \int_r^\theta F^{n-1}(s)\,ds$$

and

$$dv(\theta) = dF(\theta)$$

so that the second double integral can be written as

$$F(\theta)\int_r^\theta F^{n-1}(s)\,ds\Big|_r^1 - \int_r^1 F^n(\theta)\,d\theta =$$

$$\int_r^1 F^{n-1}(\theta)\,d\theta - \int_r^1 F^n(\theta)\,d\theta.$$

Recombine this with the first integral to get

$$n\left\{\int_r^1 \left\{F^{n-1}(\theta)\,\theta\right\} dF(\theta) - \int_r^1 F^{n-1}(\theta)\,d\theta + \int_r^1 F^n(\theta)\,d\theta\right\} =$$

$$n\int_r^1 F^{n-1}(\theta)\left[\theta - \frac{1-F(\theta)}{F'(\theta)}\right] F'(\theta)\,d\theta.$$

1.4 Using Revenue Equivalence - First Price Auction

- in the first lecture, we showed an example of a first price auction that possessed an equilibrium in increasing bidding rules. Now let us just suppose that such an equilibrium exists more generally. Then the expected payment is equal to the bid multiplied by the probability of winning, i.e.,

$$P_i(\theta) = Q_i(\theta)\, b_i(\theta) = F^{n-1}(\theta)\, b_i(\theta)$$

so

$$b_i(\theta) = \theta - \frac{\int_r^\theta F^{n-1}(s)\, ds}{F^{n-1}(\theta)} \tag{6}$$

for each $\theta \geq r$

- if all bidders use this bid function, the bidder with the high valuation will win because this function is increasing, it satisfies incentive compatiblity, so no bidder using it would prefer to act like a bidder with another valuation. Check for yourself that is doesn't pay to bid prices that no other bidder would ever bid and that a buyer of valuation r gets zero expected payoff

- in other words, the revenue equivalence theorem can be used to calculate the equilibrium bidding strategy in a first price auction.

1.5 Second Price Auction

- in a second price auction, there is an equilibrium in which each buyer bids his true valuation. This bidding strategy is increasing, so the buyer with the highest valuation will trade in a second price auction - a buyer who bids the reserve price will only win if no other buyers bid, but then he gets zero surplus

- thus from the revenue equivalence theorem a first and second price auction in which the reserve price is the same give the seller the same expected revenue.

- furthermore, the expected payment made by a bidder of type θ in the second price auction is equal to the probability of winning multiplied by the expectation of the second highest valuation conditional on θ being the highest valuation.

- the from (5), it follows that the equilibrium bid in the first price auction for a bidder of type θ is equal to the expected value of the second highest valuation or r whichever is higher, conditional on θ

being the highest valuation (just integrate by parts)

$$\frac{\int_0^r (n-1) F(s)^{n-2} f(s) r ds}{F^{n-1}(\theta)} + \frac{\int_r^\theta (n-1) F(s)^{n-2} f(s) s ds}{F^{n-1}(\theta)} =$$

$$\frac{\int_0^r (n-1) F(s)^{n-2} f(s) r ds}{F^{n-1}(\theta)} + \frac{\int_r^\theta (n-1) F(s)^{n-2} f(s) s ds}{F^{n-1}(\theta)} =$$

$$\frac{rF(r)^{n-1}}{F^{n-1}(\theta)} +$$

$$\frac{F(s)^{n-1} s\Big|_r^\theta - \int_r^\theta F(s)^{n-1} ds}{F^{n-1}(\theta)}$$

$$= \theta - \frac{\int_r^\theta F^{n-1}(s) ds}{F^{n-1}(\theta)}$$

1.6 All-Pay Auction

- let us use the technique we employed for first price auctions to compute the equilibrium in the all pay auction, suppose there is an equilibrium in increasing bidding strategies so that the equilibrium outcome is always that the buyer with the high valuation ends up trading.

- since everyone pays whether or not they win the object, the expected payment is equal to the bid, i.e.,

$$b(\theta) = F^{n-1}(\theta)\,\theta - \int_r^\theta F^{n-1}(s)\,ds$$

- Notice that in each of these applications, we know the allocation rule q_i but we don't know the rules p_i or p_i' because they have to be derived from equilibrium play which differs in each of the three auctions. The theorem says that this equilibrium play, whatever it is, must support an expected price P that satisfies

$$P(\theta) = F^{n-1}(\theta)\,\theta - \int_r^\theta F^{n-1}(x)\,dx.$$

- thats why we were able to derive the equilibrium bidding functions from each of the last equation. Once we have those functions, we can deduce the outcome functions associated with each.

- For each of the three auctions we have

$$q_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \theta_i > \theta_j \forall \text{j} \neq \text{i} \\ 0 & \text{otherwise.} \end{cases}$$

- or if you think that F has atoms, $q(\theta_i) = \frac{1}{\#\{j: j \geq j' \forall j' \neq j\}}$.

- In the second price auction, we have $p_i'(\theta_i, \theta_{-i}) \equiv 0$ while

$$p_i(\theta_i, \theta_{-i}) = \max_{j \neq i} \{\theta_j\}$$

- In the first price auction p_i' is always 0 while,

$$p_i\left(\theta_i,\theta_{-i}\right)=\theta_i-\frac{\int_r^\theta F^{n-1}\left(s\right)ds}{F^{n-1}\left(\theta\right)}.$$

- Finally for the all pay auction

$$p\left(\theta_i,\theta_{-i}\right)=p'\left(\theta_i,\theta_{-i}\right)=F^{n-1}\left(\theta\right)\theta-\int_r^\theta F^{n-1}\left(s\right)ds.$$

1.7 Identification

- bidder types are unknowns from the perspective of an outside observer, however they are associated with whatever the outside observer can see

- the theory starts with two pieces of information, and auction format, call it $\mathcal{A}$ and a family $\mathcal{F}$ of type distributions that the modeler believes are possible. In this lecture and second price auction is an auction format, while we believe that types are i.i.d and with each individual bidders type described by some distribution function F on $[0,1]$ for which F has a strictly positive density at each point in $[0,1]$

- an outside observer might have historical data on winning bids in a first price auction, or maybe the observer can record all the bids in an auction. This historical data involves some distribution G of observable information.

- Bayesian equilibrium play in the auction $\mathcal{A}$ induces some distribution on the observables denoted by $\Psi_{\mathcal{A}} : \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is the image of $\mathcal{F}$ induced by the transformation $\Psi_{\mathcal{A}}$ - in words, $\mathcal{G}$ is all the distributions of observables that you could possibly get through equilibrium play for some distribution $F \in \mathcal{F}$.

- Identification is the problem of working backwards from G to F.

- Formally, the type distribution associated with some auction $\mathcal{A}$ is said to indentifiable if for every $G \in \mathcal{G}$ there is a unique type distribution F such that $G = \Psi_{\mathcal{A}}(F)$.

- in this section we consider two questions - are type distributions identifiable in the second price auction if you have observed a historical series of transactions prices, and, are type distributions identifiable in the first price auction if you have observed all the bids.

- to start, suppose you have observed a sequence of auctions and recorded the bids that were made by different bidders, i.e., if you have held T auctions, each of which had n bidders, then you have observations on nT different bids. You believe the values behind these bids are independently drawn from some unknown distribution F.

- an econometrician has estimated that the observed distribution of bids is given by G, a cumulative distribution function that we'll assume has a density. The problem is to tell him/her what the unknown distribution of values is assuming that the bidders are all playing equilibrium strategies.

- As you now know, the equilibrium strategies depend on the type of auction the data comes from

- for a second price auction this is easy if $G(b)$ is the proportion of all bids that were less than or equal to b then that is also the proportion of values that are less than or equal to b - the distribution of bids is the distribution of values

- next time the econometrician arrives you learn they misinterpreted the data. The nT observations that were used to estimate G weren't all the bids, they were just nT observations on the prices that people paid after winning the auction - they are prices from a second price auction not the bids. Can you still identify the distribution of values?

