An Ascending Double Auction

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Abstract

We show why the failure of the affiliation assumption prevents the double auction from achieving efficient outcomes when values are interdependent. This motivates the study of an ascending price version of the double auction. It is shown that when there is a sufficiently large, but still finite, number of sellers, this mechanism has an *approximate* perfect Bayesian equilibrium in which traders continue bidding if and only if their true estimates of the 'value' of the object being traded exceed the current price. This equilibrium is ex post efficient and has a *rational expectations property* in the sense that along the equilibrium path traders appear to have made the best possible trades conditional on information revealed by the trading process.

1 Introduction

This paper studies a *dynamic ascending-price* double auction mechanism in an interdependent value environment, and shows that this mechanism supports allocation that become ex post efficient as the number of traders gets large. Rustichini, Satterthwaite, and Williams (1994) have shown that in private value environments, an equilibrium of a standard (one-shot) double auction quickly converges to an efficient allocation as the number of traders get large (see also Gresik and Satterthwaite (1989)). Recently, Perry and Reny (2002), continuing the research agenda of Milgrom (1981) and Pesendorfer and Swinkels (1997) and (2000), have shown that, in some interdependent value environments where trader types are affiliated, double auction supports an equilibrium price that converges in probability to the

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full information market clearing price as the number of traders gets large. In particular, this implies that, when the number of traders is large, the double auction realizes almost all gains from trade with a high probability and provides strategic foundation for *rational expectations equilibrium*.

It is quite surprising that the static double auction (where each trader submits only one bid or ask) supports anything close to the efficient competitive allocation, because such mechanism imposes strong restrictions on the way in which individual trading decisions depend on other traders' types. To see this, observe that a bid or an ask in a double auction is, in fact, a contingent trading plan. For example, a seller's ask is a plan that says that the seller wants to sell if the value of the appropriate order statistic of the other traders' bids and asks is above his ask price. This is a very restricted contingent plan since the trading decision can only depend on the realization of the said order statistic of the bids and asks, but not on the distribution of bids and asks below that order statistic. Furthermore, a seller has to sell *whenever* the value of this order statistic exceeds his ask price. These restrictions on the traders' contingent trading plans are very natural in the private value case, but are very restrictive in the interdependent value case, as we illustrate with an example below.

This motivates us to study an ascending price version of the double auction. Our mechanism works as follows. Initially, the price is set sufficiently low that all buyers would want to buy at that price and no seller would want to sell. At this initial price, all traders simultaneously declare whether they wish to continue bidding. All traders who wish to continue bidding are said to be active at the initial price. Traders who declare that they do not want to continue bidding are considered to be *inactive* bidders who have dropped out of the bidding at the initial price. If the number of active bidders exceeds the number of units for sale, the price is increased according to the procedure that will be described below, and the process is repeated. A trader can become *inactive* at any price and her/his decision to do so is publicly observable. Once the number of active traders is less than or equal to the number of units of output for sale, the auction ends. Each buyer who is active at the final price pays that price and receives a unit of the good.[1] Sellers who are inactive when the auction ends trade and are paid the price at which the auction has ended. Sellers who are active at the final price leave the auction without trading.

We show that this mechanism has an *approximate* perfect Bayesian equilibrium in which traders remain active only so long as their values conditional on the information made public by the bidding process exceed the current price. Since traders' who drop out of the bidding reveal their types in this equilibrium, other traders can condition their bidding decisions much more finely on the distribution

[1]Some buyers who drop out at the final price may also receive units of the good if too many traders drop out of the bidding at the final price. We discuss this in more detail below.

of types of the others. In particular, since the bidding process reveals the types of the traders with the lowest values, it ensures that the traders with the highest values end up with the good even though their types are not fully revealed.

The ability to condition the decision whether to remain active on the dropout decisions of other traders is good for efficiency reasons, but bad for strategic ones. This is why the strategies we describe constitute only an approximate perfect Bayesian equilibrium. In particular, sellers may encounter information sets where they can exploit the possibility that they may be pivotal without taking any significant chance of losing a profitable trade. We describe these information sets, then provide conditions under which the expected payoff associated with pushing up the trading price becomes arbitrarily small when the number of traders is large.

Our approximate equilibrium supports an outcome which is ex post efficient with probability 1. This allows us to address some questions that are left open in the existing literature. For example, Perry and Reny (2002) show that the equilibrium price in a double auction converges in probability to the full information market clearing price. They do not directly address what happens to the equilibrium allocation, or to the posterior beliefs (both of which are important in the rational expectations story that they are interested in) when the number of traders is large but finite. Furthermore, convergence in probability also leaves open the possibility that unusual arrays of types (i.e. the ones that occur with a small probability) lead to prices that are far from full information prices. However, in the ascending-price double auction these issues do not arise.

Also, the convergence results of Perry and Reny (2002) rely on an affiliation assumption. This assumption is common in auction theory. Nonetheless, it is a restriction on beliefs.[2] It is interesting to identify how the double auction breaks down without affiliation. It is also desirable to have a trading mechanism which supports ex post efficient allocations with weaker restrictions on beliefs, as we provide in this paper.

As we illustrate below by example, in the absence of affiliation an indirect mechanism can support an outcome close to a full information or a rational expectations equilibrium only if traders are given an opportunity to respond to the same price differently in different situations. Our mechanism accomplishes this because traders, including sellers, make trading decisions after they have observed some of the decisions of other traders. This feature of our mechanism allows traders to react in a flexible way to information about the types of the other traders.

At the same time, when the number of traders gets large, traders (almost) lose their ability to manipulate the trading price. As a result, our equilibrium has a 'rational expectations' property. Specifically, the trading decision for each trader

[2] Gresik (1991) has shown that it is impossible to achieve ex-post efficiency using static mechanisms in the interdependent value environment with valuation functions linear in traders' types and independent type distributions.

is the best outcome that is feasible given the equilibrium trading price, and given all the information that is revealed along the equilibrium path associated with the trading process. Traders appear to be making optimal choices in all situations conditional on information revealed by the final equilibrium trading price and their own equilibrium outcome.

It is worth noting that models where traders observe the actions of other traders often provide negative results regarding convergence of equilibrium. The best example might be Wolinsky (1988) where traders are given repeated opportunities to observe others' behavior. Wolinsky shows that this will prevent trade from occurring at the right price even when there are many traders with almost costless opportunities to interact.

A similar result in a different setting is provided by Horner and Jamison (2004) who analyze an infinitely repeated sequence of auctions in which bidders have private (but unchanging) information about the common value of a good being sold. Bidders are repeatedly given the opportunity to observe the bids being made by others, and could potentially use this information to learn the common value. They give examples of equilibria in which no private information is *ever* revealed. Gottardi and Serrano (2002) analyze a series of models somewhat similar in structure to the one analyzed in Wolinsky (1988) and show that aggregation failures are closely related to the traders' market power. Gottardi and Serrano (2002) point out that traders know their actions are being observed, and this provides them with an additional opportunity to manipulate the outcome of the mechanism in their favor. Their behavior becomes less informative as a result.

