### Position Auctions

One application of auctions which has become increasingly important over the last few years is their application to online advertising. Hallerman\(^1\) estimates that online advertising revenues in the US in 2007 were at least $21 billion. (Varian 2006) estimates that revenues associated with the auction of search terms by industry leaders Google and Yahoo in 2005 were at least $11 billion. (Athey and Ellison 2007) provide a less specific and more conservative estimate of $10 billion. Either way, a very large chunk of total advertising revenue on the internet.

Auctions are used to determine which urls are displayed in the “Sponsored Links” column of the google search page, and the “Sponsored Results” column on Yahoo. These auctions are typically referred to as Position Auctions. The technology behind these is remarkable. When a search word is entered, a page is returned with a set of search results (sometimes referred to as ‘organic links’) along with a series of sponsored links. The primary difference between the two is that the sponsored links almost always offer to sell you something. The organic links may or may not involve sales. The sponsored links are useful provided they contain items the person looking at the page is likely to want to buy.

The way the urls are made relevant is to associate with each search term, a set of positions.\(^2\) The first position is the place at the top right of the page where the first sponsored link will be placed, the second position is the second highest link that will be displayed, and so on. Advertisers who are interested in a particular search term can then bid for that position in an auction. When you enter your search term, google checks for bids associated with that search term, then puts the link associated with the advertiser who submitted the highest bid in the first position\(^3\), the second

---


\(^2\)It is even more sophisticated than this since Google has quite a bit of information about the viewer who has entered the search term. When the viewer sends his search request to Google, his browser passes a cookie to the google server. The cookie doesn’t identify the viewer directly, but it provides an id that can be used to view a database that contains information about past searches that have been made from the same browser. By viewing these, Google can make a good guess about whether the viewer is male or female, rich or poor, young or old, etc. The ip adress of the browser can be used to determine where the viewer is located geographically. Finally, Google knows how likely it is that the viewer using the current browser will actually follow a sponsored link. Bids that advertizers submit can be conditioned on all these things.

\(^3\)This isn’t quite correct. In fact what they do is to give each advertizer a score based on how much revenue they expect to earn from that advertizer given the bid they have submitted. The winner of the auction is the advertizer whose bid generates the highest expected revenue given his score. We use the simpler high bid auction to make things a bit simpler below.
highest bidder in the second highest position, etc, before sending the page back to you. In other words, every Google page view results in a new auction being held. In this way, we are all involved in a large number of auctions.

The payment that a bidder makes depends on the number of viewers who actually click on the ad. To give some idea of what is involved, Google estimates that bidders who want to have their link displayed in the third position of the auction associated with the search phrase “luxury hotels in London” would have to pay about $3.04 per click.

Given the enormous number of times Google and Yahoo are used to conduct internet searches each day, even a tiny probability of clicking on a sponsored ad will result in large revenues for the search sites. Tweaking the design of the auction by modifying reserve prices and limiting search slots could make a lot of difference to total revenue. Squeezing revenue is not the only consideration that is important of course. The search engines also want to attract bidders and viewers who will click on their ads. We deal with some of these issues later in the chapter on competition among auctions. This chapter focuses on the case where the search site faces an exogenous set of advertisers bidding for positions whose value to them is also exogenous.

1. Complete Information

To begin, we will simply assume that there are $K$ positions which have associated with them exogenously given values $x_1 > x_2 > \cdots > x_K > 0$. The idea is that the slot in the first position is the one that consumers are most likely to click. There are two reasons for this. The first is simply that it is closest to the cursor which has just clicked the search box in the top right corner of the browser. A more compelling argument is a sort of self fulfilling prophecy that we will explain in more detail later. If consumers think that the most useful ads are in the first position, then firms will bid more aggressively for that position. However, the firms that are most likely to win the auction for that position are the ones who are most likely to make a sale. So at least when consumers want to buy, their expectation that the firms from whom they are most likely to buy will be in the first position are going to be correct. Interpret the $x_j$ as the expected number of clicks at position $j$.

