Limits of Exact Equilibria for Capacity Constrained Sellers with costly Search*

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Abstract

The paper contrasts the exact equilibria of games where sellers compete in price with the *rational expectations* equilibria of these games. It is shown that the distribution of prices offered by sellers in both the exact and rational expectations equilibrium converge weakly to the same limit as the number of buyers and sellers grows large. Furthermore, the payoffs that sellers face in both kinds of equilibrium have the *market utility property* in the limit.

1. Introduction

Models in which buyers face a price-probability of trade trade-off have found many applications recently in economics. One application has capacity constrained firms who charge high prices but offer more reliable service or a lower rationing probability in return.[1] Perhaps a more common application is to labor markets where

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[1]Originally, [14, 15]. More recent examples include McAfee [10], Denekere and Peck [5] or Burdett, Shi and Wright [3].

high wage firms attract bigger queues of applicants, so that workers who apply to such firms face a lower probability of landing a job, or a longer wait until an offer comes their way.[2]

These models provide a convenient device for modelling the general equilibrium effect of various shocks since they generate equilibrium price and wage distributions in a simple way. When contracts are complete, the equilibrium distribution of outcomes will generally be efficient in these models ([11] or [7, 6]). Even when contracts are incomplete, these models provide a convenient method for modelling the externalities that arise as a result because of the fact that they so readily account for the entire distribution of contract offers. Acemoglu and Shimer [1], for example, use this approach to explain why unemployment insurance might enhance efficiency with incomplete contracts. In their model, contracts are incomplete because firms cannot contract on the level of investment they make to increase the productivity of a match. In the absence of unemployment insurance, risk averse workers in their model are willing to trade off wages in order to attain higher employment probabilities in the sense described above. Firms respond by offering low wage jobs but suffer high vacancy rates. As a result the firms tend to under-invest, a problem that is mitigated by the introduction of unemployment insurance.

The analytical simplicity of these models arises from an implicit *competitive* assumption. Each seller believes that the level of utility that buyers get by selecting among sellers in the search equilibrium is independent of the price that the seller offers. A natural terminology for this is to say that the seller's payoff satisfies the *market utility property*. When this property holds, a seller who is trying to determine how buyers will respond to a price cut simply finds the new flow of customers such that all customers who choose the seller at his new price will get exactly the market level of utility.

This competitive assumption circumvents a thorny analytical problem. Sellers attract buyers by making specific price offers designed to influence their search behavior. So a low price is supposed to make it more likely that any particular buyer will choose to visit the firm. To calculate the profitability of any price offer, the seller needs to determine the continuation equilibrium in buyer search strategies that results from any particular offer that he makes. These continuation

[2] One of the earliest applications of this idea in labour economics is Montgomery [12]. The article by Moen [11] shows that equilibria with this kind of price competition is efficient, contrary to what occurs in standard wage search models. Other interesting applications include Shi [18], and Acemoglu and Shimer [1]. [16], , Denekere and Peck [5].

equilibria can be complex. In general, they do not have to be continuous (the basic problem is presented in [17]). But even when they are continuous, they do not induce well behaved profit functions for firms in the first stage of the game. The induced profit functions are not quasi-concave, meaning that pure strategy equilibria do not typically exist.[3] They certainly do not have simple characterizations.

These problems are all resolved by the *market utility property*. A full equilibrium occurs when sellers expectations about this level of utility are rational, in the sense that the level of utility that sellers believe that buyers receive is equal to the level of utility that buyers actually get in the continuation equilibrium associated with the sellers' best replies.[4] To be consistent with this terminology, we refer to this kind of equilibrium as a *rational expectations equilibrium* in what follows.

The usual justification for assuming the market utility property is that it should be approximately true provided the number of buyers and sellers participating on the market is very large. This approach is adopted in Shi [18] who assumes an infinite number of buyers and sellers. The infinite numbers approach is also discussed at some length in Burdett, Shi and Wright [3]. Equilibrium for the case where there are countably infinite numbers of buyers and sellers has been analyzed formally in [16]. There, the payoff that any seller receives against a fixed distribution of prices offered by the other sellers is defined by taking limits of the payoffs the seller gets against carefully chosen finite approximations of the same distributions. These payoffs are shown to have the market utility property.

Since the payoff functions for the countably infinite case are calculated by taking limits with respect to arbitrary distributions of competitor prices, these results provide little insight into the way that these limit equilibria are related to the exact subgame perfect equilibria that they are supposed to approximate. This paper is concerned with this issue. It is shown that the actual distribution of prices offered in any rational expectations equilibrium converges weakly to the distribution of prices in the limit equilibrium. This is not surprising, as sellers' payoffs have the market utility property in both solution concepts. A more surprising result is that the expected distribution of prices in the exact subgame

[3] McAfee [10] gives an example where two sellers compete for two buyers by offering auctions with variable reserve prices and shows that this simplest possible problem has no pure strategy equilibrium precisely because the sellers' profit functions are not concave.

[4] This technique has also been used with some effectiveness in the study of competition among mechanism designers. See McAfee [10], Peters [13]. This terminology as well as the characterization of equilibrium as a *rational expectations equilibrium* are both due to Gale [7, 6] though the ideas had been used much earlier, for example [4].

perfect equilibrium of large finite game converges weakly to the distribution of prices in the (unique) limit equilibrium. Furthermore the actual distribution of prices offered in the exact equilibrium will be close to the expected distribution with high probability[5]. A corollary of this latter result is that sellers' payoffs in the exact equilibria converge (pointwise) to payoff functions that satisfy the market utility property.