The ability to directly manipulate others' beliefs is a key reason for the failure of convergence discovered by Wolinsky (1988) and the other authors cited above. One of the advantages of our mechanism is that the impact of such behavior becomes small in a large market.

Our paper also contributes to the design of ascending-price auctions and understanding of their incentive properties. One-sided ascending-price (English) auctions have been studied by Krishna (2003), Izmalkov (2003), Birulin and Izmalkov (2003) and others. Ausubel (2004) constructs an ascending-bid auction for multiple items. In a context considerably different from ours, Ausubel points out that an ascending-bid auction may retain efficiency with interdependent values, while static one-shot auctions typically suffer from winner's curse. This result relates his paper to ours, to the extent that they both point at the advantages of dynamic ascending-price auctions.

The rest of the paper is organized as follows. In section 2 we show why double auctions cannot achieve efficiency without affiliation. In section 3 we present our model. Section 4 contains our main result. In section 5 we discuss the implications. Section 6 concludes. All proofs are relegated to the Appendix.

2 Double Auctions

First, let us consider why double auctions are restrictive. Suppose that traders' bids and asks are all monotonically related to their 'types'. Then any ask price announced by a seller is equivalent to a contingent plan according to which this seller agrees to trade if the value of the appropriate order statistic -the m-th lowest value among m bids and n asks- is above this seller's ask price. Suppose that for some array of types, this order statistic is equal to p and it is ex-post efficient for our seller to trade. By the rules of the double auction, the seller will end up trading under any array of other trader types which gives rise to a higher value of this order statistic. So if the double auction supports an efficient outcome, it must be efficient for this seller to trade when this order statistic has any value higher than p. An increase in the value of this order statistic will increase the price at which the seller trades, which certainly makes the seller more willing to trade. However, a higher value of this order statistic also signals that other traders have higher private types. Under standard interdependence assumptions this will mean that the seller will also assign a higher value to the good, so her opportunity cost of trade will be higher. Some condition needs to hold to ensure that the former effect outweighs the latter.

The condition that does this in Perry and Reny (2002) is affiliation. In their formulation, a trader's value for the good depends on her own type and on some common quality q. Quality q is unknown and random, and the traders' types are distributed identically and independently conditional on q. Let F_q denote the probability distribution from which each trader's type is drawn conditional on q. When the number of traders is very large (infinite), the equilibrium price in the double auction coincides with the full information market clearing price, and the latter reveals the actual quality.

Now fix a quality q, and let p_q be the corresponding full information price. Consider a buyer whose type x_q is such that he bids p_q in the double auction. Since the double auction price is equal to the full information market clearing price, the outcome should be ex post efficient. So the buyer of type x_q must be just indifferent between trading and not trading at price p_q, and the measure of the set of traders whose types are higher than x_q (who get the good in an ex-post efficient outcome) must be exactly equal to the measure of the set of available goods.

Next, consider a lower 'quality' q', i.e. $q' < q$. The full information price $p_{q'}$ corresponding to quality q' and hence the equilibrium price in the double auction must be lower than p_q. But the buyer of type x_q bids p_q irrespective of the actual quality, which she is uncertain of. So he will win a unit of output at the new price $p_{q'}$.

Hence, to maintain ex post efficiency, the reduction in price from p_q to $p_{q'}$ in the double auction has to at least compensate the buyer of type x_q for the reduction in the quality of the good from q to q'. This is ensured by the affiliation assumption, because it implies that $p_{q'}$ cannot exceed the price p^* at which the buyer of type x_q is indifferent between trading and not trading the good of quality q'. To understand why this is so, note that under affiliation the distribution of types conditional on q first-order stochastically dominates the distribution of types conditional on q'. Therefore, the measure of the set of traders whose types are above x_q is at least as large when quality is q as it is when quality is q'. So when the quality is q', the equilibrium price in the double auction must fall below p^* to ensure that some buyers with valuations below x_q bid above this equilibrium price and end up with the good. Otherwise, the market would not clear, as the set of buyers who want to purchase good at the final price would be smaller than the set of sellers who would like to sell at that price.

Generally speaking, a double auction works well if an increase in quality causes the m^{th}-lowest order statistic of traders' types to rise faster than traders' full information values. What follows is a finite example that shows how things can go wrong. This example will also be used below to illustrate how the ascending auction procedure eliminates this problem.

There are 6 traders. Only one of them is a seller, so this is a simple auction environment except for the fact that the seller is strategic and privately informed. Types are commonly known to be integers between 3 and 10. The ex post value function of trader i is given by

$$u\left(x_i, x_{-i}\right) = x_i + W\left[\frac{\sum_{j \neq i} x_j}{5}\right] \tag{1}$$

where $W[x]$ denotes the 'whole part' function, i.e. the largest integer that is less than or equal to x.

The lowest value that a trader can have in this environment is 6 (when all traders have type 3), and the highest value is 20 which occurs when all traders have type 10. Of particular interest are two states of the world corresponding to the following two type profiles:

$$\text{State 1:} \quad \{9, 8, 8, 8, 8, 8\}$$

and

$$\text{State 2:} \quad \{9, 10, 10, 3, 3, 3\}.$$

In each of these two states, it is the seller who has type 9. The profiles of traders' values in these two states are given by

$$\{17, 16, 16, 16, 16, 16\}$$

and

$$\{14, 15, 15, 8, 8, 8\}$$

respectively.

In state 1 all the buyers have high types, which raises the seller's value because of interdependence. In state 2, only the seller and the first two buyers have high types, while the other buyers have rather low types. The full information price can be anything between 16 and 17 in state 1, but must be equal to 15 in state 2. Importantly, for the outcome of the auction mechanism to be efficient, the seller of type 9 would need to sell in state 2, but keep the good in state 1. That is, she needs to sell the good if price does not rise above 15, and to keep it if a *higher* price of 16 is reached. Thus, this example has the plausible but unusual (at least in auction theory) property that the seller's 'supply' of the good has to be inversely related to price.

Now suppose that the joint distribution of types is such that any trader of type 8 believes that with a very high probability the true state is a permutation of state 1 i.e., 4 of 5 other traders have type 8 while one trader has type 9. Similarly, traders with types 3 or 10 believe that, with a very high probability, the true state is a permutation of state 2. Finally, any trader of type 9 believes that with very high probability, the true state is either a permutation of state 1 or a permutation of state 2, and that each of the two configurations is equally likely. Types are distinctly not affiliated in this example. A trader whose type rises from 3 to 8 believes that with a very high probability the types of some of the other traders will rise, while some of them will fall.

The double auction cannot support an efficient outcome in both states 1 and 2. A trader of type 8 believes that her/his value is very close to 16, and is almost sure that there is another bidder who has this same value and belief. Therefore, in any equilibrium, a trader of type 8 must bid close to 16 with a high probability. A trader of type 10 is in a similar position. He believes that his value is very close to 15 and is almost sure that there is another trader with the same value and beliefs. As a consequence, such trader's bid must be very close to 15 with high probability. This argument implies that traders' bids will not be monotonically increasing in their types as would be necessary for ex post efficiency.