There are $m > K$ firms bidding for the various positions on the web page. Firm $i$ has characteristic $v_i$ that describes its profit per click. For simplicity, assume that $v_1 \geq v_2 \geq \cdots \geq v_m$. The expected revenue of a firm who successfully places their ad in position $j$ is $v_i x_j$. Every firm wants the first position, but not all firms will be
willing to pay the same amount for it. All firms, on the other hand, still earn profits by placing their ads in lower positions on the web page.

The position auction works as follows: each firm places a bid \( b_i \). Let's order the bids so that \( b_1 \) is the highest bid, \( b_2 \) the second highest, and so on.

The first and most desirable position is awarded to the bidder who submits the highest bid \( b_1 \). He or she then pays the second highest bid \( b_2 \) for each click. Similarly, the bidder who submits the \( k^{th} \) highest bid wins the \( k^{th} \) position, and then pays the \( k+1^{st} \) highest bid for each click he receives. Let the bids be ordered from highest to lowest. If firm \( i \) wins the \( k^{th} \) best position (with a bid \( b_k \)) and pays \( b_{k+1} \), then his or her revenue per click is \( (v_i - b_{k+1}) \). Notice that the firm that wins the \( K^{th} \) position (i.e., the lowest or worst position) pays the highest bid of among bidders who did not win a slot. Given some array \( b_1, \ldots, b_m \) of bids, bidder \( i \)'s total payoff when he wins the \( k^{th} \) slot is

\[
(v_i - b_{k+1}) x_k,
\]

while his payoff is zero if he doesn’t win a slot.

As for any problem in which we want to use Nash equilibrium to predict the outcome, we need to specify the entire game. In other words, we have to describe the payoff a player earns for any array of bids, and for any bid that he might submit instead. This is a little complex here because we have to figure out what will happen when there are ties. The first complication we have to deal with is that we can’t simply order the bids \( b_1 > b_2 \) etc, we need to tie a bid to a bidder. The reason for this complication is that we want to allow a bidder \( i \) to consider submitting bids that are higher than all the others, higher than all but one of the others, lower than all the others, etc. The rank of a bidder \( i \)'s bid is something we are going to determine as part of our equilibrium.

The strategies of the players are just their bids. So let’s make the switch and imagine that the player whose name is \( i \) submits a bid \( b^i \). The rank that this bid has depends on the bids of the other bidders. So let’s write the bids submitted by all the players as \( b = \{b^1, \ldots, b^m\} \). So let’s write \( b_k (b) \) to mean the \( k^{th} \) highest bid in the array of bids \( b \). For example, \( b_1 (b) \) is the highest bid submitted by any player, \( b_2 (b) \) is the second highest bid in the array \( b \), and so on.

We could also write the bids \( b \) in a slightly different way. For example, we could write \( b^{-i} \) to mean the array of bids submitted by the players other than \( i \), then \( b_k (b^{-i}) \) represents the \( k^{th} \) highest bid of the bidders other than bidder \( i \). Also notice that
the vector \((b^i, b^{-i})\) is the same vector as \(b = (b^1, \ldots, b^{i-1}, b^i, \ldots, b^m)\) except that the elements have been re-ordered. So \(b_k(b) = b_k(b^i, b^{-i})\).

Now we can write the payoff to each player for every array of bids in a position auction where the \(k^{th}\) position is awarded to the \(k^{th}\) highest bidder who is then charged the \(k + 1^{st}\) highest bid for each click. I’ll take a shortcut here to make the formula somewhat simpler.

\[
\pi^i(b^i, b^{-i}) = \begin{cases} 
(v_i - b_1(b^{-i})) x_1 & b' > b_1(b^{-i}) \\
(v_i - b_k(b^{-i})) x_k & \exists k \leq K : b_{k-1}(b^{-i}) > b' > b_k(b^{-i}) \\
0 & \text{otherwise.}
\end{cases}
\]