2. The Model

2.1. Basics

Consider an economy consisting of a finite number J of sellers and a finite number I of buyers. Each seller has a single unit of output to trade and each buyer wishes to purchase exactly one unit. The ratio of buyers to sellers is k. To begin, we consider the simplest possible case in which each buyer has a valuation 1. Sellers on the other hand, may have different costs. The cost of supply for seller j is given by c_j. If a buyer and seller trade at a price p, the seller gets payoff p and the buyer gets payoff $1 - p$. Buyers and sellers are risk neutral expected payoff maximizers.[6]

Prices are determined according to the following two stage process. First, sellers simultaneously and publicly announce the price at which they are willing to trade. Then, after observing all the price offers, each buyer simultaneously selects one and only one seller as a potential trading partner. Finally the seller randomly selects one of the buyers who have offered to trade with him and trades at the price that he announced at the beginning of the game.

The Perfect equilibria of this model are derived in the usual way by having sellers compute the symmetric continuation equilibria for the buyers' selection game that accompanies any particular price offer that they make, then computing their profits in this continuation equilibrium. An *exact equilibrium for the sellers' game* is simply a perfect equilibrium in which each seller's (possibly mixed) pricing strategy in the first stage is a best reply to the pricing strategies used by the

[5]The actual distribution is a random variable because of the fact that sellers will generally play mixed strategies in the exact equilibria.

[6]All of the arguments in this paper can be extended in various directions allowing firms to produce multiple units, and allowing buyers to purchase more than a single unit. The current formulation however can be more easily extended to the case where buyers have private information about their willingness to pay.

other sellers, given that each seller understands how his own prices affect the continuation equilibrium.

The analytically simpler alternative is to posit a market payoff function for buyers that is perceived to be independent of the actions of any single seller. In the *rational expectations equilibrium for the seller's game* each seller offers a price and then calculates the trading probability for each buyer type that yields that buyer type the *market* payoff. Sellers then choose the price probability combination that maximizes their expected profits. The true continuation equilibrium is then calculated relative to the prices that sellers have selected. A new market payoff for buyers is generated by this new continuation equilibrium. A market equilibrium is a fixed point of this mapping.

The rational expectations equilibrium is a 'subjective equilibrium' ([8]) for the first stage of the pricing game. In equilibrium, sellers' expectations about the payoff that buyers get from other firms is correct. However, any deviation from the equilibrium path will break this equality. For example, when there are a finite number of sellers, if one seller deviates to a higher price, he will drive some of his customers away. As they seek out new sellers they create an externality and lower buyer payoff with all sellers. The deviating seller does not realize this, and so cannot take advantage of it.

The question addressed in this paper is whether these approaches give similar answers when the number of buyers and sellers is large.

2.2. Exact Equilibria

Consider an array $p = \{p_1, \ldots p_J\}$ of price offers. Without loss of generality, suppose that they are ordered so that $p_1 \leq p_2 \leq \ldots p_J$. We focus henceforth on *symmetric* continuation equilibria and suppose that each buyer chooses a seller according to the same selection strategy $\pi = \{\pi_1, \ldots, \pi_J\}$ satisfying the restriction that $\sum_{j=1}^{J} \pi_j \leq 1$. The interpretation is that π_j is the probability with which each buyer selects seller j.

Under these conditions, the payoff that each buyer gets by choosing firm j is the same, and is given by

$$\sum_{n=0}^{I-1} \binom{I-1}{n} \pi_j^n (1-\pi_j)^{I-1-n} \frac{(1-p_j)}{n+1}$$

$$= (1-p_j) \frac{1-[1-\pi_j]^I}{I\pi_j} \qquad (2.1)$$

It is straightforward but tedious to verify that this is a decreasing function of π.

A *continuation equilibrium* is a vector of choice probabilities π^E satisfying

$$\pi^E \in \arg\max_{\pi} \sum_{i=1}^{J} \pi_j \left[(1-p_j) \frac{1-\left[1-\pi_j^E\right]^I}{I\pi_j^E} \right]$$

It is straightforward to show that the continuation equilibrium is unique except in the case where all firms offer the price 1. Assuming that buyers choose each seller with equal probability in this extreme event makes it possible (with a slight abuse of notation) to refer to the continuation equilibrium $\pi_j^E(p_j, p_{-j})$as a continuous function of seller's prices.

The profit attained by any seller whose cost is c and who offers a price p against the $J-1$ other offers p_{-j} is given by

$$\Pi_J^E(p, p_{-j}, c) \equiv (p-c) \left\{ 1 - \left\{ 1 - \pi_j^E(p, p_{-j}) \right\}^I \right\} \tag{2.2}$$

An exact equilibrium is a Nash equilibrium to the normal form game defined by (2.2). As the probability with which the seller trades varies continuously with the prices offered by all the firms, one should expect at least mixed strategy equilibria for this process. In fact this model is a special case of [14]. Thus we have

Theorem 2.1. *There exists an exact equilibrium (typically in mixed strategies) for the seller's game.*

2.3. Rational Expectations Equilibria

We turn now to an alternative approach that has proved easier to work with in practise. The idea is to begin with a level of utility that sellers expect buyers to receive from the market. When sellers adjust their price offers, they expect buyers to change their selection strategy in a way that maintains this expected payoff. This procedure generates an array of price offers from sellers all of which are best replies to the utility level that is available in the market. The continuation equilibrium relative to this set of offers generates the actual payoff that buyers receive. Equilibrium prevails when the payoff that sellers expect buyers to receive is equal to the payoff that they actually do receive when sellers choose their best replies.

The payoff that sellers expect is given by

$$\Pi_J^R(p, \pi, c) \equiv (p-c) \left\{ 1 - \{1-\pi\}^I \right\}$$

where π is the probability with which the seller thinks that buyers will choose him. In this profit function we maintain the assumption that the number of buyers is k times the number of sellers, so that we can continue to index the profit function using only the number of sellers. The payoff that buyers get from any seller charging the price p depends on the probability with which all the other buyers are expected to choose the seller and this is given as before by

$$V_J(p,\pi) \equiv (1-p)\frac{1-[1-\pi]^I}{I\pi}$$

In the competitive equilibrium sellers conjecture a market payoff to buyers equal to u and choose their prices to maximize Π^r subject to the constraint that $V_J(p,\pi) \geq u$ whenever it is possible for them to satisfy this constraint at some price above their cost. For simplicity we will assume that if $u \geq 1-c_j$ (so that sellers cannot satisfy the market payoff constraint profitably) then p and π are chosen to be c and 0 respectively.