In particular, the seller will submit the same bid in both states. He could submit a bid above 16 and keep the good in both states. Alternatively, he could bid below 15 and sell in both states. Each of these two strategies would produce ex post inefficiency in one of the two states. Finally, if the seller submits a bid between 15 and 16 he will sell in the wrong state.

The failure of double auction in this example stems from the fact that a seller has to make a bid and a trading decision completely independently of the realized profile of types of the other traders. This problem is mitigated in an ascending double auction.

3 The Model

3.1 Fundamentals

There are n sellers and m buyers trading in a market. Let $N \cup M$ denote the set of traders (both buyers and sellers). Each seller has one unit of a homogeneous good, while each buyer has an inelastic demand for one unit of this good.

A trader's type lies in a compact subset $\Omega \subset \mathbb{R}$. Below we will restrict the set of feasible types to be finite. Trader i's valuation for the good is given by $u(x_i, x_{-i})$ where x_i is trader i's privately known type, and x_{-i} is the profile of types of all other traders. The value function $u(x_i, x_{-i})$ is assumed to be continuous in x_i and non-decreasing in each of its arguments. Note that *all* traders have the same value function, with the first argument of the function denoting the type of the trader her/himself, and the second argument standing for the profile of the other traders' types. The value function in Perry and Reny (2002) is a special case of this. They assume that the full information value of trader i is given by $v(x_i, q)$ where x_i is the trader's own type, and q is the unobservable quality of the good being traded. This can be supported as a special case of our formulation by setting

$$u(x_i, x_{-i}) = \mathbb{E}_{x_i, x_{-i}} v(x_i, q)$$

as long as v is increasing in both its arguments, and x_i and q are affiliated.[3]

A buyer's payoffs is equal to her expected value less the price that she pays for the good. A buyer gets zero payoff if she does not buy and pays nothing. A seller's payoff is equal to the price that she receives less her expected value. A seller gets zero payoff if she does not sell and receives nothing.

Assumption 1 *(Single Crossing Condition) If $x_i > x_j$ (where x_j is the j^{th} component of x_{-i}), then $u(x_i, x_j, x_{-i-j}) \geq u(x_j, x_i, x_{-i-j})$.*

This assumption requires that, starting from any profile of types in which two traders have the same types, an increase in the type of one of these traders has more impact on this trader's value than the same increase in the other trader's type.[4] In Perry and Reny (2002) this restriction holds because an increase in x_i improves i's perception of the quality of the good q just as much as an increase in x_j does, but an increase in x_i also improves i's valuation of every quality.

Assumption 1 along with the assumption that all traders have the same value function implies that the trader with the highest type also has the highest value.

The value function is also assumed to possess the following continuity property:

[3]Here, the expectation is taken using posterior beliefs about q conditional on the types of all traders. Affiliation between x_j, for all j, and q is needed to ensure that the expectation is increasing in the type of every trader.

[4]We do not need to strengthen the single-crossing conditions along the lines of Krishna (2003) or Birulin and Izmalkov (2003) because bidders are symmetric here.

Assumption 2 *For any* $\triangle > 0$*, there is* N *such that if* $m \geq N$ *and* $n \geq N$*, then*

$$\left| u\left(x_i, x_j, x_{-i-j}\right) - u\left(x_i, x_j', x_{-i-j}\right) \right| < \triangle$$

for every pair $x_j, x_j' \in \mathcal{X}$*, for every* $x_i \in \mathcal{X}$ *and* $x_{-ij} \in \mathcal{X}^{n+m-2}$*.*

This assumption ensures that when the number of traders is very large, the influence of any single trader's type on any other trader's valuation becomes small. However, it does not imply that the aggregate influence of the profile of the other trader's types on a trader's valuation becomes small.

The traders' types are drawn from the joint probability distribution F_{mn} which is common knowledge. We assume that F_{mn} is symmetric[5] and that the marginal distributions of traders' types are identical and have a *finite* support $\mathcal{X} \subset \Omega$ which is independent of m and n. For $x \in \mathcal{X}$, let x^+ denote the next higher type in $\mathcal{X}$. We make the following *full support* assumption:

Assumption 3 *There exists an* $\varepsilon > 0$ *such that for any* $x_i \in \mathcal{X}$ *and any* m *and* n*,* $\Pr_{F_{mn}} \{\tilde{x}_i = x_i | x_{-i}\} \geq \varepsilon$ *for all* x_{-i}*.*

This condition is borrowed from Peters and Severinov (2005) and resembles a condition in Cripps and Swinkels (2006). Among other things, it ensures that all the conditional expectations that are used in the sequel are well defined. Conditional independence with a common and full support (as in Perry and Reny (2002)) is consistent with this assumption. Note that we do not impose any of the affiliation assumptions used by Perry and Reny (2002).

3.2 Ascending-Price Double Auction Mechanism.

Below we provide the description of our ascending-price double auction mechanism. In the course of this auction, all traders (sellers included) bid for the right to own (keep) one of the n objects being auctioned. The auction starts with a price equal to

$$\underline{q} = \mathbb{E}\left[u\left(\underline{x}, x_{-i}\right) | x_{-i(1)} = \underline{x}, \cdots, x_{-i(m)} = \underline{x} \right] \tag{2}$$

where $\underline{x}$ is the lowest type in the support $\mathcal{X}$. Here, and in the rest of the paper, the notation $y_{(k)}$ refers to the k^{th} lowest element in the type vector y. Similarly, $y_{-i_{(k)}}$ refers to the k^{th} lowest element in the vector y_{-i} which is obtained from vector y by excluding the type of the i-th trader.

[5]This symmetry assumption, like the assumption that all traders have the same valuation function, is used to construct an equilibrium in which all traders use the same bidding rule. The analytical complexities associated with asymmetric bidding rules in the interdependent value environment are daunting, so we do not know whether our results can be extended to asymmetric traders with different prior beliefs. Peters and Severinov (2005) show that in the private value case, an equilibrium exists in which buyers use a common bidding rule similar to the one constructed in this paper, even without common priors.

All traders simultaneously declare whether they want to remain *active* at price $\underline{q}$, or whether they want to drop out of the bidding at this price and become *inactive.* If the number of active bidders exceeds the number of units for sale, the auctioneer raises the price according to a formula that will be described below, and this process is repeated. The price increases until the number of active bidders is less than or equal to the number of units of the good for sale.[6] All trades are executed at the price attained at this terminal point. Active buyers are given a unit of the good at this trading price. Inactive sellers are given the trading price for the unit that they have for sale. Active sellers leave the market without trading. If the number of active traders is below the number of units for sale, the unsold units are randomly awarded to the bidders who dropped out of the bidding at the final trading price.

The history of the game at each price p consists of a list of traders who have dropped out of the bidding, and the prices at which they have dropped out. It will be convenient to summarize this history by assigning types to the traders who have already dropped out. This assignment of types is also necessary to describe the price adjustment rule. Suppose that after some history (including the null history) h, the current price is p and $k \geq 0$ bidders have dropped out and have been assigned types $\hat{x}_h \equiv \{\hat{x}_1, \ldots, \hat{x}_k\}$ ordered from the lowest to the highest. Any bidder who drops out at the current price p is assigned type $\hat{x}^p$ where $\hat{x}^p$ is the solution for x in the following equation:

$$\mathbb{E}\left[u\left(x, \tilde{x}_{-i}\right) | \tilde{x}_{-i_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-i_{(k)}} = \hat{x}_k, \tilde{x}_{-i_{(k+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = x\right] = p \quad (3)$$

The solution $\hat{x}^p$ to this equation exist by induction given the price adjustment rule described below. Also, by the monotonicity of the utility function, $\hat{x}^p > \hat{x}_j$ for all $j = 1, ..., k$.