- The result is as follows, if $G(p)$ is the distribution of trading prices in the second price auction (the proportion of auctions for which the winning bidder paid something less than or equal to p, then, the proportion of bidders in the auctions whose values are less than of equal to p is given by solving the following equation for $\tilde{F}$

$$G(p) = n(n-1)\int_0^{\tilde{F}} t^{n-2}(1-t)\,dt.$$

Notice that since the right hand side is strictly increasing in $\tilde{F}$, it has a unique solution for every p.

- to see this, start with the observation that in any particular second price auction, the trading price is the second highest value among all the bidders who were there. If the distribution of values were equal to F, then the probability that the second highest value is equal to some winning price b is

$$n(n-1)F(b)^{n-2}(1-F(b))f(b)$$

So the distribution F supports a cumulative distribution

$$G(b) = n(n-1)\int_0^b F\left(\tilde{b}\right)^{n-2}\left(1-F\left(\tilde{b}\right)\right)f\left(\tilde{b}\right)d\tilde{b}$$

- $G(b)$ is given by the data we have, so we have to try to find an F that will support this relationship.

- now do the integration on the right hand side by a change of variable in which $t = F\left(\tilde{b}\right)$ and $dt = f\left(\tilde{b}\right) d\tilde{b}$ so it equals

$$n(n-1)\int_0^{F(b)} t^{n-2}(1-t)\,dt$$

- This gives the property that F has to satisfy so that it would support the observed distribution of bids - which is the result we want.

- Case 2: G is the distribution of bids in a first price auction.

- In this case the result is that if the proportion of bids that is less than or equal to p in a series of identical first price auctions with N bidders then, F must satisfy

$$G(p) = F\left(p + \frac{G(p)}{g(p)(n-1)}\right).$$

- what this says is that you find $F(x)$ by finding p_x such that

$$x = p_x + \frac{G(p_x)}{g(p_x)(n-1)}$$

then taking $F(x)$ as $G(p_x)$. Notice that the function $p + \frac{G(p)}{g(p)(n-1)}$ is completely determined by the data you have.

- Here is the argument: the expected payoff to a bidder in a first price auction is

$$(\theta - b(\theta')) F^{n-1}(\theta')$$

where θ' is the value that the bidder pretends to be.

- if b is an equilibrium bidding strategy, then this expected payoff will be maximized at $v' = v$, which gives the first order condition

$$b'(\theta) F^{n-1}(\theta) = (n-1)(\theta - b(\theta)) F^{n-2}(\theta) f(\theta)$$

which you could write as

$$1 = \frac{(\theta - b(\theta)) f(\theta)}{F(\theta) b'(\theta)} (n-1) \tag{7}$$

- now for any bid b the data says that the probability that any bidder will bid something less than or equal to $\tilde{b}$ is

$$G\left(b\left(\tilde{\theta}\right)\right) = F\left(b^{-1}\left(b\left(\tilde{\theta}\right)\right)\right) = F\left(\tilde{\theta}\right)$$

where $\tilde{\theta}$ is the type of bidder who bids $\tilde{b}$ so that $g\left(b\left(\tilde{\theta}\right)\right) b'\left(\tilde{\theta}\right) = f\left(\tilde{\theta}\right)$ or

$$g\left(\tilde{b}\right) = \frac{f\left(\tilde{\theta}\right)}{b'\left(\tilde{\theta}\right)}$$

evaluated at $\tilde{\theta} = b^{-1}\left(\tilde{b}\right)$.

- Now we can substitute these last two observations into (7) and evaluate them at $\tilde{\theta}$ to get

$$1 = \left(\tilde{\theta} - \tilde{b}\right) \frac{g\left(\tilde{b}\right)}{G\left(\tilde{b}\right)} (n-1)$$

which means that

$$\tilde{\theta} = \tilde{b} + \frac{G\left(\tilde{b}\right)}{g\left(\tilde{b}\right)(n-1)},$$

This expression just says that if we observe a bid $\tilde{b}$, then the type of the player who submitted it must be $\tilde{b} + \frac{G(\tilde{b})}{g(\tilde{b})(n-1)}$. This is the inverse function for the bidding rule expressed in terms of the observables G rather than the unobservables.

- the implication of this is that the proportion of bids that are less than or equal to $\tilde{b}$, $G\left(\tilde{b}\right)$ is the same as the measure $F\left(\tilde{b} + \frac{G(\tilde{b})}{g(\tilde{b})(n-1)}\right)$.

1.8 Position Auctions

- An search site has a webpage with a lot of traffic (like google). This webpage has two 'slots' at the top for ads. There are three firms who have their own websites with ads on them. Consumers *click through* the ad slots on the search company's web to look at these firms' ads. Any consumer who visits a firm web page will either decide to buy the firm's good, in which case the payoff to both the consumer and the firm is 1 (price plays no role here). If the consumer doesn't buy, the payoff to both the consumer and firm is 0.

- each firm has a quality v which measures the probability that a consumer will choose to buy the product after seeing the web page. Each firm knows its own quality, otherwise information is incomplete. Ex ante each consumer believes that each firm's quality is independently drawn from some distribution F with support $[0, 1]$.
- The search site holds an auction in which each of the three firms bids the amount they are willing to pay per 'click'. A click occurs when a consumer clicks through the link in the search sites slot and looks at an individual firm's webpage.
- the highest bidder's link is placed in the top slot - the high bidder

pays the second highest bid for each click on its ad. The second highest bidder's ad is placed in the lower slot, the second high bidder pays whatever the third highest bidder bid.

- Clicking on an ad is assumed to be costly - the search cost for consumer i is s_i drawn using a distribution G with support on $[0, 1]$.

- the process then goes like this - each of the three firms submits a bid, say b_i. Suppose $b_1 > b_2 > b_3$. Firm 1 (who bids b_1) has their link placed in the top slot, firm 2 has their link placed in the lower slot.

- each consumer decides whether or not to click on one of the links on the search site's web page.

- when a consumer clicks through the link to firm i and views firm $i's$ webpage, firm i makes a payment to the search firm equal to whatever price it won in the auction, each consumer buys from the firm with probability v_i.

- a consumer who clicks through a link and fails to buy can try again at the second slot
- let T_1 and T_2 be the proportion of consumers who click on the top slot. Then the profit of firm 1 is

$$T_1 (v_1 - b_2)$$

- the profit of firm 2 is

$$T_2 (v_2 - b_3)$$

- firm 3 earns 0
- The payoff to a consumer with search cost s who clicks through slot i is

$$v_i - s$$

  while the payoff to a consumer who clicks through both slots is $v_1 - s + (1 - v_1)(v_2 - s)$

- Result: There is an equilibrium in which each of the three firms bids less than their value in the auction, both T_1 and T_2 are positive with $T_1 > T_2$.

- Lets assume, as we did with standard auctions, that the bidding rule that firms use is a strictly increasing function $b(v)$ (whose range is contained in $[0,1]$). If so, the probability that a firm with value v wins the top slot is

$$F^2(v)$$

which is exactly the same as in the first and second price auctions we looked at previously. The probability the firm wins the second slot is

$$2F(v)(1-F(v))$$

- Without knowing what the equiilbrium bidding function is, we can still address an interesting question. As we look at the page with two ads displayed, what should we expect the quality of the link in the first ad to be?

- We have to begin with the fact that when the firm starts her search she already has a bunch of information, in particular, she can see the names of the firms in each of the slots, and presumably knows the identify of the firm that didn't win a spot. Call the firms A B and C. The probability that things work out this way is

$$\int_0^1 \int_0^v \int_0^{v'} f(v) f(v') f(\tilde{v}) \, d\tilde{v} dv' dv =$$

$$\int_0^1 \int_0^v f(v') F(v') \, dv' f(v) \, dv =$$

$$\int_0^1 \frac{1}{2} \int_0^v dF^2(v') = \frac{1}{2} \int_0^1 F^2(v) f(v) \, dv =$$

$$\frac{1}{6} \int_0^1 dF^3(v) = \frac{1}{6}.$$

So all the probability calculations that follow should be conditional on that. This condition will cancel out when we do conditional probability calculations.