This is a complicated expression, so well go over it. When a bidder considers a bid \(b'\), she first checks the bids of the other players (i.e., \(b^{-i}\)) and puts them in order \(\{b_1(b^{-i}), \ldots, b_{m-1}(b^{-i})\}\). She then looks to see if her bid \(b'\) is higher than one of the \(K\) highest bids that the others have submitted. For example, if there are four bidders (as there will be in the graphical illustration below), and they submit the bids \(\{2, 4, 3, 6\}\) (where 2 is submitted by the bidder with identity 1, 4 by bidder with identity 2, etc, then bidder 3 first takes the other bids \(\{2, 4, 6\}\) and puts them in descending order. This gives her the vector \(b_1(\{2, 4, 6\}) = 6, b_2(\{2, 4, 6\}) = 4\) and \(b_3(\{2, 4, 6\}) = 2\). Suppose there are \(K = 3\) positions to be won in the auction. She sees that there is a bid by bidder 1, i.e., 2, which is less than the third highest bid (her bid of 3). That means she will win the third highest spot, and pay \(b_3(\{2, 4, 6\}) = 2\) for it. Her profit is then the difference between her value and the price 2 multiplied by the average number of clicks associated with slot 3. The math expression above is certainly a more concise way of saying this. It even works when there are more bidders than slots once you think it through.

Now that we have the profit function (1.1), we basically know the payoffs that each player will get for every array of bids. That is like knowing what payoffs we put in each cell of the matrix in a bimatrix game. A Nash equilibrium of this game is an array of \(m\) bids such that no player can improve his payoff given the bids of the other players.

We can’t really do the gambit thing and check every profile of bids for profitable deviations. There is a continuum of profiles to worry about. Solving games like this usually means making a good guess about what the equilibrium looks like, then trying to verify your guess is an equilibrium. The educated guess here comes from the observation that if you win a slot at all, you won’t pay what you bid. Instead, you’ll pay what one of the other players bid. So your only real choice is to decide which slot, if
any, you want. The others’ bids completely determine what you pay when you win it.

The second part of the guess comes from the intuition that it is likely to be the case that the bidder with the highest value will want the best slot, the bidder with the second highest value will want the second best slot, and so on. Why? Well, at this stage in your reasoning you can’t really articulate why, but is just seems right. Nash equilibrium is going to give us a conceptual tool that will make it crystal clear why that intuition will work.

So let’s suppose that the bidder with the highest value per click submits the highest bid, the bidder with the second highest value per click submits the second highest bid, and so on. The auction would then award the top slot to the bidder with the highest value per click. The bidder who wins the $k^{th}$ highest slot would pay the $k^{th}$ highest bid among the other bidders as we described.

As in a second price auction, a bidder can’t change the price they pay for any particular slot, but they can change the slot that they win by outbidding someone else. For example, the bidder who wins the second highest slot could get the best slot by raising his or her bid until it is as large as the bid of the highest bidder. If the bidder decides to do this, then he or she could win the top slot, but to get it they would pay the bid of the high bidder for it according to the rules of the position auction.\footnote{Perhaps you can see why exactly matching the bid of the high bidder (a tie) doesn’t make sense. If the second highest bidder finds this deviation profitable, then instead of winning the top slot with probability $\frac{1}{2}$, they could do strictly better by adding a bit to their bid so that they are the highest bidder instead of one of the highest.}

So from the perspective of an individual bidder who thinks he knows the bids of the others, he really only needs to choose one of the positions whose ‘prices’ are all defined by the bids of the others. In particular, a set of bids will constitute a Nash equilibrium whenever

\begin{align}
(v_i - b_i (b_{-i})) x_i &\geq (v_i - b_j (b_{-i})) x_j \\
\text{for each } i \text{ who wins a slot and each } j; \text{ and in addition for any bidder } i \text{ who doesn’t win a slot}
\end{align}

\begin{align}
(v_i - b_j (b_{-i})) x_j &\leq 0.
\end{align}

This is still kind of complicated, so I’ll show how to construct an equilibrium using a diagram and a simple algorithm. We illustrate for the case where there are three slots.