Suppose that l bidders drop out and become inactive at price p. Each of these (now inactive) bidders is assigned type $\hat{x}^p$ and an order number between $k+1$ and $k' = k + l$ (the ordering is arbitrary). Thus, $\hat{x}_i = \hat{x}^p$ for $i = k+1, ..., k'$.

If there are still more active bidders than there are goods for sale, the price is adjusted upwards to p^+ which satisfies

$$\mathbb{E}\left[u\left(\hat{x}^{p+}, \tilde{x}_{-i}\right) | \tilde{x}_{-i_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-i_{(k')}} = \hat{x}_{k'}, \tilde{x}_{-i_{(k'+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \hat{x}^{p+}\right] = p^+ \quad (4)$$

Recall that $\hat{x}^{p+}$ stands for the next element on the grid $\mathcal{X}$ which is higher than $\hat{x}^p$. If some bidders drop out at p^+, then the auction ends in case the number of

[6]The mechanism could be modified to allow inactive traders to re-enter the bidding as in Izmalkov (2003). Since the equilibrium that we construct achieves an efficient outcome without reentry and, moreover, reentry would never be optimal for a trader i even if it was allowed - provided that other traders use the equilibrium strategies, we assume re-entry away to simplify the presentation.

goods exceeds the number of remaining active bidders. Otherwise, the described price adjustment procedure is performed again. If no bidders drop out at p^+, then the price is raised to the next level p^{++} which is given by the same expression as in (4) except that $(\hat{x}^{p+})$ is replaced by the next element on the grid $(\hat{x}^{p+})^+$.

We summarize the history of the game via a tuple $h \equiv \{\hat{x}_1, \ldots, \hat{x}_k, p\}$ where each element in the tuple (except the price p) corresponds to the type assigned to some trader who has dropped out on the path of the game. These types are calculated by iteratively applying (3). Finally, let H be the set of all such tuples with $k \leq m$. A strategy for each bidder is a map from H into the set of probability distributions over the set $\{a, i\}$ (where a stands for active, and i stands for inactive). These descriptions of the price adjustment procedure and the strategy sets complete the definition of a sequential game of incomplete information.

Definition 1 *A δ-perfect Bayesian equilibrium is a set of strategies and beliefs such that no trader can increase his or her payoff by more than δ by deviating from her equilibrium strategy given her beliefs, and such that beliefs satisfy Bayes rule on the equilibrium path.*

4 A Strategy Rule for the Ascending-Price Double Auction

We will assume that after any history h all traders (except the trader whose type is the subject of beliefs) believe that with probability 1, an inactive trader with order number i has the type $\hat{x}_i$ which was assigned by the price adjustment rule. Also, all traders believe that each active trader (except the trader whose type is the subject of beliefs) has type at least as high as $\hat{x}^p$ given by the solution to (3). We are now ready to describe equilibrium strategy rule for the traders in this auction mechanism. For every history h, a strategy rule specifies whether or not an active trader should continue bidding. To describe the strategy rule we start with the following definition:

Definition 2 *Active trader i's willingness to pay after history h is equal to:*

$$\mathbb{E}\left[u\left(x_i, \tilde{x}_{-i}\right)|h; \tilde{x}_{-i_{(k+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \hat{x}^p\right] \tag{5}$$

where $h = \{\hat{x}_1, \ldots, \hat{x}_k, p\}$ for some k and $\hat{x}^p$ is the solution to (3).

Conditioning on h in this expectation means conditioning on the event $\tilde{x}_{-i_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-i_{(k)}} = \hat{x}_k$.

An active trader's willingness to pay is her expected value conditional on the event that just enough traders drop out at the current price p so that the auction ends. Note that, by the full support assumption, the expectations in this definition are always well-defined.

Definition 3 *The strategy σ^* is defined as follows: each bidder remains active (continues bidding) if his willingness to pay is strictly higher than the current price, and becomes inactive otherwise.*

Observe that, according to σ^*, any bidder whose type is strictly higher than $\hat{x}^p$ will continue bidding at price p. It remains to characterize the outcome of the auction when traders use the strategy σ^*. This outcome depends on the types of all the traders.

Definition 4 *Consider some array of types $x = \{x_1, \ldots, x_n\}$ and suppose that trader i has the r^{th} lowest type in this array. Let trader i's* perceived value $v_i[x]$ *under the array x be equal to*

$$\mathbb{E}\left\{u(x_i, \tilde{x}_{-i}) | \tilde{x}_{-i_{(1)}} = x_1, \ldots, \tilde{x}_{-i_{(r-1)}} = x_{r-1}, \tilde{x}_{-i_{(r)}} = \cdots = \tilde{x}_{-i_{(m)}} = x_i\right\} \quad \textit{if } r \leq m \tag{6}$$

$$\mathbb{E}\left\{u(x_i, \tilde{x}_{-i}) | \tilde{x}_{-i_{(1)}} = x_1, \ldots, \tilde{x}_{-i_{(r-1)}} = x_{r-1}\right\} \quad \textit{if } r > m \tag{7}$$

Also, let $v_{(m)}[x]$ (or simply $v_{(m)}$ when the array of types is clear from the context) be the perceived value of the trader with the m^{th} lowest type in x.

Trader i's *perceived value* is roughly his expected valuation of the good, conditional either on the knowledge of m lowest types in the array x, or on the knowledge of the types lower than his and the estimate that the m-th lowest type in the array x does not exceed her type x_i. By construction, $v_{(m)}$ is the m^{th}-lowest such value. Our main theorem can now be stated.

Theorem 1 *Let $\delta > 0$ and suppose that Assumptions 1-3 hold. Then there is some N such that there exists an δ-perfect Bayesian equilibrium where all traders use the strategy rule σ^* if the number of sellers and hence the goods being sold is larger than N. For each array of types x that occurs with a positive probability, all trades occur at price $v_{(m)}[x]$. A trader whose type is above $x_{(m)}$ will win a unit of the good for sure, a trader whose type is below $x_{(m)}$ will not win a unit of the good. A trader whose type is $x_{(m)}$ may or may not win a unit - in either case his expected value for the good will be the same as the equilibrium trading price.*

Proof: *See the appendix.*

Two observations are in order at this point. First, in the proof we show that the traders with n highest perceived values always end up with the good. By symmetry and the single-crossing Assumption 1, these are the traders with n highest types.

So the equilibrium outcome of the mechanism is ex post efficient with probability 1.