- The expected quality of firm A is then

$$V_1 \equiv \frac{\frac{1}{2}\int_0^1 \tilde{v}F^2(\tilde{v})\,f(\tilde{v})\,d\tilde{v}}{\frac{1}{6}}.$$

- the reason this is interesting is that consumers will only click on a slot if the expected payoff exceeds their search cost. V_1 represents the expected payoff to searching the top slot. This determines the click through rate - $T_1 = G(V_1)$ - this is the measure or proportion of consumers whose search costs are low enough that they will be willing to click through the link in the top slot.

- the click through rate on the lower slot is more complicated. Only consumers whose search cost is below V_1 will ever click on the second slot - and only if they fail to trade with firm A. Even then, they won't know the quality of firm A but will become pessimistic because the know that it is the best of the two firms.

- this requires figuring out the expected quality of firm B conditional on having failed to find a trade in the upper slot.

- we'll do this by calculating the conditional density for each value v of firm B - recall from the rule for conditional probability

$$\Pr(v|\text{failed to trade with } A) = \frac{\Pr(v \cap \text{failed to trade with } A)}{\Pr(\text{failed to trade with } A)}$$

- both probabilities should be conditional on the outcome with A in the top slot, but the condition $\frac{1}{6}$ will be in both the numerator and denominator, so we'll leave it out. The probability in the numerator is

$$\int_v^1 (1-\tilde{v}) f(\tilde{v}) \, d\tilde{v} f(v) F(v)$$

- The event the consumer didn't trade with A has probability

$$\frac{1}{2}\left(\int_0^1 (1-\tilde{v}) F^2(\tilde{v}) f(\tilde{v}) \, d\tilde{v}\right)$$

  which by the rule for conditional probability gives

$$\Pr(v|\text{failed to trade with } A) = \frac{\int_v^1 (1-\tilde{v}) f(\tilde{v}) \, d\tilde{v} f(v) F(v)}{\frac{1}{2}\int_0^1 (1-\tilde{v}) F^2(\tilde{v}) f(\tilde{v}) \, d\tilde{v}}.$$

- Notice that if we integrate across all possible values v for firm B we get

$$\int_0^1 \int_v^1 (1-\tilde{v}) f(\tilde{v})\, d\tilde{v} f(v) F(v)\, dv =$$

$$\frac{1}{2}\int_0^1 \left\{ \int_v^1 (1-\tilde{v}) f(\tilde{v})\, d\tilde{v} \right\} dF^2(v)$$

integrating by parts gives

$$\frac{1}{2}\left\{ \int_v^1 (1-\tilde{v}) f(\tilde{v})\, d\tilde{v} \cdot F^2(v) \,|_0^1 \right\} +$$

$$\frac{1}{2}\int_0^1 F^2(v)(1-v) f(v)\, dv$$

which shows that this conditional probability distribution integrates to 1.

- Now that we have the density, we can find the expectation of v at firm B conditional on failing to trade with A from the formula

$$V_2 \equiv \int_0^1 \frac{v \int_v^1 (1-\tilde{v}) f(\tilde{v})\, d\tilde{v} F(v) f(v)\, dv}{\frac{1}{2}\int_0^1 (1-\tilde{v}) F^2(\tilde{v}) f(\tilde{v})\, d\tilde{v}}$$

- Now the click through rate on the lower slot is just the proportion of all the buyers whose search costs are less than V_2 who fail to trade when they visit the firm in the top slot, i.e.,

$$T_2 = (1 - V_1)\, G(V_2)$$

- Now we'll use the generic method to work out the bidding rule

- The payoff to the seller - there are a bunch of possibilities - for a start, we only need to worry about the cases where the seller is the high bidder, or the second high bidder. This bid is submitted at the very beginning before anything is learned about the other bidders, so being the high bidder involves two cases where each of the other two bidders is the second high bidder.

- for example, the values of the bidders might be v, v' and v'', in descending order. The probability this happens is $f(v') \cdot f(v') \cdot f(v'')$ and in this case the payoff of the high bidder is

$$T_1(v - b(v')).$$

- In order to take the expectation across all possibilities, we need to start with these, so we'll get

$$\int_0^v \int_0^{v'} T_1(v - b(\tilde{v})) f(\tilde{v}) \, d\tilde{v} f(v') \, dv' =$$

$$\int_0^v T_1(v - b(v')) F(v') f(v') \, dv'$$

since each of the other bidders could play the role of the second high bidder, we need to take 2 of these, one for each of the other bidders.

- then there is the case where our bidder is the second high bidder, say $v' > v > v''$

- now, take one of the other two bidders to play the role of v' and sum to get

$$\int_v^1 T_2 (v - b(v'')) f(v') \, dv' f(v'') \, dv'' =$$

$$(1 - F(v)) \int_0^v (v - b(v')) f(v') \, dv'.$$

- Again, there are two different bidders who could play the role of the high bidder.

- Now we can put it together to make the payoff 2 times

$$\int_0^v T_1 (v - b(v')) F(v') f(v') \, dv' +$$

$$(1 - F(v)) \int_0^v T_2 (v - b(v')) f(v') \, dv'$$

This has to be larger than

$$\int_0^{\tilde{v}} T_1 \left(v - b\left(v'\right)\right) F\left(v'\right) f\left(v'\right) dv' +$$

$$\left(1 - F\left(\tilde{v}\right)\right) \int_0^{\tilde{v}} T_2 \left(v - b\left(v'\right)\right) f\left(v'\right) dv'$$

for all $v$and $\tilde{v}$. The first order condition for bidding is

$$T_1 \left(v - b\left(v\right)\right) F\left(v\right) f\left(v\right) + \left(\left(1 - F\left(v\right)\right) T_2 * \left(v - b\left(v\right)\right)\right) f\left(v\right) =$$

$$f\left(v\right) \int_0^{v} T_2 \left(v - b\left(v'\right)\right) f\left(v'\right) dv'$$

which gives the following result

$$\left(v - b\left(v\right)\right) = \frac{\int_0^v T_2 \left(v - b\left(v'\right)\right) f\left(v'\right) dv'}{F\left(v\right) T_1 + \left(1 - F\left(v\right)\right) T_2}$$

1.9 Applications outside auction theory

- (Klemperer) suppose two parties are involved in a lawsuit. The party that wins the lawsuit gets utility that is θ_i higher than it does when it loses (the payoff to losing is normalized to zero). These payoffs are private and drawn from a common monotonic distribution F

- each party spends b_i defending itself and the party that spends the most wins the lawsuit

- under standard rules (in the US) each party pays its own legal fees, so the winner gets $\theta_i - b_i$

- could expenditures on legal fees be reduced by forcing the losing party to pay the winner some fraction of the loser's expenses? what about forcing the loser to pay some fraction of the winner's legal expenses, should there be a minimum legal expenditure required to win the case?

- the key insight is that the existing legal system is equivalent from a strategic viewpoint to an all pay auction - the party who spends (bids) the most wins the case (trades) but all parties pay what they spent (bid).