A *rational expectations equilibrium* is an array of prices $p^R = \left\{p_1^R, \dots p_J^R\right\}$, choice probabilities $\pi^R = \left\{\pi_1^R, \dots \pi_J^R\right\}$ and a utility level u^R such that

(i) for each j, the pair $\left(p_j^R, \pi_j^R\right)$ maximizes $\Pi_J^R(p,\pi,c_j)$ subject to the constraints $\pi\left[V_J(p,\pi)-u\right] \geq 0; p \geq c_j$;

(ii) for all $m = 1, \dots J$

$$u^R = \sum_{i=1}^{J} \pi_j^R\left(p_j^R, p_{-j}^R\right)\left[V_J\left(p_j^R, \pi_j^R\left(p_j^R, p_{-j}^R\right)\right)\right] \geq V_J\left(p_m^R, \pi_m^R\left(p_m^R, p_{-m}^R\right)\right)$$

Notice that the second part of the condition ensures that the choice probabilities constitute a continuation equilibrium relative to the prices that the sellers offer on the equilibrium path. It is precisely in this sense that the sellers' expectations are correct in a rational expectations equilibrium. The fist condition requires that the seller maximize his profit relative to both his price and the probability with which he is chosen. The probability with which the seller expects to be chosen in this approach declines as his price rises much as the seller's quantity would fall with a price increase as buyers move a long a (compensated) demand curve.

The properties of this equilibrium are studied in [3]. The analytical convenience of this solution concept is demonstrated in the following result.

Theorem 2.2. *For each J suppose that there are at least two sellers whose costs c_j are less than 1. Then the rational expectations equilibrium exists and is unique.*

Proof. appendix ■

2.4. Limit Equilibrium

Finally, we consider the case where the set of buyers and sellers are both countably infinite. As in the finite case, it is assumed that there are k buyers for every seller. To see the appropriate limit theory, let $\mathcal{F}$ denote the set of probability distribution functions with support $[0,1] \subset \mathbf{R}$. Let $\mathcal{F}^J$ denote the subset of $\mathcal{F}$ consisting of all probability distributions satisfying the following two properties

(i) $\operatorname{supp} F$ is finite and $\# \operatorname{supp} F \leq J$; and

(ii) for all $x \in [0,1]$, $F(x) = j/J$ for some $j = 0, 1, \ldots J$.

The set $\mathcal{F}^J$ consists of all of the distribution functions that can be used as finite approximations to distributions in $\mathcal{F}$ in the sense that $\mathcal{F}^J$ is the set of all distribution functions delivered by price offers by J sellers. Endow $\mathcal{F}$ with the topology of weak convergence, so that each $\mathcal{F}^J$ is a compact subset of $\mathcal{F}$ which itself is compact.

For any distribution $F \in \mathcal{F}$ let $\tilde{p}_j(F,J) \equiv \inf\{p : F(p) = j/J\}$. Then $\tilde{p}(F,J) = \{\tilde{p}_1(F,J), \ldots, \tilde{p}_J(F,J)\}$ is the array of price offers at the J-tiles of the distribution F. Define $\tilde{\Pi}_J(p,F,c)$ to be the profit function faced by a firm who offers the price p against the J other price offers $\tilde{p}(F,J)$ assuming that his cost is c. We could alternatively write this as $\tilde{\Pi}_J\left(p, \tilde{F}^J, c\right)$ where $\tilde{F}^J \in \mathcal{F}^J$ is the finite approximation to the distribution F delivered by the prices $\tilde{p}(F,J)$. Now we have the following result.

Theorem 2.3. *For each distribution $F \in \mathcal{F}$, there exists a payoff u_F such that*

$$\lim_{J\to\infty} \tilde{\Pi}_J(p,F,c) \equiv \tilde{\Pi}(p,F,c) = \begin{cases} 0 & \text{if } 1-p < u_F \\ \left(1-e^{-k'}\right)(p-c) & \text{otherwise} \end{cases}$$

where k' is the solution to

$$\frac{1-e^{-k'}}{k'}(1-p) = u_F$$

u_F is the solution to

$$\int b(p,u)\,dF(p) = 1$$

and

$$b(p,u) = \begin{cases} 0 & \text{if } 1-p<u \\ b : \frac{1-e^{-bk}}{bk}(1-p) = u & \text{otherwise} \end{cases}$$

Proof. [13, Theorem 3,pp 254, Lemma 8,pp 261] ■

In the limit, this profit function is very simple. As a firm adjusts it's price offer p, it should expect the probability with which it trades to adjust in such a way that buyers continue to receive the market payoff u_F. In this sense the payoff function that sellers face converges pointwise in p and F to a function that satisfies the market utility property since deviations by a single firm do not change the market payoff u_F.

It is natural to use these limit results to *define* a payoff function in the limit game when sellers play against a distribution of prices by the other sellers. Suppose that there are a continuum of buyers and sellers. The measure of the set of sellers is 1 and the measure of the set of buyers is k. The distribution of sellers costs is given by G. The limit game for the sellers is then defined by the profit function $\tilde{\Pi}(p,F,c)$ defined in Theorem 2.3. An equilibrium for the limit game is given by a measurable function $p^{\infty}:[0,1]\to\mathbf{R}$ and a distribution F^{∞} of prices. The corresponding level of utility given by Theorem 2.3 is $u_{F^{\infty}}$. The function $p^{\infty}(c)$ is interpreted as the price offered by a seller whose cost is c. This pair must satisfy the two conditions

$$\tilde{\Pi}(p^{\infty}(c), F^{\infty}, c) \geq \tilde{\Pi}(p', F^{\infty}, c)$$

for all $p' \in [0,1]$; and

$$F^{\infty}(x) = G\left[(p^{\infty})^{-1}(x)\right]$$

for (Lebesque) almost all x, where $(p^{\infty})^{-1}(x) = \{c : p^{\infty}(c) \leq x\}$.