The second observation is that the bidding decisions of traders on the equilibrium path completely reveal m lowest types of traders. So once the auction ends, the expected value of the good conditional on all information revealed by the bidding for a trader i who ends up with a unit of the good is equal to his perceived value $v_i[x]$. So the traders who consume the good have perceived values at least as high as the trading price $v_{(m)}[x]$. A trader who ends up without the good has expected value that is at least as high as her perceived value. However, his expected value is monotonic in his own type and therefore cannot exceed $v_{(m)}[x]$. Consequently, the equilibrium outcome is the best one for every trader conditional on the equilibrium price and the information that traders have at the end of the bidding process. We refer to this as the *rational expectations property.*

However, unlike in a standard rational expectations equilibrium where traders only condition their beliefs on price, the traders' beliefs at the end of the bidding process are conditioned on all the information revealed in the course of bidding. The ascending auction procedure we study here can never reveal the highest trader types, so the *equilibrium* trading price will not generally be fully revealing, and in particular the equilibrium trading price will not coincide with the full information market clearing price.[7]

Yet since our rational expectations property holds uniformly for all trader types, it holds for every trader under any array of types that generates the same equilibrium outcome for the trader. Thus the ascending double auction has a stronger rational expectations property: the outcome of the mechanism will appear to be the best one for each trader conditional on the information conveyed by her own trading outcome (i.e., the final trading price and whether or not she has won the good). We show by an example below that this property may not hold if a trader conditions her belief only on information conveyed by the final price.

5 Discussion

In the example studied in section 2, our bidding rule σ^* supports ex post efficient trade in the problematic states that we described above. Recall that in our discussion we have focused on two type profiles/states of the world:

$$\text{State 1:}\quad \{9, 8, 8, 8, 8, 8\}$$
$$\text{State 2:}\quad \{9, 10, 10, 3, 3, 3\}$$

[7]The equilibrium price in the double auction described in Perry and Reny (2002) does not coincide with the full information price in this sense either. They show that the equilibrium price in the double auction will be close to the full information price with high probability.

with full information values equal to $\{17, 16, 16, 16, 16, 16\}$ and $\{14, 15, 15, 8, 8, 8\}$ in states 1 and 2 respectively.

Given the utility function (1) and the assumption that the lowest trader type is 3, we can use (2) to compute the starting (reserve) price in our mechanism - it is equal to the lowest full information value of 6 which occurs when all traders have type 3.

Let us, first, focus on state 2 with the array of types $\{9, 10, 10, 3, 3, 3\}$. All bidders need to compute the type assigned to any bidder who drops out at price 6. This type is given by the solution to (3), and is equal to 3. At this price, the seller (whose type is 9) and the buyers whose types are 10 have willingness to pay exceeding 6. However, the three buyers with types equal to 3 would drop out immediately according to σ^*. Successive application of the price adjustment rule causes the price to rise to 14. At price 14, the seller (whose type is 9) would drop out because her willingness to pay no longer exceeds the current auction price. The two high-value buyers with types 10 will then continue to bid until the auction price reaches 15, at which point both of them will drop out. One of them will be chosen at random to trade. Hence, the final trading price will be equal to the full information market clearing price 15.

Now let us consider state 1. Again, bidding will start at price 6. At this point, the seller's willingness to pay is 12, while the buyers' willingness to pay is 11. Each trader's willingness to pay (which changes as the price increases) remains above the price and all traders remain active, until the price reaches 16. At this price, each buyer's willingness to pay is also 16. So, all buyers will drop out of the bidding at 16 and will be assigned type 8. The seller, as the only trader who remains active at this price, wins the auction and ends up keeping the good. Again, the final trading price is equal to the full information market clearing price.

The seller's "odd" desire to keep the good when the price is high and trade it when the price is low is accommodated by the fact that the seller can reserve his decision whether or not to remain active when the price exceeds 15 until after he observes the bidding decisions of the other traders. When he observes three buyers drop out at price 6, he concludes that there is no point holding out for a high price once bidding reaches 15. When all buyers continue to bid until the price reaches 15, the seller concludes that the value of the good is even higher, and bids more aggressively to hold on to it.

The same example, but with a different state of the world, can be used to illustrate why σ^* is only an approximate equilibrium and why we need a large number of sellers. For example, in the state

$$\{7, 7, 3, 3, 3, 3\}$$

the trading price is 10 if traders use σ^*. When the auction price reaches 10, both the seller and the buyer of type 7 have to decide whether to continue bidding.

According to σ^*, both of them should drop out at price 10. If they do so, the seller may or may not sell.

The problem is that the seller knows that he is pivotal when the price reaches 10, since at this point only he and another buyer are active. If the seller drops out at this point, the process ends and he earns zero profit. If he continues bidding, there are two possibilities. One is that the buyer will drop out, which will happen if the buyer's type is 7, as in the current type profile. The seller will then win the auction and earn zero profit. The other possibility is that the buyer will remain in the bidding -which would happen if the buyer's type is 8 or greater. In that case, the price will rise to at least 12 (by (4)). The seller might win the auction at price 12 if the buyer drops out, but this does not create a problem because her surplus is zero in that case, just as it is if he drops out at price 10. Alternatively, if the seller drops out at price 12 before the buyer does, the seller will get a higher price from a buyer - whose type in this case must be at least 8. So dropping out of bidding at price 10 is suboptimal for the seller.

In this example, the expected gain to the seller from continuing to bid is significant. However, when the number of sellers and hence the number of goods being auctioned is large, an active seller becomes pivotal when there are exactly n other active traders each of whom has a willingness to pay exceeding the current price. When n is large, the full support Assumption 3 guarantees that with a high probability *at least* one of the other active bidders has willingness to pay equal to the current price. This makes it very unlikely that the seller can prolong the auction by continuing to bid.

We can also use this example to illustrate the rational expectations property of our mechanism. This property was described in the previous section. Consider the outcome in state 1 where the profile of types is $\{9, 8, 8, 8, 8, 8\}$. When all the buyers drop out at price 16, the seller believes that each of the buyers has type 8. Bidding ends and the seller fails to trade. Ex post, the fact that all the buyers dropped out simultaneously at price 16 reveals the true state to the seller. Conditional on this belief about the state, the seller's demand correspondence at price 16 consists of only one outcome, not trading, which is what happens in the auction. So the outcome is in the seller's 'demand' correspondence conditional on all the information the outcome reveals. In state 2 where the profile of types is $\{9, 10, 10, 3, 3, 3\}$, the seller also learns the state once bidding ends. Conditional on this information he would prefer to trade at price 15. Once again, the outcome is in his demand correspondence given the price and posterior beliefs.

However, it would be wrong to think that the seller can choose the best outcome if she conditions her decision only on information revealed by price. For, consider the states $\{8, 7, 7, 7, 7, 7\}$ and $\{8, 9, 9, 3, 3, 3\}$ which may also arise in the example which we have considered. Recall that a trader's utility function is equal to $u(x_i, x_{-i}) = x_i + \left[\frac{\sum_{j \neq i} x_j}{5}\right]$, and so the full information values in these two states are $\{15, 14, 14, 14, 14\}$ and $\{13, 14, 14, 9, 9, 9\}$, respectively. The equilibrium trading price predicted by Theorem 1 is 14 in both states but the seller keeps the good in the first state and sells in the second. Our ascending-price auction mechanism can deliver this outcome, because the seller learns a lot more about the types of the others in the course of bidding. In contrast, the described outcome would be infeasible in a static rational expectations equilibrium where the seller only gets to see the price.

