Competitive Equilibrium

Matching with Transferable Utility

Again imagine a bilateral matching problem. Unlike the matching problem with deferred acceptance, what we want to do here is to allow traders on each side of the market to use money to try to influence the match. A common convention is to assume that utility is measured in money so that if a buyer has to pay some price p to trade with a particular seller, then the price is just subtracted from the monetary value of trade to get the final payoff. This is referred to as *transferable utility.*

The we proceed just as with deferred acceptance. We think of a *match* as a pairing of two *traders.* One one side of this matching is is a set of individuals called I who act as buyers. Individuals in this set are indicated with lower case i. The other side is J is the set of sellers, with individuals identified as lower case j. Instead of working with *ordinal* preferences $\succeq_i$ or $\succeq_j$ as we did with deferred acceptance, we'll assume that buyer i is indifferent between trading with seller j and receiving $u_i(j)$ dollars. Similarly, in order for seller j to trade with buyer i, seller j must be paid $u_j(i)$ dollars.

So when i would rather match with j instead of j' we write

$$u_i(j) \geq u_i(j').$$

Similarly

$$v_j(i) \geq v_j(i')$$

means that absent any monetary payment j prefers to trade with i' instead of with i. That means it is more costly for j to trade with i than with j.

If i and j are matched, we say they *trade* with each other at a price p. If they do $i's$ payoff is

$$u_i(j) - p$$

while j's payoff is

$$p = v_j(i).$$

If either i or j decide they don't want to trade their payoff will be zero.

As with deferred acceptance, we'll define a *matching* as a function $\mu : I \cup J \to I \cup J$ satisfying

- for each $i \in I$
$$\mu(i) \in J \cup \emptyset;$$
- for each $j \in j$
$$\mu(j) \in I \cup \emptyset;$$
- if for some pair i and j, $\mu(i) = j \implies \mu(i') \neq j$ and $\mu(j) = i \implies \mu(j') \neq i$;
- for any pair (i, j) $\mu(i) = j \implies \mu(j) = i$.

This leads to the oldest matching method known to theory, a competitive equilibrium. In elementary textbooks, a competitive equilibrium is a price (or a relative price when there are two goods) that ensures that demand and supply are equal. In a matching problem, what each buyer and seller care about is the partner they receive, as in deferred acceptance. To make this look like what you have studied before, we can imagine that each buyer in I wants to sign a contract to work with a seller in J. Every seller is different and valued differently by each buyer.

Then we can set prices for these contracts. For example p_j is the price it will cost any buyer to hire a particular seller j. Once an agreement is reached the buyer pays the seller the price p_j and the seller provides whatever service they provide. What we want to do is to provide a set of prices $\{p_j\}_{j\in J}$ and then create a matching μ such that for every i for which $\mu(i) \neq \emptyset$

$$u_i(\mu(i)) - p_{\mu(i)} \geq \max\left[\max_j [u_i(j) - p_j], 0\right]$$

for all j and for each j such that $\mu(j) \neq \emptyset$. While for sellers

$$p_j - v_j(\mu(j)) \geq \max\left[\max_i [p_j - v_j(i)], 0\right].$$

for each j. A competitive equilibrium is a set of prices and a matching that satisfy these conditions.

If we could find this pair, then given the prices in a competitive equilibrium the outcome is *stable* since, conditional on prices every buyer and seller has their favorite partner already.

So far competitive equilibrium is not an algorithm like deferred acceptance, but a conceptual trading system. The people who devised the idea didn't know what an algorithm was. Yet that gave them an advantage because it forced them to defend their invention by thinking through what their system could accomplish if you could just find the right prices and the matching.

Properties. Suppose you manage to find a competitive equilibrium.