The first condition simply requires that each seller play a best reply against the distribution F^{∞} of prices offered by the other firms. The second condition requires that the proportion of sellers who offer prices less than or equal to x must be equal to the proportion of sellers whose best replies are less than or equal to x for almost all x.

The limit equilibrium has many desirable properties. The most useful is the following

Theorem 2.4. *The limit equilibrium exists, is unique, and generates a strictly positive level of expected utility for buyers provide G does not have all it's mass concentrated at the point 1.*

Proof. appendix ■

3. Convergence Results for Rational Expectations Equilibria

The rational expectations equilibria and the limit equilibrium are closely linked. The key to the relationship is the fact that sellers' payoff functions have the market utility property in both cases. Let $c_j^J \equiv \inf\{c : G(c) = j/J\}$ denote the costs for seller j in a market consisting of exactly J sellers. It is assumed here (and throughout the rest of the paper) that there is a point $x < 1$ such that $G(x) > 0$. In addition to the J sellers who participate in the finite version of this market, there are also kJ identical buyers. Let the triple $\left(p_J^R, \pi_J^R, u_J^R\right)$ denote the rational expectations equilibrium for this market. Since G assigns probability weight to costs below 1, there will always be at least two sellers who have costs below 1 provided J is large enough. Thus by Theorem 2.2, the rational expectations equilibrium will be unique provided J is sufficiently large.

Theorem 3.1. *Let u_J^R be any convergent sequence of rational expectations equilibrium payoffs with $\lim_{J\to\infty} u_J^R = u^0$. Let $F_J^R(\cdot)$ denote the distribution of prices generated by the rational expectations equilibrium prices p_J^R. Then $F_J^R(\cdot)$ converges weakly to F^∞ the limit equilibrium distribution of prices, and $u^0 = u_{F^\infty}$.*

4. Convergence Results for Exact Equilibria

One of the shortcomings of Theorem 2.3 is that a very special sequence of approximations to limit distribution of prices is used. The result can be extended so that a similar result applies when any sequence of approximating distributions is used, provided only that this sequence converges weakly to the distribution of interest. Define F^1 to be the degenerate distribution that assigns all of it's probability mass to the point 1.

Theorem 4.1. *Suppose that $F \neq F^1$. Let F^J be any sequence of approximating distributions from $\mathcal{F}^J$ that converges weakly to F. Then*

$$\lim_{J\to\infty} \tilde{\Pi}_J\left(p, F^J, c\right) = \begin{cases} 0 & \text{if } 1-p < u_F \\ \left(1-e^{-k'}\right)(p-c) & \text{otherwise} \end{cases}$$

where k' is the solution to

$$\frac{1-e^{-k'}}{k'}(1-p) = u_F$$

and u_F is the solution to

$$\int b(p,u)\, dF(p) = 1$$

where

$$b(p,u) = \begin{cases} 0 & \text{if } 1-p < u \\ b : \frac{1-e^{-bk}}{bk}(1-p) = u & \text{otherwise} \end{cases}$$

Proof. appendix

■

Now consider the normal form game defined by the profit functions $\Pi_J\left(p, p_{-j}, c_j^J\right)$ where $c_j^J \equiv \inf\{c : G(c) = j/J\}$. Interpret this as the profit function faced by the seller j who offers the price p against the array p_{-j} of J price offers made by his opponents. In other words this is the payoff function in a game consisting of $J+1$ players. The array of price offers p_{-j} generates a distribution $F^J \in \mathcal{F}^J$ and for any such distribution we can write $\Pi_J(p, p_{-j}, c) \equiv \tilde{\Pi}_J\left(p, F^J, c\right)$.

From Theorem 2.1 these games all have equilibria in mixed strategies. Let $\phi^J \equiv \left\{\phi_1^J, \ldots \phi_{J+1}^J\right\}$ be the equilibrium mixed strategy distributions for each of the sellers in the game consisting of $J+1$ sellers. The empirical distribution of prices generated by these mixed strategies in any play of the game is a random variable given by $\tilde{\Phi}^J$ with expectation $\frac{1}{J+1}\sum_{j=1}^{J+1}\phi_j^J$. Let $\psi_J(\cdot)$ denote the probability measure that this induces on $\mathcal{F}$ in the game consisting of $J+1$ players. The payoff that a seller of cost c faces in any such equilibrium can be written $\int \tilde{\Pi}_J\left(p, \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right)$.

Theorem 4.2. *Let $\left\{\phi^J\right\}$ be any sequence of equilibria having the property that $\frac{1}{J+1}\sum_{j=1}^{J+1}\phi_j^J$ converges weakly to some distribution $\tilde{F} \in \mathcal{F}$. Then for each $p \in [0,1]$, $\lim_{J\to\infty}\int \tilde{\Pi}_J\left(p, \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right) = \tilde{\Pi}\left(p, \tilde{F}, c\right)$.*

Proof. appendix ■

This means that the payoff function that sellers face in any exact subgame perfect equilibrium has the market utility property in the limit.

Theorem 4.3. *Let* $\left\{\phi_j^J\right\}$ *be any (sub)sequence such that* $\frac{1}{J+1}\sum_{j=1}^{J+1}\phi_j^J$ *converges weakly to* $\tilde{F}$. *Then there exists a function* $\tilde{p}(\cdot)$ *such that* $\left(\tilde{p}(\cdot),\tilde{F}\right)$ *is a limit equilibrium.*

Since the limit equilibrium is unique by Theorem 2.4, every weakly converging subsequence of $\frac{1}{J+1}\sum_{j=1}^{J+1}\phi_j^J$ must converge to the same limit F^{∞}.