The horizontal axis in the picture measures the possible values that firms can have for slots. The values of the top four firms are marked. The dashed line with the
Figure 1.1. Recursive Construction of Slot Prices

The lowest slope (the blue line in the picture) is the graph of the function

\[(v - v_4) x_3\]

which describes the payoff to different firm types when they buy slot 3 at price \(v_4\). It is flat because the slope of the line is \(x_3\), the click through rate of the worst available slot. The advantage of drawing this is that it makes it possible to see the profits the higher valued firms would make from this slot as well. Following our construction, we choose the price \(p_2\) for slot 2 that makes the bidder with value \(v_2\) just indifferent between purchasing slot 2 at price \(p_2\), and buying slot 3 at price \(v_4\). The dashed red line in the picture represents the payoff function \((v - p_2) x_2\). This line is steeper than the line for the third slot because its slope is equal to the higher click through rate \(x_2\). Since the click through rate \(x_2\) is higher than the click through rate for slot 3, \(p_2\) is going to have to be higher than \(v_4\). Since the payoff function given by the red line is steeper than the one associated with the blue line, firm 3 is going to prefer slot 3 to slot 2 when they are priced this way. The green line (the one with the steepest slope) shows how to construct the price \(p_1\) for the most valuable slot.

Now these prices \(v_4\), \(p_2\) and \(p_1\) satisfy the inequalities (1.3) and (1.2). To see why, look at the outcome for the bidder with value \(v_3\). He is supposed to win slot 3 at price \(v_4\), so the profit he gets from that can be read as the distance up to the blue dashed line. Similarly, he could try the other slots by paying \(p_2\) or \(p_1\), and you can read the consequences of these choices by measuring the distance up to the red and
green lines - at value $v_3$, up is down to those lines, which means that bidder 3 would lose money on these other slots.

On the other hand, bidder 2 is supposed to win slot 2 at price $p_2$. His profit is the distance up to the red line above $v_2$. He could win the lower slot at price $v_4$, but if he did, he would get the same profit because of the way we chose $p_2$. Again, if he tried to win the higher slot, his profit would be negative, which you can see by reading from the graph.

So the bids, $v_4, p_2, p_1$ and $v_1$ will support the slot prices that satisfy (1.3) and (1.2).

It isn’t hard to understand why the firms do this in this example. They never expect to have to pay what they bid, so there is no reason for them to worry about their bid, beyond the fact that it secures the slot they want. Firm 3, for example, knows in this equilibrium that he will only have to pay $v_4$ for the slot.

Second, observe that we can support many Nash equilibria this way. All we need to do is to make sure that each of the top three firms makes a profit on the slot they win, and that each prefers their slot to any other slot. In the picture, just slide the red line to the left to any position in which firm 3 prefers slot 3 to slot 2 (which just means the blue line has to be higher than the red line at $v_3$). Similarly, slide the green line to the left lowering $p_1$ to any position at which the blue line is higher than the green line at $v_2$. These changes will generate new Nash equilibrium with new prices for the different slots. These prices will be lower for slots 1 and 2.

On the other hand, we could go the other way and raise revenues that are generated by the position auction. We can’t just raise bidder 3’s bid this time, because if we do, bidder 2 will switch and want to bid on the worst slot instead of on the second best slot. We could raise revenues by increasing the bid by bidder 4 however. This may be one reason that google doesn’t explain exactly how they set their prices. The Nash equilibrium leaves some money on the table for them if they can set the price of the lowest slot to the value of the third highest bidder.

A second price auction of the kind we are studying here often has the feature that bidders will bid their ‘valuations’. This won’t necessarily support on equilibrium here as Figure 1.2 illustrates.

In this Figure, each bidder bids his own value, which would give prices $v_4$ for slot 3, $v_3$ for slot 2 and $v_2$ for slot 1. The it is easy to see from the figure, that at these prices, all three of the top bidders would prefer to win the third slot. The reason
is that the higher prices that are supported by the bidders values, aren’t warranted given the marginally higher click through rates associated with the better slots. We have no problem constructing an equilibrium for this problem, however, it won’t support an outcome where bidders bid their true values.