6 Conclusions

We have provided an ascending-price double auction mechanism and a strategy rule which constitutes an δ-perfect Bayesian equilibrium for this mechanism when the number of sellers is large enough. To the best of our knowledge, dynamic double auctions have not yet been studied in the literature. The allocation supported by this equilibrium in our mechanism is ex post efficient. This property holds even when the types of the traders are not affiliated, so our mechanism delivers an efficient outcome in cases where a standard double auction would not. This is so because in our mechanism traders acquire information about the types of the other traders in the course of the bidding.

The equilibrium we describe is only an approximate equilibrium. It is difficult to say whether there is a way around this problem. The same property of the ascending-price mechanism that supports the revelation of information in the course of bidding also allows sellers to realize whether they are pivotal and attempt to manipulate the price. When the market is large, there is little for sellers to gain from this information, so their incentive to deviate becomes arbitrarily small.

Appendix - Proof of Theorem 1

To prove the theorem, we first specify the traders' beliefs. We then show that σ^* induces traders to behave as if they had private values equal to their *perceived values* defined by (6) or (7). This allows us to show that buyers cannot gain by deviating from σ^*. Sellers, on the other hand, can increase their payoff by deviating from σ^* in some information sets. In the final section, we show that the extra gains that the sellers get by deviating in these special information sets become arbitrarily small as the number of traders get large, provided that Assumption 3 holds.

6.1 Beliefs

The traders' beliefs in the ascending-price double auction are determined by the prior type distribution and observations of the dropout decisions of the other traders. Specifically, the beliefs are constructed recursively using the procedure described in section 2.

Condition 1 *Consider some history* $h = \{\hat{x}_1, \ldots, \hat{x}_k, p\}$*. Then the type of a bidder who drops out after history* h *is believed to be* $\hat{x}^p$ *with probability 1, where* $\hat{x}^p$ *is defined by (3). If a bidder remains active after history* h*, then her type is believed to be at least as large as* $\hat{x}^p$*.*

Lemma 1 *If all traders use strategy* σ^**, then beliefs constructed according to Condition 1 satisfy Bayes rule after any history that occurs with a positive probability on the equilibrium path of the auction.*

Proof: Follows immediately from the definition of σ^*. *Q.E.D.*

We will say that i believes that the array of types x_{-i} is possible after history h if the beliefs given by Condition 1 put a positive probability on x_{-i}.

6.2 Buyers Act as if they have Private Values

The following lemma provides a result that is central to the logic of the proof of the theorem. Fix an array of types $\{x_1, \ldots, x_{m+n}\}$ and calculate the corresponding array of perceived values $\{v_1, \ldots, v_{m+n}\}$ defined by (6) or (7). Then trader i with type x_i who uses σ^* and believes that other traders follow σ^* will act just as if he had private value equal to v_i. If trader i deviates from σ^*, then she acts in the same way as a trader with some type x_i' following σ^*. So, trader i can figure out the impact of his deviation by calculating the new array of perceived values associated with $\{x_1, \ldots, x_i', \ldots, x_m\}$ then applying σ^* to obtain the new outcome.

Lemma 2 *Let* h_0 *be a bidding history. Suppose that all traders' beliefs satisfy 1 and that all traders other than i follow strategy* σ^* *in the auction mechanism after history* h_0*. Let* h_1 *be a successor to* h_0 *that i believes occurs with a positive probability given i's strategy, and let* x_{-i} *be an array of types of traders other than i that i believes is possible conditional on the history* h_1*. Then there is* x_i' *such that each active trader j's (*$j \neq i$*) willingness to pay in* h_1 *is larger than the price p associated with* h_1 *if and only if trader j's perceived value under array of types* (x_i', x_{-i}) *is larger than p. If i is active in* h_1*, then* $x_i' = x_{-i_{(m)}}$*. Otherwise,* $x_i' \leq x_{-i_{(m)}}$*.*

Proof: **"Only if" Part: If active trader j's willingness to pay after history h_1 is larger than p, then trader j's perceived value under the array of types (x_i', x_{-i}) is larger than p.**

Let $h_1 = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$. If i is active at h_1 and the auction has not yet ended at this point, then the belief that i's type is at least as large as $x_{-i_{(m)}}$ is consistent with 1, so let us take $x_i' = x_{-i_{(m)}}$. If i is inactive at h_1, let x_i' be the type i was assigned by (3) when he became inactive.

Consider some trader j who is active after history h_1. Then j's willingness to pay at this point is given by:

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_{-j_{(r+1)}} = \cdots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right] \tag{8}$$

where $\hat{x}^p$ is defined by the solution to (3). Suppose that trader j's willingness to pay is larger than p. Since by definition of $\hat{x}^p$, $\mathbb{E}\left[u\left(\hat{x}^p, \tilde{x}_{-j}\right)|h_1; \tilde{x}_{-j_{(r+1)}} = \cdots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right] = p$, and the utility function is monotonically increasing in trader's type, we have $x_j > \hat{x}^p$.

Suppose that j's type x_j is such that $x_j = (x_i', x_{-i})_{(t)}$. Let $t' = \min\{t, m\}$ and let $x_j' = \min[x_j, (x_i', x_{-i})_{(m)}]$. Then by Definition 4, j's perceived value is equal to

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right)|\tilde{x}_{-j_{(1)}} = (x_i', x_{-i})_{(1)}, \ldots, \tilde{x}_{-j_{(t'-1)}} = (x_i', x_{-i})_{(t'-1)}, \tilde{x}_{-j_{(t')}} = \cdots = \tilde{x}_{-j_{(m)}} = x_j'\right] \tag{9}$$

Since x_{-i} is an array of types of traders other than i that i believes is possible conditional on the history h_1, and i acts as if her type is x_i' and she follows σ^*, and j is active after history h_1, we have $r < t'$ and $\hat{x} = ((x_i', x_{-i})_{(1)}, ..., (x_i', x_{-i})_{(r)})$. Therefore, j's perceived value in (9) is at least as large as the following:

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right)|\tilde{x}_{-j_{(1)}} = (x_i', x_{-i})_{(1)}, \ldots, \tilde{x}_{-j_{(r)}} = (x_i', x_{-i})_{(r)}, \tilde{x}_{(r+1)} = \cdots = \tilde{x}_{(m)} = \hat{x}^p\right]$$
$$= \mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_{-j_{(r+1)}} = \cdots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right]$$

The last expression is trader j's willingness to pay.

"If" Part: If active trader j's willingness to pay in h_1 is less than or equal to p, then j's perceived value under the array of types (x_i', x_{-i}) is also less than or equal to p.