5. Conclusion

The convergence results presented here indicate that the distribution of prices offered in a rational expectations equilibrium converge weakly to the distribution of prices in the exact equilibrium as the number of traders gets large.

Two assumptions seem important in deriving these results. First, it is assumed that firms compete in price. In the model presented here, the restriction to price is not important. One could imagine the price as a proxy for the level of utility that sellers offer to buyers when they provide them with more complex contracts. A model in which sellers compete in these utility levels would have identical properties. The more problematic part of this assumption is probably the fact that the price contract is complete in the sense that the buyer knows the level of utility he would receive from the contract without having to infer anything about the sellers behavior before the contract is struck (as in Acemoglu and Shimer [1]), or after (as would occur if there were moral hazard on the seller's part, or some kind of asymmetric information). It seems likely that the convergence results could be extended to the incomplete contracting case but the results presented here do not apply in that case.

A second important assumption is that all buyers are identical. Limit equilibrium concepts have been developed for the case where buyers differ and have private information (for example [13]). However, the properties of exact equilibria are much more difficult to pin down in this case. The key property of exact equilibria in this model is that seller's payoff functions are continuous. This guarantees the existence of equilibria in mixed strategies. The independence of sellers' actual prices when the use mixed pricing strategies is used to prove the convergence theorem. In the simplest case where sellers compete in auctions (and buyers have private information about their types) the continuation equilibrium is not typically continuous in the sellers' reserve price offers, so this same approach cannot be applied.

6. Appendix

6.1. Proof of Theorem 2.2

Proof. Let

$$\left[\phi_p^J(c,u),\phi_\pi^J(c,u)\right]\equiv\arg\max\left\{\Pi_J(p,c):\pi\left[V_J(p,\pi)-u\right]\geq 0;\pi=0\Rightarrow p=c\right\}$$

Quasi-concavity of the payoff functions ensures that this correspondence is single valued because of the fact that p is set equal to c whenever $\pi=0$ is optimal.

Lemma 6.1. $\left[\phi_p^J(c,u),\phi_\pi^J(c,u)\right]$ *is continuous. Furthermore,* $\phi_\pi^J(c,u)$ *is monotonically declining in* u *provided it is strictly positive.*

Proof. From

$$\Pi_J(p,c)=(p-c)\left\{1-\{1-\pi\}^I\right\}$$

and

$$V_J(p,\pi)=(1-p)\frac{1-[1-\pi]^I}{I\pi}$$

both functions are strictly quasi concave when $\pi>0$. Thus if $u<1-c_j$ the optimal solution is unique. When $u\geq 1-c_j$ the solution is unique by construction. Since both the objective and the constraint set are continuous in (c,u), $\left[\phi_p^J(c,u),\phi_\pi^J(c,u)\right]$ is continuous by the maximum principle. The first order conditions are necessary and sufficient by quasi-concavity. It is straightforward but tedious to verify monotonicity of $\phi_\pi^J(c,\cdot)$ from these conditions. ■

It is immediate that if we set $u=0$, then $\left[\phi_p^J(c,0),\phi_\pi^J(c,0)\right]=(1,1)$, while if $u\geq 1-c_j$ then $\left[\phi_p^J(c_j,u),\phi_\pi^J(c_j,u)\right]=(c,0)$. Otherwise $\phi_\pi^J(c,u)$ is continuous and monotonically declining in u by Lemma 6.1. A necessary and sufficient condition for equilibrium is that $\sum_{j=1}^J\phi_\pi^J(c_j,u)=1$. Since $\sum_{j=1}^J\phi_\pi^J(c_j,0)\geq 2$ (since there are at least two firms whose costs are below 1) and $\sum_{j=1}^J\phi_\pi^J(c_j,1)=0$, the existence and uniqueness of the rational expectations equilibrium follows from the mean value theorem. ■

6.2. Proof of Theorem 2.4

Proof. First consider the profit function defined in Theorem 2.3. Each seller wishes to maximize the function

$$(p - c)\left(1 - e^{-k}\right)$$

subject to the constraint that

$$(1 - p)\,\frac{1 - e^{-k}}{k} = u$$

if such a solution exists and $k = 0$ otherwise. If the optimal k for the firm is strictly positive, then maximization requires that the constraint hold with equality. This will be true if

$$p = \frac{\left(1 - e^{-k}\right) - ku}{\left(1 - e^{-k}\right)}$$

The the seller's problem is to choose k to maximize

$$(1 - c)\left(1 - e^{-k}\right) - ku$$

The first order condition for this problem is

$$(1 - c)\, e^{-k} = u$$

which gives the optimal solution for k strictly declining in u whenever $(1 - c) > u$, otherwise, the optimal solution for k is constant and equal to zero.

Let $k^*(c, u)$ denote this optimal choice for k for a seller of type c when the market payoff is u. In equilibrium, the average buyer seller ratio[7] must be k so

$$\int k^*(c, u)\, dG(c) = k$$

[7]To see this suppose there are a finite number of buyers and sellers with the ratio of buyers to sellers equal to k. π_j is the probability with which each of these buyers chooses each firm j and

$$\sum_{j=1}^{J} \pi_j \cdot g_j = 1$$

where g_j is the number of sellers of type j. Let $b_j/J \equiv \pi_j$. Then $k\pi_j = kb_j/J$ is the expected number of buyers who select on single firm of type j. Then multiplying both sides of the expression above by k gives

$$\sum_{j=1}^{J} kb_j \cdot \frac{g_j}{J} = k$$

From the previous argument, it is apparent that the integral on the left hand side of this equation has value ∞ when $u = 0$, has value 0 when $u = 1$ and is otherwise continuous and monotonically decreasing in u provided G does not have all it's mass concentrated at the point 1. This ensures that there is a unique utility level consistent with equilibrium. Since the seller's optimal price is unique for any level of market utility, this ensures a unique distribution of prices consistent with the limit equilibrium. ■