**Exercises.**

1. Can you give an algebraic condition (i.e., and inequality) that will ensure that there is a Nash equilibrium for this game in which each bidder bids their value?
2. Using the diagrammatic method, show how to compute prices for the case where the bidder with the fourth highest value bids $v_3$ instead of $v_4$.
3. Write down a constrained maximization problem that shows how google would maximize its ad revenues from this action by setting slot prices satisfying (1.3) and (1.2). Can you suggest an algorithmic method to solve this problem? Could you code it?

## 2. Consumer Search

Apart from the fact that firms bid more than a slot is truly worth to them in the position auctions, there are a number of characteristics of this model that seem
unrealistic. First, each firm knows exactly what the values of the other firms are. This is what allows them to confidently bid way more for a slot than it is worth to them. Perhaps more important, the click through rates that the firms are bidding on are unrelated to the characteristics of the firms that are actually making the bids. Most consumers will only click on sponsored ads if they feel that these ads will provide them useful information. Whether they do or not must surely depend on the characteristics they believe the bidding firms have. Secondly, there is a strong connection between click through rates and the position of the ad on the web page. Presumably this reflects the fact that consumers think that ads in a higher slot are more informative. It would be nice to know exactly why this might be.

As in the first section, we’ll assume there are lots of consumers. Consumers just want to buy a product. If they succeed in finding the product they want, they get a payoff of 1. Of course, not any product will do. If consumers follow a link to a firm with value $v$, we’ll assume that $v$ represents the same thing for them as it does for the firm - the probability that they will find the good they want. This means the expected payoff to consumers when they visit the website of a firm with quality $v$ is just $v$. What will be different this time is that we’ll assume that consumers bear some cost when they click on a sponsored link. There are two reasons this is a reasonable assumption. First, consumers interest in following a commercial link differ. Some consumers have a direct an immediate need, while others may be viewing search results for other reasons. A consumer whose primary interest is to locate a fact may find it very costly to follow a sponsored link that may or may not lead him to something that he can buy. Secondly, even consumers with immediate needs may find sponsored links costly. The link itself probably won’t directly provide the price and product information they want, causing them to have to search a specific website while they may be able to find the information they want more effectively by searching organic links.

Suppose the search cost for each consumer is given by $s_i$ which we assume is independently drawn from a distribution $G$ with support on $[0, 1]$. We suppose there are 3 firms bidding on two slots, the highest or top spot, and the lower spot. Firms are interested entirely in making a sale. Like consumers, they earn a payoff 1 when there is a sale. The firm’s quality is $v$ - the probability with which it makes a sale to any consumer who clicks on their ad. So $v$ also represents the firms’ value for a click. Lets assume that the value per click of each of three firms is independently drawn from some distribution $F$ with support on $[0, 1]$. The auction is exactly what it was in section 1 - each firm submits a bid for the slots. The highest bid wins the top slot and pays the second highest bid. The second
highest bid wins the lower slot and pays the third highest bid. The low bidder doesn’t get an advertising slot (or any clicks).

As before, the firms are interested in the click through rates $x_1$ and $x_2$ associated with each of the slots. Now we can actually find the click through rates as part of the Bayesian Nash equilibrium. If $p_1$ is the expected price to be paid by the winner of the top slot, and $p_2$ is the expected price to be paid by the winner of the lower slot, then a firm with value $v$ has payoff

$$\pi(v) = \begin{cases} 
  x_1(v - p_1) & \text{wins top slot} \\
  x_2(v - p_2) & \text{wins lower slot} \\
  0 & \text{otherwise.}
\end{cases}$$

Our job now is to try to figure out what $x_1$, $x_2$, $p_1$, and $p_2$ are. This model (which is based on (Athey and Ellison 2007) is relatively simple, but captures a number of interesting features. First, the value of the different slot positions to firms is endogenous since the click through rate will depend on how many consumers have search costs low enough to induce them to follow the link. The identity of these consumers is also endogenous since whether or not they want to follow a link will depend on how likely it is the link will provide them a transaction. If the firms who are most likely to be able to supply consumers (that is, firms with high $q$) think that consumers are clicking the top ad first, then they will be willing to bid more for the first slot. If they do, then consumers will be quite rational in thinking that they are more likely to find a transaction if they click the first link than if they click the second.