- total expected legal expenditures are equivalent to the seller's revenue in the auction problem

- the minimum expenditure requirement (for example forcing litigants to be represented by lawyers) is equivalent to the reserve price in the auction

- suppose first that there is an equilibrium in which parties expenditures are increasing functions of their gains θ, under the existing rules a party whose gain is exactly equal to the minimum expenditure requirement cannot gain by litigating, nor can they lose since they receive the default payoff 0 by spending nothing, so the revenue equivalence theorem implies that expected legal expenditures are

$$2\int_r^1 F(\theta)\left[\theta - \frac{1-F(\theta)}{F'(\theta)}\right]F'(\theta)\,d\theta \tag{8}$$

while parties expected expenditures are equal to the equilibrium legal expenses

$$F(\theta)\,\theta - \int_r^\theta F(s)\,ds$$

- legal expenditures go to lawyers, so assuming that objective of the legal system is to maximize lawyer income, the optimal expenditure requirement is to set r such that

$$r - \frac{1 - F(r)}{F'(r)}$$

  as in the optimal selling mechanism,

- on the other hand, if the objective is to maximize expected gains to litigation less expected expenditures (and assuming the virtual valuation is increasing), the reserve price 0 satisfies at least the necessary condition for optimization (just check the derivative of the payoff evaluated at $r = 0$.

- what about having the loser pay a portion of his own expenses to the winner as an additional penalty

- for simplicity assume the loser pays the winner whatever the loser actually spent litigating the case

- assume that the equilibrium bidding strategy is increasing, then the party with the highest valuation will win the case. Let s the minimum expenditure required to litigate the case. A litigant who spends exactly s will win and get his value θ (without any transfer from the other player) if the other player decides not to contest

- on the other hand, if the other player decides to contest, the litigant who makes the minimum expenditure s will lose for sure and be forced to pay $2s$, so the expected payment is

  $$F(r)\, s + (1 - F(r))\, 2s = s + (1 - F(r))\, s$$

  which should equal $F(r)\, r$ in order that the marginal participant get exactly 0 surplus

- then by the revenue equivalence theorem, a legal system with minimum expenditure $s = \frac{F(r) r}{2 - F(r)}$ in which the loser pays his own expenditure to the winner yields the same expected expenditures as a system where each litigant pays his own costs and where minimum expenditures are r

2 Multi-Unit Private Value Auctions

- maintaining the assumption that each bidder wants only a single unit and that each bidders' valuation is independently drawn from the distribution F
- suppose that the seller has $n > K > 1$ units to offer for sale.
- analogously to the case with a single unit, there are a number of different ways that the good could be allocated

  1. goods could be allocated to the K highest bidders at the highest rejected price
  2. again the K highest bidders at the lowest accepted price
  3. K highest bidders are allocated, each pays the price that he or she bids

4. K objects could be auctioned one at a time to the K highest bidders

- *Revenue Equivalence Theorem for Multiple Units:* Suppose that the auction rules and equilibrium are such that for every vector $\theta \in \Theta^n$ of valuations, the buyers with the K highest valuations trade if and only if their valuations are at least r, while buyers whose valuations are exactly equal to r get zero expected payoff. Then the expected payment by a buyer of type θ is given by

$$\int_r^\theta s \begin{pmatrix} n-1 \\ K \end{pmatrix} F^{n-K-1}(s)\,(1-F(s))^{K-1} f(s)\,ds \tag{9}$$

- **Proof:** By assumption, only the highest K valuation buyers will trade in equilibrium, so if the K^{th} highest valuation among the other buyers exceeds θ the buyer will fail to trade. Conversely, if this K^{th} highest valuation is less than θ, then the buyer will be one of the winning bidders provided his own valuations exceeds the seller's reserve price r. The density for the K^{th} highest valuation among the other *N-1* bidders is

$$\binom{n-1}{K} F^{n-K-1}(s)\,(1-F(s))^{K-1} f(s)$$

(there is one bidder among the other $n-1$ who has valuation exactly equal to s, $K-1$ whose valuations exceed s, and $n-1-(K-1)-1$ left over whose valuations are lower than s. Then there are $\binom{n-1}{K}$ different groups of bidders among the other n-1 bidders who we could use as high bidders). So the probability with which the buyer trades is

$$Q(\theta) = \int_0^\theta \binom{n-1}{K} F^{n-K-1}(s)\,(1-F(s))^{K-1} f(s)\,ds$$

when the buyers' valuation exceeds r and $Q(\theta) = 0$ otherwise.

- incentive compatibility gives

$$P_i(\theta) = \int_0^\theta Q(s)'\,s ds$$

as above, so substituting for Q' gives the result.

2.1 Highest Rejected Bid Auction (uniform price auction)

- suppose the K highests bidders trade and pay the price bid by the $K+1$ highest bidder (that is the highest bid that fails to win) and that the seller sets reserve price r.

- then the price paid by a buyer who trades is independent of the price that he bids, and bidding true valuation is a weakly dominant strategy, and a Bayesian equilibrium. Since this bidding function is monotonically increasing, the buyers with the highest n valuations will trade as required by the revenue equivalence theorem

- since a buyer with valuation r will only trade when the K^{th} highest bid among the other buyers is below r a buyer with this valuation will pay r when he wins and nothing if he doesn't so his expected payoff is zero as required by the revenue equivalence theorem to expected payments are given by (9) and seller revenue is just n times the expectation of this payment over θ

2.2 Pay your own bid (Discriminatory Price Auction)

- suppose K highest bidders trade, and each pays his own bid, again with reserve price r

- assume for the moment that this indirect mechanism and the equilibrium associated with it satisfy the assumptions of the revenue equivalence theorem

- then the expected payment is equal to the bid multiplied by the probility of winning, or

$$b(\theta)\int_0^\theta \begin{pmatrix} n-1 \\ K \end{pmatrix} F^{n-K-1}(s)\,(1-F(s))^{K-1} f(s)\,ds =$$

$$\int_r^\theta s \binom{n-1}{K} F^{n-K-1}(s)\,(1-F(s))^{K-1} f(s)\,ds$$

by (9), and this can be solved for the equilibrium bidding rule

3 Approximation

- return to a simple first price auction with asymmetric bidders. Values are again independently distributed on $[0,1]$, but in this section they are not identically distributed, so F_i is the probability distribution for bidder i's value.
- As we have discussed before, equilibrium bidding rules are different in this case
- example bidder 1 has distribution $F_1(x) = x$ (i.e. U$[0,1]$) while bidder 2 has distribution $F_2(x) = \frac{x}{2}$ (i.e, $U[0,2]$).

- verify for yourself that the rules

$$b_1(\theta) = \frac{2}{3\theta}\left(2 - \sqrt{4 - 3\theta^2}\right)$$

and

$$b_2 = \frac{2}{3\theta}\left(\sqrt{4 + 3\theta^2} - 2\right)$$

are equilibrium bidding rules by verifying that these both solve the differential equations that characterize the equilibrium.

- Since $b_1(\theta) \geq b_2(\theta)$ bidder 2 can lose the auction even though he has a higher value than bidder 1 - so the auction equilibrium is inefficient in the sense that with strictly positive probability, the auction will give the good to the wrong person.

- the auction is also not revenue maximizing. Player 2 wins the auction when

$$\theta_2 > \frac{1}{\sqrt{\frac{1}{\theta_1^2} + \frac{3}{4}}}$$

while revenue maximization requires that player 2 be given the good when $\theta_2 > \theta_1 + \frac{1}{2}$ (use the virtual valuations).

- so even though the auction could be made efficient by reverting to a second price auction, this might involve a loss in revenue (I am not sure, one could probably do the calculation numerically).

- This leads to an unpleasant situation in which you know that the auction you have designed isn't right, but the only alternatives you can think of are just different (better in some ways worse in others).