Proof of Theorem 3.1

Proof. Since u_J^R converges to u^0, then by construction $V_J\left(p, \pi_J^R\right) \to u^0$ or

$$\frac{1-\left(1-\pi_J^R\right)^{kJ}}{kJ\pi_J^R} \to \frac{u^0}{(1-p)}$$

for any seller who sets a price below 1. Define b_J to be the solution to $\pi_J^R = b_J/J$ so that

$$\frac{1-(1-b_J/J)^{kJ}}{kb_J} \to \frac{u^0}{(1-p)}$$

If b_J converges to $b \neq 0$, then the left hand side of this last expression converges to

$$\frac{1-e^{-kb}}{kb} > 0$$

This is monotonic in b. Since $\left\{u_J^R\right\}$ has a unique limit, b_J cannot have two distinct limits. If b_J diverges, then the left hand side goes to zero. Thus $\{b_J\}$ either has a non-zero limit, or it diverges. In either case the profit function

$$(p-c)\left[1-(1-b_J/J)^{kJ}\right] \to (p-c)\left(1-e^{-kb}\right) \tag{6.1}$$

pointwise, where b satisfies

$$\frac{1-\left(1-e^{-bk}\right)}{kb} = \frac{u^0}{(1-p)}$$

.

Rewriting the sum as an integral with respect to the distribution defined by g_j/J gives

$$\int k'(c)\, dG(c) = k$$

It remains to show that this limit coincides with $\tilde{\Pi}(p, F^{\infty}, c)$. To see this, consider the distribution of prices in the rational expectations equilibrium. It is given by

$$F_J^R(p) =$$
$$\#\left\{c_j^J : \arg\max_{p'}\left\{\left(p - c_j^J\right)\left[1 - (1 - \pi_J)^{kJ}\right]; V_J(p', \pi_J) \geq u_J^R\right\} \leq p\right\}/J \leq$$
$$G\left\{c : \arg\max_{p'}\left\{(p - c)\left[1 - (1 - \pi_J)^{kJ}\right]; V_J(p', \pi_J) \geq u_J^R\right\} \leq p\right\}$$

The inequality follows from the fact that $c_j^J \equiv \inf\{c : G(c) = j/J\}$. Now define $\tilde{p}(c) \equiv \arg\max_{p'}\left\{(p - c)\left[1 - e^{-k'}\right]; (1 - p)\frac{1 - e^{-k'}}{k'} \geq u^0\right\}$. Taking the lim sup of both sides of the last inequality and using (6.1) gives

$$\limsup F_J^R(p) \leq G\left(\tilde{p}^{-1}(p)\right)$$

so that $F_J^R(p)$ converges weakly to the distribution on the right hand side of this inequality. Together $\tilde{p}(c)$ and $G(\tilde{p}^{-1}(p))$ constitute a limit equilibrium. Since the limit equilibrium is unique by Theorem 2.4, we must have $G(\tilde{p}^{-1}(p)) = F^{\infty}(p)$. ■

6.3. Proof of Theorem 4.1

Proof. This Theorem is proved in three steps. First, we rewrite the probability with which buyers choose any particular seller as a function $h_J\left(\frac{u}{1-p}\right)/J$ of the level of utility that buyers get in equilibrium as a function of the number of sellers (the number of buyers is always k times the number of sellers. Step 1 is to show that the family of functions $\{h_J\}$ is equi-continuous.. Step 2 is to show that whenever the sequence $\left\{F^J\right\}$ converges weakly, the payoff u^J that buyers get in the resulting continuation equilibrium converges. Finally in step 3, convergence of u^J and equi-continuity together imply joint continuity of the seller's payoff function.

Rewrite the payoff to a buyer who selects seller j as

$$(1 - p_j)\frac{\left[1 - (1 - b_j/J)^{kJ}\right]}{b_j k}$$

Now fix a payoff u. The variable b_j should satisfy the condition

$$\frac{\left[1 - (1 - b_j/J)^{kJ}\right]}{b_j k} = \frac{u}{1 - p_j}$$

whenever that is possible. Consider the family of functions $\{g_J(\cdot)\}$ from $[0, J]$ to $\mathbf{R}$ defined by the left hand side of this equation. Every function in this family is bounded (by 0 and 1) continuous and monotonic. Together these imply that the functions have continuous monotonic inverse functions when the domains of the inverse functions are restricted to $[1/Jk, 1]$. Let h_J denote the function whose range is $[0, \infty]$ that is defined by extending g_J^{-1} as follows

$$h_J(x) = \begin{cases} g_J^{-1}(x) & \text{if } x \in [1/kJ, 1] \\ 0 & \text{if } x > 1 \\ J & \text{if } x < 1/kJ \end{cases}$$

The function $h_J\left(\frac{u}{1-p_j}\right)$ gives the value of b (if there is one) that yields buyers the expected payoff u whenever the price p_j is smaller than $1-u$. It gives 0 if $u > 1-p$ and it gives J if $(1-p_j)/kJ > u$.

Lemma 6.2. *(i) the family of functions $\{h_J(x)\}$ is equi-continuous at each $x > 0$; and*

(ii) For any fixed $u^0 > 0$, the family of functions $\left\{h_J\left(\frac{u^0}{1-p}\right)\right\}$ mapping price $p \in [0,1]$ into $\mathbf{R}$ is equi-continuous..