Formally, the fact that j's willingness to pay in $h_1 = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$ does not exceed p can be written as

$$\mathbb{E}\left[u\left(x_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_{-j_{(r+1)}} = \ldots = \tilde{x}_{-j(m)} = \hat{x}^p\right] \leq p$$

where $\hat{x}^p$ is defined by (3) as the solution to

$$\mathbb{E}\left[u\left(\hat{x}^p, \tilde{x}_{-j}\right)|h_1; \tilde{x}_{-j_{(r+1)}} = \ldots = \tilde{x}_{-j(m)} = \hat{x}^p\right] = p$$

By monotonicity of the value function in own type, $x_j \leq \hat{x}^p$. Since all bidders other than i follow σ^* in history h_1, it follows that $\hat{x} = ((x_i', x_{-i})_{(1)}, ..., (x_i', x_{-i})_{(r)})$.

So,

$$\mathbb{E}\left[u\left(x_j,\tilde{x}_{-j}\right)|h_1;\tilde{x}_{-j_{(r+1)}}=\ldots=\tilde{x}_{-j_{(m)}}=\hat{x}^p\right]\geq$$
$$\mathbb{E}\left[u\left(x_j,\tilde{x}_{-j}\right)|\tilde{x}_{-j_{(1)}}=\min[x_j,(x_i',x_{-i})_{(1)}],\ldots,\tilde{x}_{-j_{(m)}}=\min[x_j,(x_i',x_{-i})_{(m)}]\right]$$

Since $x_j \leq \hat{x}^p$, the last expression is equal to j's perceived value, so j's perceived value is less than or equal to p. *Q.E.D.*

6.3 The outcome associated with σ^*

Let $h = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$ be a history and x_{-i} be a set of types that i thinks is possible in h. Let $h' = \{\hat{x}_1, \ldots, \hat{x}_{r'}, p'\}$ be a successor to h that will be realized when the traders other than i have a profile of types x_{-i} and follow strategy σ^* while i follows some strategy. By Lemma 2, when the profile of other trader types is x_{-i}, there is a type x_i' such that bidder i can predict whether trader j will choose to continue bidding by calculating j's perceived value under (x_j, x_i', x_{-i-j}) and comparing it to p', for all $j \neq i$. This provides a simple necessary and sufficient condition for bidding to end at price p' when traders other than i are using σ^* - the number of traders whose perceived values exceed p should not be higher than n.

Let $\hat{v}[x_i', x_{-i}]$ (or just $\hat{v}$ when the implied array of types is clear from the context) be the vector of perceived values associated with types (x_i', x_{-i}).

Lemma 3 *Let h be a history and x_{-i} a profile of types that trader i believes is possible after history h. Suppose that all traders other than i use strategy σ^* in the continuation following h. Suppose the bidding ends after some history $h' = \{\hat{x}_1, \ldots, \hat{x}_m, q\}$ and that i wins a unit of output. Then q cannot exceed $\hat{v}_{(m)}[x_{-i_{(m)}}, x_{-i}]$.*

Proof: Write $\hat{v}_{(m)} = \hat{v}_{(m)}[x_{-i_{(m)}}, x_{-i}]$. Suppose $q > \hat{v}_{(m)}$. Since all traders other than i use strategy σ^* there must be some predecessor history $h'' = \{\hat{x}_1, \ldots, \hat{x}_r, q''\}$ of h' with $r < m$ from which the price is increased to q. Note that $\hat{x}^q$ solves

$$\mathbb{E}\left[u\left(\hat{x}^q,\tilde{x}_{-i}\right)|h'';\tilde{x}_{-i_{(r+1)}}=\cdots=\tilde{x}_{-i_{(m)}}=\hat{x}^q\right]=q.$$

Since $r < m$, at least $n+1$ trader types in the array of types (x_i', x_{-i}) are at least $\hat{x}^q$. Each of these traders has a willingness to pay of at least q which strictly exceeds $\hat{v}_{(m)}$. Then by Lemma 2 each of these traders has a perceived value that strictly exceeds $\hat{v}_{(m)}$. But by definition, $\hat{v}_{(m)}$ is the m^{th} lowest perceived value, so at most n perceived values can be equal to or higher than $\hat{v}_{(m)}$, a contradiction. *Q.E.D.*

Lemma 4 *Suppose that all traders other than* i *use strategy* σ^* *in the continuation after history* h*, the type profile of traders other than* i *is given by* x_{-i} *and buyer* i *thinks that profile* x_{-i} *is possible after history* h*. Also, suppose that buyer* i *is awarded a unit of the good at price* q *when the auction ends. Let* $x_i' = x_{-i_{(m)}}$*. Then* $q \geq \hat{v}_{(m)}[x_i', x_{-i}]$.

Proof: Suppose that the trading price at the end of the auction is equal to $p < \hat{v}_{(m)}[x_i', x_{-i}]$. By Lemma 2, traders with perceived values $\hat{v}_{(m)}[x_i', x_{-i}]$ and higher will all remain active at price p since they are using strategy σ^*. There are at least n such traders other than i. So the auction can end at p only if buyer i drops out, but in this case i will not win the good. *Q.E.D.*

We summarize the results obtained so far in the following Corollary:

Corollary 1 *Suppose that the type profile of traders other than* i *is given by* x_{-i}*. Also, suppose that after history* h *trader* i *believes that the type profile* x_{-i} *is possible, and all traders other than* i *follow the strategy* σ^* *in the continuation. Then*

1. *If trader* i *drops out of the bidding so that all traders believe that* i*'s type is* x_i'*, then all trades will occur at price* $\hat{v}_{(m)}[x_i', x_{-i}]$.
2. *Suppose that* i *is a buyer who is active at* h *and, like all other traders, follows* σ^* *in the continuation. Let* v *be the array of perceived values corresponding to the type profile* (x_i, x_{-i})*. Then buyer* i *will win a unit of the good at price* $v_{(m)}$ *if his perceived value exceeds* $v_{(m)}$*;* i *will not win a unit if his perceived value is less than* $v_{(m)}$*. Equivalently,* i *will trade if his type exceeds* $(x_i, x_{-i})_{(m)}$*, and will not trade if his type is less than* $(x_i, x_{-i})_{(m)}$.

To complete the proof, we establish the following Lemma:

Lemma 5 *If Assumptions 2 and 3 hold, and the number of sellers* n *is large enough, the strategy rule* σ^* *for all players, and belief system constructed according to Condition 1 constitute a* δ*-perfect Bayesian equilibrium in the ascending double auction.*

Proof: **Buyer's Part.** Let us establish that a buyer cannot gain by deviating from σ^*. Let x_{-i} be an array of types that buyer i thinks is possible after some history. By Lemmas 3 and 4, buyer i will trade at price $v_{(m)}[x_i, x_{-i}]$ under this array of types if all traders follow σ^* in the continuation and $x_i > x_{-i_{(m)}}$. By Lemma 4, i cannot lower this trading price by deviating from σ^* unless he fails to trade.

So it only remains to show that buyer i cannot get a higher payoff by deviating from σ^* in his decision to drop out of the bidding. Suppose that $x_i < x_{-i_{(m)}}$. Then by Corollary 1, if buyer i follows σ^*, then he will not trade and hence obtain zero payoff. If he deviates from σ^* and ends up trading, then by Lemmas 3 and 4, he will pay the price of at least $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$. But $v_{(m)}[x_{-i_{(m)}}, x_{-i}] \geq v_{(m)}[x_i, x_{-i}] > v_i[x_i, x_{-i}]$. By Lemma 2, i's willingness to pay is also below the trading price, so such deviation will not be profitable.