- the computer scientists figured out that you could actually quantify the loss without having to figure out the optimal mechanism

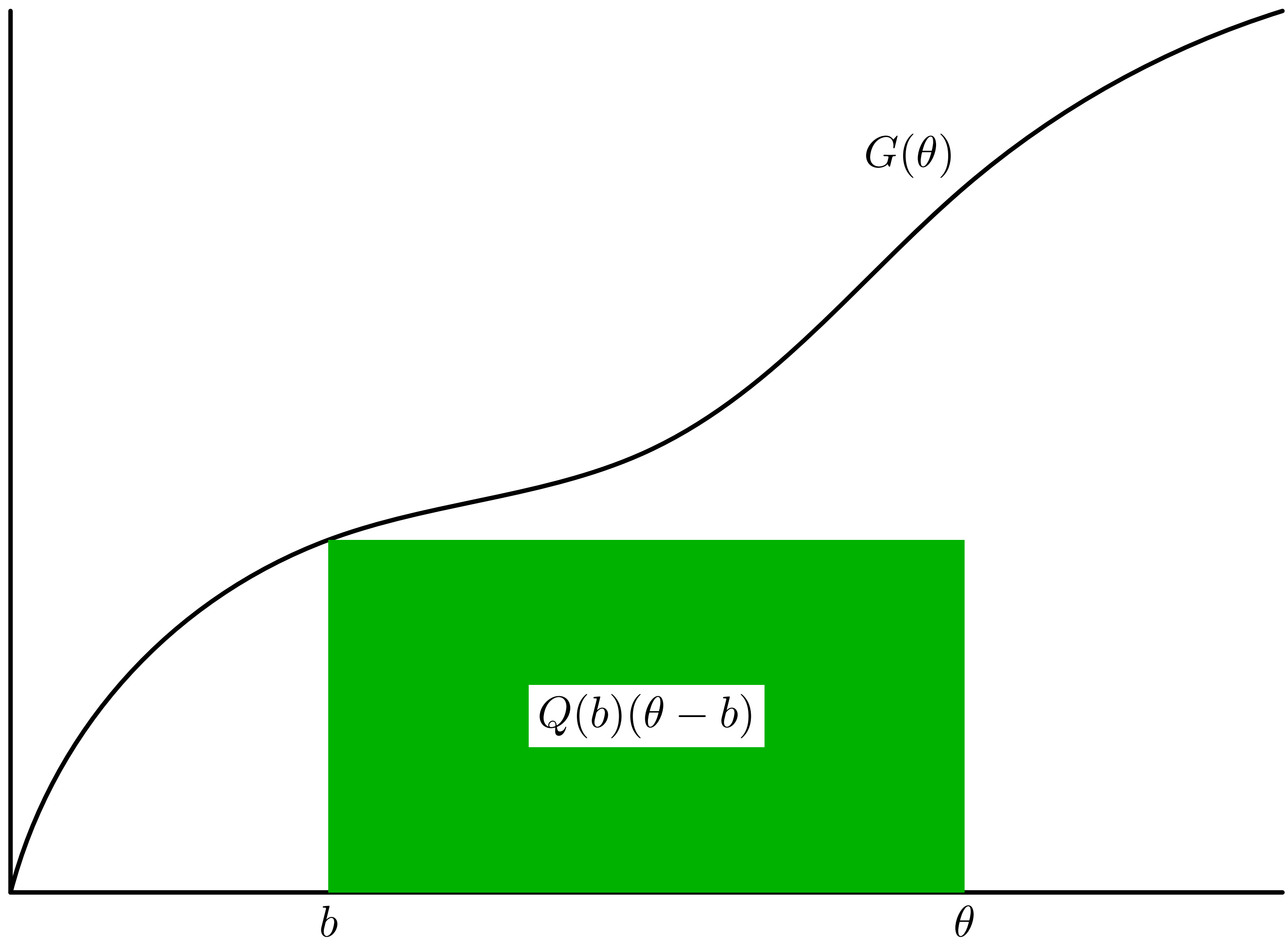
- there is a large literature on this (see the article by Jason Hartline in the readings) - I'll illustrate the methods with surplus and the first price auction. Here is the theorem

- **Theorem** For any n player first price auction with values independently distributed according to the profile $\{F_1, \ldots, F_n\}$, the expected surplus generated by any Bayesian equilibrium of the auction is at least $\frac{e-1}{e}$ as large as the maximal expected surplus.

- for reasons I don't understand, the computer scientists like to say the first price auction is a $\frac{e}{e-1}$ (or a 1.58) approximation of maximal social surplus.

- To see how they prove this, start with an arbitrary bidder in the auction whose value is θ. From the equilibrium she believes is being played, she has some belief about the probability distribution of the *highest bid* of her opponents. Call this distribution function G_i - and notice that it is different for every bidder.

- Suppose she bids b in the auction. Then the probability distribution over the bids of the others, along with her own bid determine her expected payoff as shown in the following diagram:



- now observe a second fact - the expected value of the highest bid of the others is