Proof. The derivative of g_J is given by

$$\frac{1}{b}\left[(1-b/J)^{kJ-1} - \frac{\left[1-(1-b/J)^{kJ}\right]}{bk}\right] =$$

It is straightforward to show that this derivative is equal to -1 at $b=0$; $-1/J^2$ at $b = J$, and is otherwise decreasing in J. Thus the inverse function $h_J(\cdot)$ from $[0,\infty] \to \mathbf{R}$ is differentiable and has a uniformly bounded derivative on any interval that excludes the point 0. This implies that for any strictly positive pair of numbers $x' > x$, $|h_J(x) - h_J(x')| = \int_x^{x'} h_J'(s)\,ds \leq (x'-x)\triangle$ for some bound $\triangle$. This is sufficient to ensure that the family $\{h_J\}$ of functions from $[0,\infty] \to \mathbf{R}$ is equi-continuous on any compact subset of $[0,\infty]$ that excludes the point 0.

For any payoff $u^0 > 0$, and price $p^0 \leq 1-u^0$, let $x^0 \equiv \frac{u^0}{1-p^0}$. Take $\varepsilon > 0$. By the equi-continuity of the functions h_J when the argument is different from 0, there is a $\delta > 0$ such that $|x'-x^0| < \delta$ implies $|h_J(x') - h_J(x^0)| < \varepsilon$ for all J. By the continuity of the function $\frac{u}{1-p}$ at prices different from 1, there is a $\gamma > 0$ such that $|p'-p^0| < \gamma$ implies $\left|\frac{u}{1-p'} - \frac{u}{1-p^0}\right| < \delta$. Together this implies that for any $\varepsilon > 0$ there is a $\gamma > 0$ such that $|p' - p^0| < \gamma \Rightarrow \left| h_J\left(\frac{u^0}{1-p'}\right) - h_J\left(\frac{u^0}{1-p^0}\right)\right| < \varepsilon$ for all J. Equi-continuity of h_J at prices above $1 - u^0$ follows trivially from the fact that the function h_J is constant and equal to zero for such prices. Since the family of functions h_J is equi-continuous at each value for p, it is equi-continuous in p. ■

The function $h_J(x)$ gives the value for b that satisfies

$$\frac{\left[1 - (1 - b/J)^{kJ}\right]}{bk} = x$$

Since b/J is the probability with which buyers choose the seller in question, and since these probabilities must sum to one, the equilibrium level of utility is given by the solution to the equation

$$\sum_{j=1}^{J} h_J\left(\frac{u}{1-p_j}\right) \cdot \frac{1}{J} = 1$$

Since the array of prices $\{p_j\}$ delivers a distribution of prices in $\mathcal{F}^J$, and since each distribution function in $\mathcal{F}^J$ can be thought of as giving probability mass $1/J$ at each of the J different points in its support, this condition can be rewritten as

$$\int h_J\left(\frac{u}{1-p}\right) dF_J(p) = 1 \tag{6.2}$$

Lemma 6.3. *Let F_J be any sequence converging weakly to some distribution F. Suppose that $F \neq F^1$. Then the sequence of solutions u_J^* to equation (6.2) converges to the solution u^* to the equation*

$$\int h\left(\frac{u}{1-p}\right) dF(p) = 1$$

Proof. Let $F^+(1)$ denote the size of the atom (if there is one) at the point 1 associated with the distribution F. Since $F \neq F^1$, $F^+(1) = \alpha < 1$. By weak convergence $\limsup_J F_J^+(1) \leq \alpha$. Then for any $\varepsilon > 0$ there is some J^* such that for all $J > J^*$, $1 - F_J^+(1) > 1 - \alpha - \varepsilon$. In other words, since F_J is a distribution with a finite support, there is some J large enough so that the number of sellers who offer prices strictly less than 1 in the distribution F_J is at least $J(1 - \alpha - \varepsilon)$.

Observe that $h_J(0) = J$. It follows that for $J > J^*$

$$\int h_J\left(\frac{0}{1-p}\right) dF_J(p) = \sum_{j=1}^{J} h_J\left(\frac{0}{1-p_j}\right)\frac{1}{J} \geq \frac{2J}{J} = 2$$

On the other hand $h_J\left(\frac{1}{1-p}\right) = 0$. So $\int h\left(\frac{1}{1-p}\right) dF(p) = 0$ is a continuous non-increasing function with $\int h_J\left(\frac{0}{1-p}\right) dF_J(p) > 1$ and $\int h\left(\frac{1}{1-p}\right) dF(p) < 1$. Since h_J is monotonically decreasing whenever it is strictly positive, it follows that for large enough J there is a unique solution $u_J^* > 0$ to 6.2.

Now choose any converging subsequence of $\{u_J^*\}$ and suppose $\lim_{J\to\infty} u_J^* = u^0$. As all the u_J^* are strictly positive, we can consider a closed subset U of $[0,1]$ that does not contain 0 and restrict $u_J^* \in U$ for all J large enough. Consider the family of functions $\left\{h_J\left(\frac{u}{1-p}\right) : J = 1, \ldots \infty; u \in U\right\}$. The family h_J is equi-continuous on any compact set excluding 0 and the function $\frac{u}{1-p}$ is uniformly continuous on U, hence this family is equi-continuous.. It follows by Ascoli's theorem that the sequence of functions $h_J\left(\frac{u_J^*}{1-p}\right)$ converges uniformly to $h\left(\frac{u^0}{1-p}\right)$. Since F_J converges weakly to F by assumption and $h_J\left(\frac{u_J^*}{1-p}\right)$ converges uniformly, $\int h_J\left(\frac{u_J^*}{1-p}\right) dF_J(p)$ converges to $\int h\left(\frac{u^0}{1-p}\right) dF(p)$ by [2, Theorem 5.2].