Proposition 1. *Every competitive equilibrium is Pareto optimal.*

Proof. Suppose not. Then there is some alternative matching, say ρ and prices q such that for some i

$$u_i(\rho(i)) - q_{\rho(i)} > u_i(\mu(i)) - p_{\mu(i)}.$$

Suppose that $\rho(i) = j$. In other words, j is i's match in the alternative allocation so that

$$q - v_j(i) \geq p_j - v_j(\mu(j))$$

Then

$$u_i(j) - q_j > u_i(\mu(i)) - p_{\mu(i)} \geq u_i(j) - p_j$$

which can only be true if $q_j < p_j$. Now looking from the perspective of seller j,

$$p_j - v_j(i) \leq p_j - v_j(\mu(j))$$

which gives

$$q_j - v_j(i) < p_j - v_j(i)$$

which is a contradiction. □

Now just add up the payoffs of all the pairs who are matched in a competitive equilibrium

$$\sum_{i\in I} u_i(\mu(i)) - p_{\mu(i)} + v_{\mu(i)}(i) + p_{\mu(i)} =$$

$$\sum_{i\in I} u_i(\mu(i)) + v_{\mu(i)}(i)$$

Proposition. *If the pair μ and p is a competitive equilibrium, then*

$$\sum_{i\in I} u_i(\mu(i)) - v_{\mu(i)}(i) \geq \sum_{i\in I} u_i(\rho(i)) - v_{\rho(i)}(i)$$

for any alternative matching ρ.

Proof. Again, we do the proof by contradiction. Suppose we find a matching ρ that produces a strictly higher sum, i.e.,

$$\sum_{i\in I} u_i(\rho(i)) - v_{\rho(i)}(i) > \sum_{i\in I} u_i(\mu(i)) - v_{\mu(i)}(i).$$

Now for each i let $\hat{t}_i = u_i(\rho(i)) - u_i(\mu(i))$ while for each j let $\hat{t}_j = v_j(\rho(j)) - v_j(\mu(j))$. Now we'll just require each i to make a payment

$$p_{\mu(i)} + \hat{t}_i$$

while each j receive a payment equal to

$$p_j + \hat{t}_j.$$

Then the payoff to any buyer in the new matching will be

$$u_i(\rho(i)) - p_{\mu(i)} - \hat{t}_i = u_i(\rho(i)) - p_{\mu(i)} - (u_i(\rho(i)) - u_i(\mu(i))) = u_i(\mu(i)) - p_{\mu(i)}$$

and for each seller

$$p_j + \hat{t}_j - v_j(\rho(j)) = p_j + (v_j(\rho(j)) - v_j(\mu(j))) - v_j(\rho(j)) = p_j - v(\mu(j))$$

so that with these required payments every buyer and seller ends up with the same payoff as they did with matching $\mu(j)$. Now if we sum up all the payments made by sellers we have

$$\sum_i p_{\mu(i)} + \hat{t}_i$$

while the total payments we have to give back to the sellers is

$$\sum_j p_j + \hat{t}_j.$$

The difference is

$$\sum_i \left(p_{\mu(i)} + \hat{t}_i\right) - \sum_j \left(p_j + \hat{t}_j\right) =$$

$$\sum_i \hat{t}_i - \sum_j \hat{t}_j =$$

$$\sum_i (u_i(\rho(i)) - u_i(\mu(i))) - \sum_j (v_j(\rho(j)) - v_j(\mu(j))) =$$

$$\sum_i \left(u_i(\rho(i)) - v_{\rho(i)}(i)\right) - \sum_i \left(u_i(\mu(i)) - v_{\mu(i)}(i)\right) > 0.$$

This additional revenue can be handed out to buyers and sellers making at least one of them strictly better off without hurting the others. This contradicts the presumption that μ, p is a competitive equilibrium since all such equilibrium must be Pareto optimal. □

So what do we learn from this? The two propositions seem somewhat 'tangential' to the main problem. The main question we need to answer is whether there is a competitive equilibrium at all, and, if there is, what is it and what are the prices.

To find stable allocations we used the deferred acceptance algorithm. For competitive equilibrium there are a number of ways to do it, but we'll use something called the Hungarian algorithm, which is discussed in the next section.