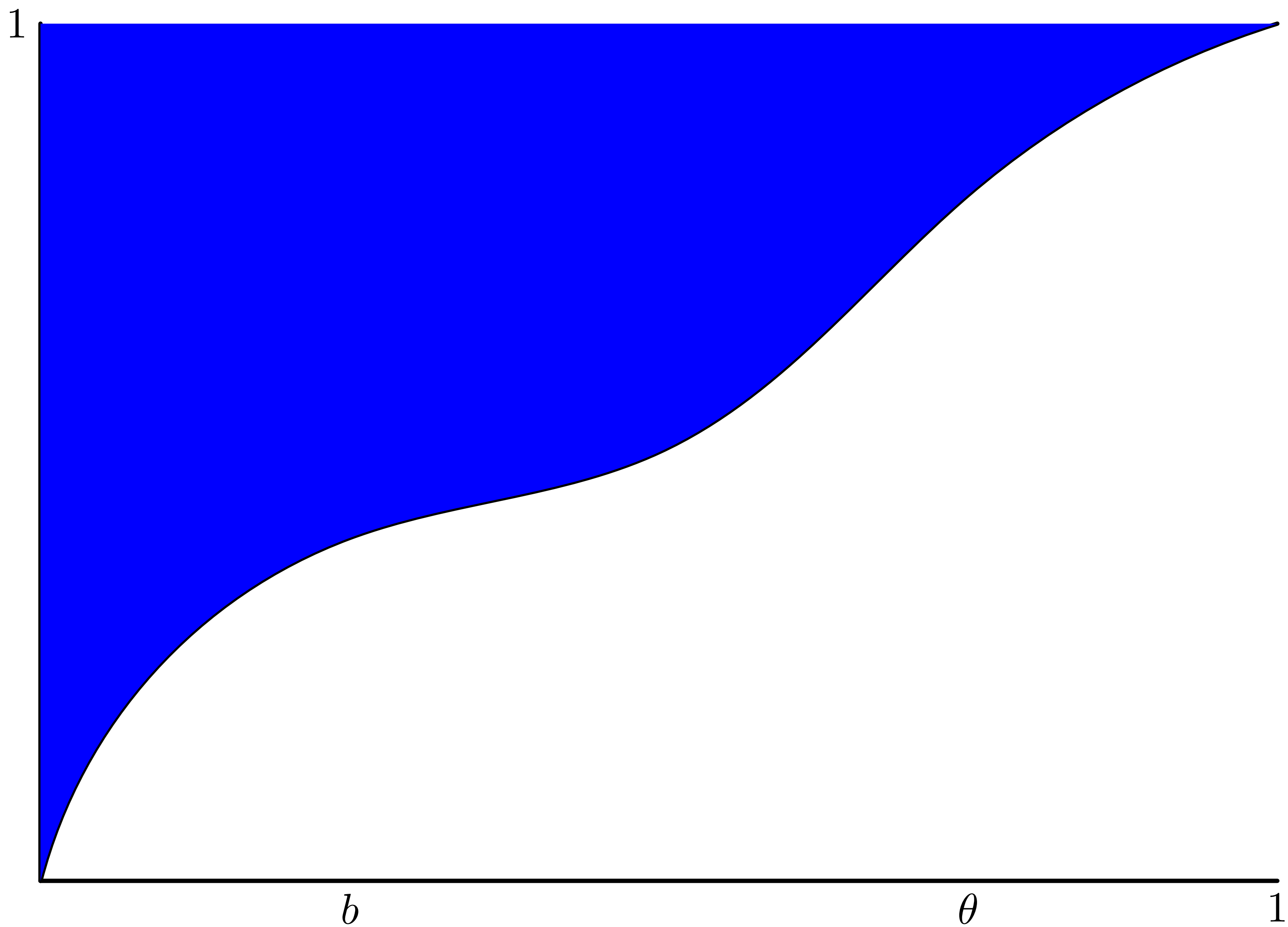
$$\hat{B} \equiv \int_0^1 b dG(b) =$$

$$bG(b)|_0^1 - \int_0^1 G(b)\, db =$$

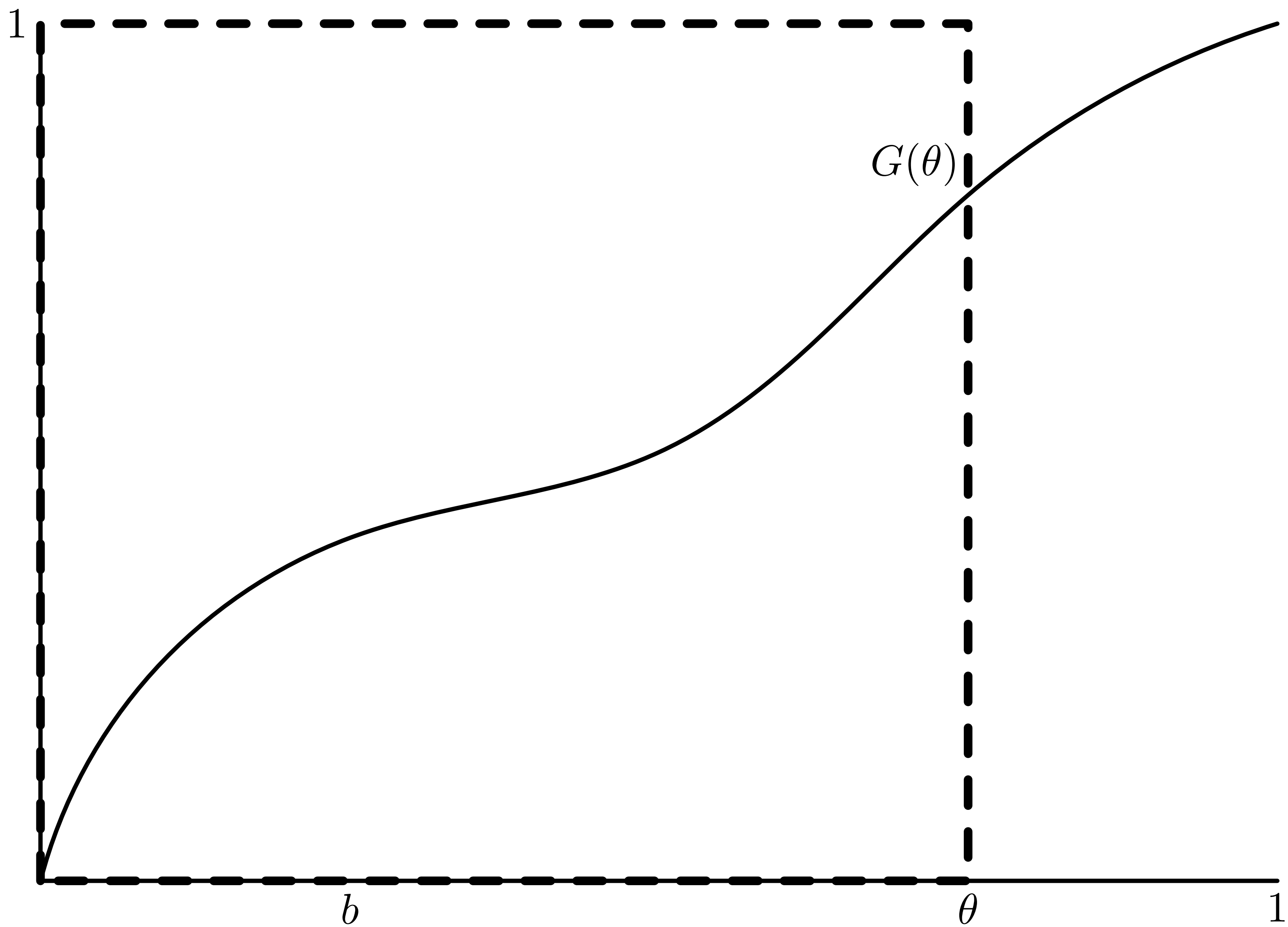
$$\int_0^1 \{1 - G(b)\}\, db$$

by the (hopefully) now familiar technique of integration by parts.

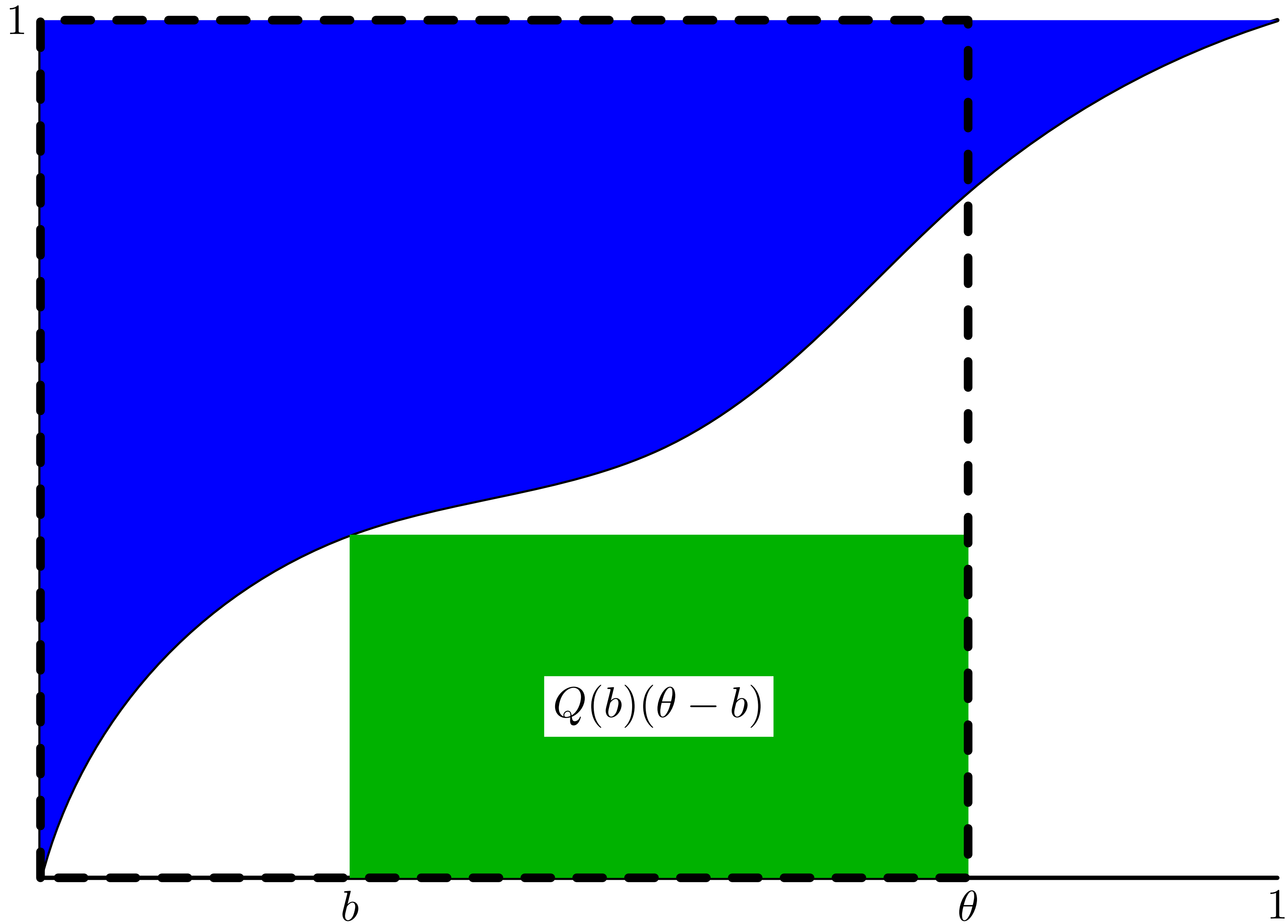
- what that means is that the expectation of the highest bid of the others is given by the area above the curve $G(b)$ as in the following figure


1
b
θ
1

- finally, the maximum possible surplus that could be earned by a bidder whose value is θ is given by the area of the following rectangle


1
$G(\theta)$
b
θ
1

- putting this all together gives a first bit of approximation


1
$Q(b)(\theta - b)$
b
θ
1

- it kind of looks like the area in blue plus the area in green together would have almost the same area as the dashed square that represents the buyer's potential surplus. This hints at the notion of approximation.