This model also resolves one of the issues associated with the position auction we discussed in the first part of this chapter, since firms won’t know exactly the values of their competitors. This helps make bidding strategies of the firms more reasonable.

2.1. Equilibrium. We’ll use two methods that we’ve used before. First we’ll assume that all three firms use the same bidding rule $b(v)$ and that this rule is monotonically increasing. Second, we’ll use the approach in which a firm deviates by bidding as if it has a different type instead of thinking of a different bid. As you’ll see, this will get us all the values we want.

If the bidding rule is increasing, then the probability that a firm with value $v$ wins the top slot is

$$F^2(v)$$

which is exactly the same as in the first and second price auctions we looked at previously. The probability that a firm with value $v$ wins the lower slot is

$$2F(v)(1 - F(v))$$
Notice that because we are using the assumption that the bidding rule is increase, we can say this without actually knowing what the equilibrium bidding rule is. With that said, the expected quality of the firm who wins the top slot is the expectation of the highest value which is

\[ V_1 \equiv \int_0^1 \tilde{v} F^2(\tilde{v}) f(\tilde{v}) d\tilde{v}. \]

Similarly, the expected quality of the firm who wins the second highest slot is

\[ \int_0^1 \tilde{v} F(\tilde{v}) (1 - F(\tilde{v})) f(\tilde{v}) d\tilde{v}. \]

It is completely intuitive that the expectation of the second highest value is going to be less than the expectation of the highest value. For example, if \( F \) happened to be uniform on \([0, 1]\), then these expectations would reduce to

\[ \int_0^1 \tilde{v} (\tilde{v}^2) d\tilde{v} = \frac{1}{4}. \]

The expected quality of the firm in the second slot is

\[ \int \tilde{v} \cdot 2\tilde{v} (1 - \tilde{v}) d\tilde{v} = \frac{1}{6}. \]

Since our consumers prefer firms with higher values, it now becomes clear why we click on the top link first. If consumers expect firms to bid for the top ad using a monotonic (in their value) bidding rule, which is our running assumption, then it is in their own interest to click on the top ad because that is where they are most likely to find the good they want.

Now we can immediately address an important question. Each consumer will click on the ad in the top position the value \( V_1 \) is larger than their search cost \( s \). Since \( s \) is distributed \( G \), the probability that the consumer clicks on the top ad is \( G(V_1) \), in other words, if we are right about the monotonic bidding strategy, the click through rate \( x_1 \) must be \( G(V_1) \).

Their search problem doesn’t end at this point, because there is still a positive probability of \( 1 - \int_0^1 \tilde{v} F^2(\tilde{v}) f(\tilde{v}) d\tilde{v} \) (\( \frac{3}{4} \) in the uniform case) that they won’t find what they want after clicking on the first ad. It is tempting at this point to jump to the conclusion that the click through rate on the lower slot would then be

\[ G \left( \int_0^1 \tilde{v} F^2(\tilde{v}) f(\tilde{v}) d\tilde{v} \right). \]

Unfortunately, life is not so simple. One of the great things that Bayesian equilibrium does for you is to re-enforce the idea that every time you see something, your beliefs will change a bit. This is what happens when you fail to trade with the seller who
wins the top slot. You don’t see his or her v, you just know that you didn’t trade. Yet if you couldn’t find something you wanted with the top seller, what are the chances with the second best seller - evidently it should be something smaller than

$$\int_0^1 \tilde{v} F^2 (\tilde{v}) f (\tilde{v}) d\tilde{v}$$

Now we can try to use some elementary probability theory. The probability that a consumer buys when she clicks on the first ad is $\int_0^1 \tilde{v} F^2 (\tilde{v}) f (\tilde{v}) d\tilde{v}$ as we discussed above. What we need to find is the expected quality of the firm with the second highest slot conditional on failing to trade with the firm at the first slot. Recall that the probability of an event $B$ conditional on an event $A$ is