Similarly, if $x_i > x_{-i_{(m)}}$ (i.e. $v_i[x_i, x_{-i}] > v_{(m)}[x_i, x_{-i}]$), then i will trade at price $v_{(m)}[x_i, x_{-i}]$ and receive a strictly positive surplus, which is better than what he could get by dropping out before the auction ends. So following σ^* is a best reply for buyers if all other traders are using σ^*.

It remains to consider the case $x_i = x_{-i_{(m)}}$. By Corollary 1, in this case i may or may not win an auction if he follows σ^*. If he does win, he will pay the price $v_{(m)}[x_i, x_{-i}]$ which is equal to both his perceived value and his willingness to pay. By Lemma 4, there is no strategy for i which allows him to trade at a price below $v_{(m)}[x_i, x_{-i}]$, so any i's deviation from σ^* is unprofitable. This completes the proof for the buyers.

Seller's Part. A different approach is needed for sellers, because a seller can get a positive payoffs only if he is inactive at the end of the auction and trades.

Fix a history h, a seller i, and an array of types x_{-i} that the seller thinks is possible conditional on h. Suppose $x_i > x_{-i_{(m)}}$ (i.e. $v_i[x_i, x_{-i}] > v_{(m)}[x_i, x_{-i}]$). If seller i follows strategy σ^* along with all other traders, then the auction will end at price equal to $v_{(m)}[x_i, x_{-i}]$ and seller i will not trade and receive zero payoff. If seller i deviates to some alternative strategy and sells as a result of this deviation, then from part 1 of Corollary 1, there is x'_i such that $x'_i < x_{-i_{(m)}} < x_i$ and the final price in the auction would be $v_{(m)}[x'_i, x_{-i}]$. Seller i's payoff in this case is equal to

$$v_{(m)}[x'_i, x_{-i}] - u(x_i, x_{-i})$$

Since $x'_i < x_i$, this is less than

$$v_{(m)}[x_i, x_{-i}] - u(x_i, x_{-i}).$$

The same relationship holds for any array to types x'_{-i} which has the same lowest m components as x_{-i}. Thus, we have:

$$\begin{aligned} &v_{(m)}[x'_i, x_{-i}] - \mathbb{E}\left\{u(x_i, \tilde{x}_{-i}) \,|\tilde{x}_{-i_{(1)}} = x_{-i_{(1)}}, ..., \tilde{x}_{-i_{(m)}} = x_{-i_{(m)}}\right\} \leq \\ &v_{(m)}[x_i, x_{-i}] - \mathbb{E}\left\{u(x_i, \tilde{x}_{-i}) \,|\tilde{x}_{-i_{(1)}} = x_{-i_{(1)}}, ..., \tilde{x}_{-i_{(m)}} = x_{-i_{(m)}}\right\} \end{aligned}$$

The last expression is less than zero because $x_i > x_{-i_{(m)}}$. So such deviation is unprofitable.

On the other hand, if $x_i \leq x_{-i_{(m)}}$ for some array of types x_{-i} that i thinks is possible when auction price reaches some level p, then i could try to raise his payoff by remaining in the bidding after the auction price reaches the level at which she should drop according to σ^*. The downside is that he may lose a profitable trade when he does this.

Let $h = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$ be a history in which trader i's willingness to pay is less than or equal to p. Then i's expected gain from continuing to bid is:

$$\begin{aligned}
&- Pr\left\{x_i' > x_{-i_{(m)}}|h\right\} \times \mathbb{E}\left\{v_{(m)}[x_i, x_{-i}] - u_i\left(x_i, x_{-i}\right)|x_i' > x_{-i_{(m)}}; h\right\} + \\
&Pr\left\{x_{-i_{(m)}} \geq x_i' > x_{-i_{(m-1)}}|h\right\} \times \mathbb{E}\left\{v_i[x_i', x_{-i}] - v_i[x_i, x_{-i}]|x_{-i_{(m)}} \geq x_i' > x_{-i_{(m-1)}}; h\right\} \\
&+ Pr\left\{x_{-i_{(m-1)}} \geq x_i'|h\right\} \times \mathbb{E}\left\{v_{(m)}[x_i', x_{-i}] - v_{(m)}[x_i, x_{-i}]|x_{-i_{(m-1)}} \geq x_i'; h\right\}
\end{aligned} \tag{10}$$

The first term is non-positive and reflects the cost to seller i of losing a trade that he would have made had he not deviated. The second term reflects the seller's gain when he is pivotal and raises the trading price by deviating. The third term represents the impact that seller i has on the price by making others think that his type is higher and thereby inducing them to bid more aggressively, even though seller i is non-pivotal. We will show that the last two terms become arbitrarily small (and in particular smaller than δ) when n becomes large.

Consider the third term in (10). We have:

$$\begin{aligned}
&\mathbb{E}\left\{v_{(m)}[x_i', x_{-i}] - v_{(m)}[x_i, x_{-i}]|x_{-i_{(m-1)}} \geq x_i'; h\right\} = \\
&\mathbb{E}\left\{u\left(x_{-i_{(m-1)}}, x_i', (x_{-i} \setminus x_{-i_{(m-1)}})|(x_i', x_{-i})_{(1)}, ..., (x_i', x_{-i})_{(m)}, x_{-i_{(m-1)}} \geq x_i'\right)\right\} - \\
&\mathbb{E}\left\{u\left(x_{-i_{(m-1)}}, x_i, (x_{-i} \setminus x_{-i_{(m-1)}})|(x_i, x_{-i})_{(1)}, ..., (x_i, x_{-i})_{(m)}, x_{-i_{(m-1)}} \geq x_i\right)\right\}
\end{aligned} \tag{11}$$

By Assumption 2, the difference between the utility values under expectation sign in (11) is arbitrarily small uniformly in x_{-i} if n is large enough. So, the expectation of the difference is also arbitrarily small when n is large.

The utility difference in the second term remains bounded from above and below as n grows. However, we have:

$$\begin{aligned}
&\Pr\left\{x_{-i_{(m)}} \geq x_i' > x_{-i_{(m-1)}}|h\right\} \leq \max_{p' \geq p} \Pr\left\{x_{-i_{(m)}} \geq x_i' > x_{-i_{(m-1)}}|h\right\} \\
&\leq \max_{p' \geq p} \Pr\{x_j > \hat{x}^{p'} \text{ for every bidder } j \text{ active at } p'\}
\end{aligned}$$

By assumption 3, $\Pr\left\{x_j \leq \hat{x}^{p'}|\forall x_{-j}\right\} \geq \varepsilon$ for every bidder j active at p' and for every $p' > p$, so $\Pr\{x_j > \hat{x}^{p'} \text{ for every bidder } j \text{ active at } p'\} \leq (1-\varepsilon)^n$. The probability that the bidder is pivotal is then arbitrarily small at every auction price level, provided that the number of sellers n is large enough.

Since the first term in (10) is negative, we conclude that a seller's gain from deviating at any price level becomes smaller than δ when n is sufficiently large. *Q.E.D.*

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