- if you wanted to prove that the blue area and the green area together were at least half the buyer's surplus, it wouldn't be so hard. Here is how you would do it

- $$Q(b)(\theta - b) + \hat{B} \geq$$

$$Q\left(\frac{\theta}{2}\right)\left(\theta - \frac{\theta}{2}\right) + \hat{B} =$$

$$\int_0^{\frac{\theta}{2}} \left\{\left(\theta - \frac{\theta}{2}\right) + b\right\} dG(b) +$$

$$\int_{\frac{\theta}{2}}^{1} b dG(b) \geq$$

$$\int_{0}^{\theta/2} \frac{\theta}{2} dG(b) + \int_{\theta/2}^{1} \frac{\theta}{2} dG(b) \geq \theta/2$$

- the first inequality follows from the fact that the distribution G is associated with a Bayesian Nash equilibrium. The equality that leads to the third line follows because it is a first price auction.

- Notice that this inequality doesn't depend on the underlying distributions of buyers' values, or what Bayesian equilibrium is being played.

- this is the method of approximation. The only complication is that the approximation isn't very good. It could obviously be tightened because the term that appears in the first integral is strictly larger than $\frac{\theta}{2}$. Since we want to use this in the real approximation we are looking for we can go a step further.

- **Theorem: (Hartline)** If a bidder has value θ and bids b as part of a Bayesian Nash equilibrium, then $Q(b)(\theta - b) + \hat{B} \geq \frac{e-1}{e}\theta$

- **Proof:** The expected payoff to the bidder in a Bayesian equilibrium is at least as large as her payoff when she uses any other strategy. Focus on the strategy that draws a random outcome from the interval $\left[0, \frac{e-1}{e}\theta\right]$ using a mixed strategy that uses a distribution with density $\frac{1}{\theta - \tilde{b}}$ (in other words, the probability with which the bidder bids something less than or equal to b using this strategy is $\int_0^b \frac{d\tilde{b}}{\theta-\tilde{b}}$. (Verify for yourself that $\int_0^{\frac{e-1}{e}\theta} \frac{d\tilde{b}}{\theta-\tilde{b}} = 1$ so that this is a proper density).
  Now following the logic above

$$Q(b)(\theta - b) + \hat{B} \geq$$

$$\int_0^{\frac{e-1}{e}\theta} \left\{ \int_0^b b \frac{1}{\theta - \tilde{b}} d\tilde{b} + \int_b^{\frac{e-1}{e}\theta} \left\{ \left(\theta - \tilde{b}\right) + b \right\} \frac{1}{\theta - \tilde{b}} d\tilde{b} \right\} dG(b) +$$

$$\int_{\frac{e-1}{e}\theta}^{1} bdG(b) =$$

$$\int_0^{\frac{e-1}{e}\theta} \left\{ \int_b^{\frac{e-1}{e}\theta} 1d\tilde{b} + b \right\} dG(b) + \int_{\frac{e-1}{e}\theta}^{1} bdG(b) =$$

$$\int_0^{\frac{e-1}{e}\theta} \left\{ \frac{e-1}{e}\theta \right\} dG(b) + \int_{\frac{e-1}{e}\theta}^{1} bdG(b) \geq$$

$$\frac{e-1}{e}\theta.$$

- this theorem provides a tool to get the result we are more interested in.

- Let $b_i(\theta)$ be an arbitrary bidding rule for bidder i in an auction where values are independently distributed according to the distribution functions $\{F_1, \ldots, F_n\}$. Suppose the F_i are all continuously differentiable, in particular so that no two θ_i can be the same with positive probability. Suppose the seller in a first price auction has zero cost. Suppose $\{q_i\}_{i=1,\ldots,n}$ is the allocation rule supported by awarding the good to the person who submits the highest bid. Then the expected surplus generated by the auction is

$$\int \cdots \int V(\theta)\, dF_1 \ldots dF_n = \int \cdots \int \sum_{i=1}^{n} q_i(\theta)\, \theta_i dF_1 \ldots dF_n$$

$$\equiv \int \cdots \int \sum_{i=1}^{n} q_i(\theta)\left(b_i(\theta_i) + (\theta_i - b_i(\theta_i))\right) dF_1 \ldots dF_n$$

$$= \int \cdots \int \sum_{i=1}^{n} q_i(\theta)\, b_i(\theta_i)\, dF_1 \ldots dF_n +$$

$$\int \cdots \int \sum_{i=1}^{n} Q_i(\theta_i)\left(\theta_i - b_i(\theta_i)\right) dF_1 \ldots dF_n$$

- In words, the expected surplus in an auction is always equal to expected revenues of the seller plus the sum of the expected payoffs of the buyers.

- Maximal expected surplus is

$$\max_{\{q_i\}_{i=1,n}} \int \cdots \int \sum_{i=1}^{n} q_i(\theta)\,\theta_i dF_1 \ldots dF_n$$

  subject to the constraint that $q_i(\theta_i) \in \mathbb{R}^+$; $\sum_{i=1}^{n} q_i(\theta_i) \leq 1$ and $q_i(\theta) = 1$ if $b_i(\theta_i) > b_j(\theta_j)\, \forall j \neq i$.

- Now we get the main approximation theorem.

- **Theorem** (Hartline) The expected surplus generated by the auction must be at least $\frac{e-1}{e}$ times the maximal expected surplus.

- **Proof:** We start with the result of the value approximation theorem given above

$$Q(b_i(\theta))(\theta_i - b_i(\theta)) + \hat{B}_i \geq \frac{e-1}{e}\theta_i$$

Suppose that $q_i^* (\theta)$ is the outcome function that maximizes expected surplus (obviously this outcome function is the one that gives the good to the buyer with the highest value - as you recall this isn't what the auction does).

- Trivially since $q_i^* (\theta)$is always less than or equal to 1, we must then also have (nothing deep, we are just shrinking one term on the left, but the whole thing on the right)

$$Q (b_i (\theta)) (\theta_i - b_i (\theta)) + \hat{B}_i q_i^* (\theta) \geq \frac{e-1}{e} \theta_i q_i^* (\theta) .$$

Now sum these terms over i and take expectations with respect to the θ_i.

$$\sum_{i=1}^{n} \int_0^1 Q (b_i (\theta_i)) (\theta_i - b) \, dF_i (\theta_i) +$$

$$\int \cdots \int \sum_{i=1}^{n} \hat{B}_i q_i^* (\theta) \, dF_1 \ldots dF_n \geq$$

$$\frac{e-1}{e}\int\cdots\int\sum_{i=1}^{n}\theta_i q_i^*\left(\theta\right)dF_1\ldots dF_n.$$

Notice that the first term on the left is the sum of the expected payoffs of the buyers, while the term on the right is the maximal expected surplus.
The last step comes from the following simple logic

$$\int\cdots\int\sum_{i=1}^{n}\hat{B}_i q_i^*\left(\theta\right)dF_1\ldots dF_n =$$

$$\int\cdots\int\sum_{i=1}^{n}\max_{\theta_j\neq\theta_i} b_j\left(\theta_j\right)q_i^*\left(\theta_i\right)dF_1\cdots dF_n \leq$$

$$\int\cdots\int\sum_{i=1}^{n}\max_{\theta_i} b_i\left(\theta_i\right)q_i^*\left(\theta_i\right)dF_1\cdots dF_n$$

where the last term is the expected revenue in the auction. The bottom line is that the sum of the expected payoffs to buyers plus the expected revenue to the seller is always at least $\frac{e-1}{e}$ of the maximal expected surplus.