$$\Pr (B|A) = \frac{\Pr (A \cap B)}{\Pr (A)}.$$ 

Think of $A$ as the event that the consumer fails to trade with the firm who won the first slot, and $B$ is the event where the seller who won the second slot has value $v$. so we can think of the event 'Fail to Trade with the top slot firm' as $A$ and 'Probability firm in the second highest spot has quality $v$' as $B$. Now we can trying filling out the information$^5$

$$\Pr (B|A) = \frac{\int_0^1 (1 - \tilde{v}) f (\tilde{v}) d\tilde{v} 2 F (v) (1 - F (v)) f (v)}{1 - \int_0^1 \tilde{v} F^2 (\tilde{v}) f (\tilde{v}) d\tilde{v}}$$

The numerator gives the probability (density) that a fails to trade with the firm who won the top slot, then trades with a seller of value $v$ in the lower slot. Dividing by the probability of the condition $A$ gives a conditional probability density that can then be used to compute the expected value to the firm in the lower slot conditional on failing with the firm in the top slot:

$$V_2 \equiv \int_0^1 v \frac{\int_0^v (1 - \tilde{v}) f (\tilde{v}) d\tilde{v} 2 F (v) (1 - F (v)) f (v)}{1 - \int_0^1 \tilde{v} F^2 (\tilde{v}) f (\tilde{v}) d\tilde{v}} dv$$

Again, notice that this doesn’t depend on the bidding rule $b$ as long as it is increasing.

So we can proceed to find $b$. First, lets use $V_2$ to compute the click through rate for the second slot. The probability that a consumer will click on the ad in the second slot is

$$G (V_1) (1 - V_1) \frac{G (V_2)}{G (V_1)} = (1 - V_1) G (V_2)$$

$^5$I should mention here that I am using $f (v)$, which is a density, as if it were a probability, which isn’t formally correct. Literally the formula here is for the density of the conditional distribution.
Notice that this is much lower than the expected quality of the second best firm at the beginning of the search process. Once again, this is because the failure to trade with the first firm suggests that the best firm has a relatively low quality. This makes the consumer more pessimistic about the quality of the firm available at the second link.

Now that we have click through rates for each of the slots, we can use exactly the same approach as we used for the standard auctions to try to figure out what the bidding rule is. The payoff to the firm whose value is \( v \) and who bids as if its value were \( v' \) is

\[
F^2(v') \int_0^{v'} G(V_1) \ast (v - b(v)) \frac{F(v) f(v)}{F^2(v')} d\tilde{v} + 2F(v') (1 - F(v')) \int_0^{v'} (1 - V_1)G(v_2) \ast (v - b(v)) \frac{f(v)}{2F(v') (1 - F(v'))} d\tilde{v} = \int_0^{v'} G(V_1) \ast (v - b(v)) F(\tilde{v}) f(\tilde{v}) d\tilde{v} + \int_0^{v'} ((1 - V_1)G(v_2) \ast (v - b(v))) f(\tilde{v}) d\tilde{v}.
\]

Now we can find the bidding rule by using the logic that if \( b \) is part of an equilibrium, then it should be optimal for each firm to bid as if their value were equal to their true value \( v \). The first order condition for maximization of \((2.1)\) with respect to \( v' \) is

\[
G(V_1) (v - b(v')) F(v') f(v') + ((1 - V_1)G(v_2) \ast (v - b(v'))) f(v') = 0
\]

which can only be satisfied if \( b(v) = v \).

Since firms bid their values, the price that the high bidder should expect to pay when its value is \( v \) is

\[
\int_0^{v} \tilde{v} 2F(\tilde{v}) f(\tilde{v}) d\tilde{v}
\]

which is what we called \( p_1 \) above. Notice that it depends on the high bidders value. For the expected price paid by the firm who wins the lower slot we have

\[
\int_0^{v} \tilde{v} f(\tilde{v}) d\tilde{v}.
\]

As we mentioned above

\[
x_1 = G(V_1)
\]

while

\[
x_2 = (1 - V_1)G(V_2).
\]
REFERENCES