Suppose now that $\lim_{J\to\infty} u_J^* = u^0 \neq u^*$ for some converging subsequence. Since U is closed, $u^0 \in U$ which implies that $u^0 > 0$. Define

$$\delta = \left|1 - \int h\left(\frac{u^0}{1-p}\right) dF(p)\right| > 0$$

δ is strictly positive by the assumption that $u^0 \neq u^*$. By the equi-continuity of the h_J there is, for any $\varepsilon > 0$, a δ such that for any $x > 0$, $|x' - x| < \delta \Rightarrow |h_J(x') - h_J(x)| < \varepsilon$ for all J. By the continuity of $\frac{u}{1-p}$ there is for any δ a γ such that $|u' - u| < \gamma \Rightarrow \left|\frac{u'}{1-p} - \frac{u^0}{1-p}\right| < \delta$ for any pair $(u, 1-p)$ with $u > 0$. Together this implies that for any $\varepsilon > 0$ there is a γ such that $|u' - u^0| < \gamma \Rightarrow \left|h_J\left(\frac{u'}{1-p}\right) - h_J\left(\frac{u^0}{1-p}\right)\right| < \varepsilon$ for all J and for all p. Choose J large enough that $|u_J - u^0| < \gamma$. Then

$$\varepsilon > \int \left|h_J\left(\frac{u_J}{1-p}\right) - h_J\left(\frac{u^0}{1-p}\right)\right| dF^J(p) \geq$$

$$\int h_J\left(\frac{u_J}{1-p}\right) dF^J(p) - \int h_J\left(\frac{u^0}{1-p}\right) dF^J(p)$$

Since ε is arbitrary, this implies that there is some J such that

$$\int h_J \left(\frac{u_J}{1-p} \right) dF^J (p) \neq 1$$

a contradiction.

Corollary 6.4. *For each price p*

$$\lim_{J \to \infty} h_J \left(\frac{u_J^*}{1-p} \right) = h \left(\frac{u^*}{1-p} \right)$$

Proof. Joint continuity follows from the equi-continuity of the family $\left\{ h_J \left(\frac{\cdot}{1-p} \right) \right\}$ and the convergence of u_J^*. Recall that $h_J (\infty) = 0$. (for example [9]) ■

This final corollary gives

$$\lim_{J \to \infty} \tilde{\Pi} \left(p, F^J, c \right) = \lim_{J \to \infty} (p-c) \left[1 - \left(1 - \frac{h_J \left(\frac{u_J^*}{1-p} \right)}{J} \right)^{kJ} \right] =$$

$$(p-c) \left(1 - e^{-h\left(\frac{u^*}{1-p} \right) k} \right)$$

which proves theorem 4.1. ■

6.4. Proof of Theorem 4.2

Proof. The prices offered by firms form a triangular system of row-wise independent random variables. Thus sup $\left| \Phi^J (x) - \frac{1}{J+1} \sum_{j=1}^{J+1} \phi_j^J (x) \right|$ converges almost surely to 0 by the Glivenko-Cantelli theorem [19, Theorem 1, p 106]. Almost sure convergence implies that the probability measure ψ_J converges weakly to a measure that assigns point mass to the distribution $\tilde{F}$.

Next, we show that $\tilde{F}$ cannot assign point mass to the price 1. Let $p < 1$ and suppose to the contrary that Φ^J converges almost surely (in the sup norm) to a point mass at 1. Then $a.s.$ $u_J^* \to 0$ and $h_J \left(\frac{u_J^*}{1-p} \right) / J \to 1$. This implies that $\tilde{\Pi}_J \left(p, \tilde{F}_J, c \right) \to^{as} p - c$. On the other hand, since not every firm have a better than average chance of trading, there must be at least one firm whose ex ante profits have a limit below $(1-c) \left(1 - e^{-k} \right) < p - c$ for p close enough to one. This implies the existence of a profitable deviation.

Now we have from Theorem 4.1 that $\tilde{\Pi}_J\left(p, \tilde{F}_J, c\right) \to \tilde{\Pi}\left(p, \tilde{F}, c\right)$ for any sequence $\tilde{F}_J \to F$ as long as F does not have point mass at 1. The distribution $\psi_J(\cdot)$ converges weakly to a degenerate distribution ψ having the property that $\psi(F^1) = 0$. Thus

$$\lim_{J\to\infty} \int \tilde{\Pi}_J\left(p, \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right) = \tilde{\Pi}\left(p, \tilde{F}, c\right)$$

by[2, Theorem 5.5, p 63] ■

6.5. Proof of Theorem 4.3

Proof. Let $\tilde{p}(c)$ be the price that maximizes the function $\tilde{\Pi}\left(p, \tilde{F}, c\right)$. From the definition and the fact that the level of utility $u\left(\tilde{F}\right)$ that buyers receive in the limit is independent of the price that the seller sets, it is straightforward to show that $\tilde{p}(c)$ is a continuous monotonically increasing function.

Now choose a cost level c and any price $p' \neq \tilde{p}(c)$. Since $\tilde{p}(c)$ is a function $\tilde{\Pi}\left(p', \tilde{F}, c\right) < \tilde{\Pi}\left(p(c), \tilde{F}, c\right)$. Furthermore by Theorem 4.2,

$$\lim_{J\to\infty} \int \tilde{\Pi}_J\left(p', \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right) = \tilde{\Pi}\left(p', \tilde{F}, c\right)$$

and

$$\lim_{J\to\infty} \int \tilde{\Pi}_J\left(p(c), \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right) = \tilde{\Pi}\left(p(c), \tilde{F}, c\right)$$

which implies that for some J large enough $\int \tilde{\Pi}_J\left(p', \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right) < \int \tilde{\Pi}_J\left(p(c), \tilde{F}_J, c\right) d\psi_J\left(\tilde{F}_J\right)$. This implies that p' does not lie in the support of this seller's equilibrium strategy for large enough J.

Now since $\tilde{p}(c)$ is continuous and monotonic, almost surely $\lim_J \tilde{\Phi}^J(p') = G\{c : \tilde{p}(c) \leq p'\}$, for if not, then some measurable group of sellers cannot be using their best replies when J is large. This implies that $\tilde{F}(p') = G\{c : p(c) \leq p'\}$ which verifies that $\left\{\tilde{F}, p(\cdot)\right\}$ is an equilibrium. ■

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