4 Interdependence

- players types Θ_i aren't their values, they are just signals that provide information about value - assume types for each player are drawn from closed intervals.

- player $i's$ value for the object being sold in the auction is

$$v_i(\theta_1, \ldots, \theta_n)$$

- in case the buyer's value also depends on information possessed by the seller, let θ_0 be the seller's type, so that

$$v_i(\theta_0, \ldots, \theta_n)$$

- In either case, v_i is assumed to be non-decreasing in each of its arguments, and strictly increasing in θ_i

- the distribution of values is given by $F(\theta_1, \ldots, \theta_n)$ with continuous and differentiable density $f(\theta_1, \ldots, \theta_n)$

- an environment is symmetric if $v_i = v_j = u$ for all i and j and if the density is symmetric in the sense that if the vectors θ and θ' are permutations of one another, then $f(\theta) = f(\theta')$

- define
$$v(\theta_i, y) = \mathbb{E}\left\{u(\theta_i, \theta_{-i}) \,|\theta_i; \max_{j \neq i} \theta_j = y\right\}$$

- **Theorem:** In a second price sealed bid auction in a symmetric environment, there is a Bayesian Nash equilibrium in which all bidders use the the bidding rule $b(\theta_i) = v(\theta_i, \theta_i)$.

- **Proof:** The payoff a buyer gets when his type is θ and his bid is b' and all other bidders are using the rule $b(\theta_i) = v(\theta_i, \theta_i)$ is given by

$$\int_0^{b^{-1}(b')} (v(\theta_i, y) - b(y))\, g(y|\theta_i)\, dy$$

where $g(y|\theta_i)$ is the density of the maximum value of the θ_{-i}. Then by substitution, this is

$$\int_0^{b^{-1}(b')} (v(\theta_i, y) - v(y, y))\, g(y|\theta_i)\, dy.$$

- Since b is monotonically increasing in a symmetric environment, b^{-1} is monotonically increasing, so raising the bid results in the integral being taken over larger values of y. Again, using monotonicity, $y > \theta_i$ if and only if $v(y, y) > v(\theta_i, y)$, so, for example, if $b' > v(\theta_i, \theta_i)$ the intergral will include an interval along which $v(\theta_i, y) - v(y, y)$ is negative, and this interval could be eliminated by reducing the bid. Exactly the same argument shows that bids below are dominated.

- In a *button auction* (often referred to as an English auction) price starts at 0 and rises continuously.

- Each bidder begins the auction with his/her finger pressing down on a button. A bidder who takes their finger off the button is considered to be out of the auction and can no longer participate. Everyone observes when someone else takes their finger off the button and the price that prevailed when this event occurs.

- The auction ends when the second last bidder takes their finger off the button, at which point the price stops rising. The winning bidder is the remaining bidder who pays whatever the price was when it stopped going up.

- This is a dynamic game in which at every instant, a bidder who hasn't yet taken their finger off the button sees a sequence of prices $\hat{p}_k, \hat{p}_{k-1}, \ldots, \hat{p}_1$ at which the previous bidders dropped out of the auction. These are ordered so that $\hat{p}_k \geq \hat{p}_{k-1} \geq \ldots$ and so on. So the k^{th} bidder who dropped out of the auction did so at price $\hat{p}_k$.

- A strategy for a bidder is a plan about when to take their finger off the button. Think of the *bid* b as the price at which the bidder plans to take her finger off the button.

- Define the following bidding rules

$$b_0(\theta_i) = v(\theta_i, \theta_i, \ldots, \theta_i).$$

- what that means is that the bidder plans to keep her finger on the button as long as the price is less than $b_0(\theta_i)$ if no one drops out in the interim.

- since $v(\theta_i, \theta_i \ldots, \theta_i)$ is strictly increasing by assumption, $b_0(\theta_i)$ is strictly increasing, so it has an inverse b_0^{-1}.

- then, if the first bidder to drop out drops out at price $\hat{p}_1$, define

$$b_1(\theta_i, \{\hat{p}_1\}) = v\left(\theta_i, \theta_i, \ldots, b_0^{-1}(\hat{p}_1)\right)$$

  Here $v\left(\theta_i, \ldots, b_0^{-1}(\hat{p}_1)\right)$ means that the last of the n arguments in $v(\theta_i, \ldots, \theta_i)$ is replaced with $b_0^{-1}(\hat{p}_1)$.

- now using these two bidding rules, we can define the other bidding rules inductively.

- suppose that k bidders have dropped out at prices $\hat{p}_1 \leq \hat{p}_2 \leq \ldots \leq \hat{p}_k$, that we have defined the monotonically increasing bidding rule $b_k(\theta_i, \hat{p}_1, \ldots, \hat{p}_k)$, and that another bidder drops out at price $\hat{p}_{k+1}$. Then define

$$b_{k+1}(\theta_i, \hat{p}_1, \ldots \hat{p}_{k+1}) = v\left(\theta_i, \ldots, \theta_i, b_0^{-1}(\hat{p}_1), \ldots, b_{k-1}^{-1}(\hat{p}_k), b_k^{-1}(\hat{p}_{k+1})\right).$$

- **Theorem:** The bidding rules b^k defined above constitute a (symmetric) perfect Bayesian equilibrium for the english auction.

- **Proof:** Consider any history $(\hat{p}_1, \ldots, \hat{p}_k)$ in which bidder i still has her finger on the button, and suppose that all the other bidders are using the strategy $\{b_k\}$ as described above. Let y_k be the k^{th} lowest type among the types of the other players. Suppose the current price is p. Then the value of the good to the bidder with value θ_i is

$$\mathbb{E}\left\{v(\theta_i, \theta_{-i}) | y_1 = b_0^{-1}(\hat{p}_1), \ldots, y_k = b_k^{-1}(\hat{p}_k), y_{k+1} \geq b_k^{-1}(p), \ldots, y_{n-1} \geq b_k^{-1}(p)\right\} \geq$$

$$v\left(\theta_i, b_k^{-1}(p), \ldots, b_k^{-1}(p), b_0^{-1}(\hat{p}_1), \ldots, b_k^{-1}(\hat{p}_k)\right)$$

where in this expression the term $b_k^{-1}(p)$ appears $n-k-1$ times.

- if $\theta_i > b_k^{-1}(p)$, then the previous expression is larger than

$$v\left(b_k^{-1}(p), b_k^{-1}(p), \ldots, b_k^{-1}(p), b_0^{-1}(\hat{p}_1), \ldots, b_k^{-1}(\hat{p}_k)\right) = p$$

so the payoff to keeping her finger on the button is larger than dropping out and getting zero. Conversely, if $\theta_i < b_k^{-1}(p)$ then the expected value is less than the current price, and dropping out is a best reply.

4.1 Affiliation

- When the joint distribution of types has a density f, a special class of distributions can be described

- **Definition:** Variables $(\theta_1, \ldots, \theta_n)$ are said to be *affiliated* if for any pair of vectors θ and θ'

$$f(\theta \vee \theta') f(\theta \wedge \theta') \geq f(\theta) f(\theta')$$

where $\theta \vee \theta' \equiv \{\max[\theta_1, \theta_1'], \ldots, \max[\theta_n, \theta_n']\}$ and $\theta \wedge \theta' = \{\min[\theta_1, \theta_1'], \ldots,$