1 Risk Aversion

- let $\mathcal{X}$ be a subset of the Real Line so that outcomes in x are interpreted as random incomes. The set of lotteries on $\mathcal{X}$ is associated with a set of probability distribution functions F whose support is contained in $\mathcal{X}$. It seems reasonable to assume that a decision maker doesn't distinguish between lotteries that support the same distribution function.

- So define $\mathcal{L}$ to be the set of probability distribution functions on $\mathcal{X}$

- as always the utility function V represents the preference ordering $\succcurlyeq$ over $\mathcal{L}$ if $V(F) \geq V(F')$ if and only if $F \succcurlyeq F'$.

- if preferences obey the independence axiom (among other things), then the utility function$V$ has the expected utility property (is linear in F) in the sense that $V(F)$ can be written as $\int u(x)\, dF(x)$ where u is a *utility for wealth* function.

- the utility for wealth function u exhibits *risk aversion on* $\mathcal{X}$ if

$$\int u(x)\, dF(x) \leq u\left(\int x dF(x)\right)$$

  for all $F \in \mathcal{L}$

- *Jensen's Inequality* A function $u : \mathbf{R} \to \mathbf{R}$ is concave on $\mathcal{X}$ if

$$\int u(x)\, dF(x) \leq u\left(\int x dF(x)\right)$$

  for all $F \in \mathcal{L}$ therefore, a risk averse individual must have a concave utility for wealth function

- the *certainty equivalent* $c[F, u]$ of a lottery u is the solution to

$$u(c) = \int u(x)\, dF(x)$$

- the Arrow Pratt measure of absolute risk aversion is given by

$$A[x] = -\frac{u''(x)}{u'(x)}$$

- the Arrow Pratt measure of relative risk aversion is given by

$$R[x] = -\frac{u''(x)}{u'(x)}x$$

1.1 Portfolio Theory

- suppose there are n assets and that asset i pays a random return $z_i \in \mathbf{R}$. The z_i are jointly distributed according to the distribution function $F(z_1, \ldots z_n)$. A *portfolio* is a vector of shareholdings $\{x_1, \ldots x_n\}$ x_i is interpreted as the number of shares (or units) of asset i that are held in the portfolio

- the (random) return to the portfolio is $\sum_{i=1}^{n} x_i z_i$. If the prices of the shares are given by q_i for $i = 1, \ldots n$, then the portfolio problem is to

$$\max \int u\left(\sum_{i=1}^{n} x_i z_i\right) dF(z_1, \ldots z_n)$$

subject to

$$\sum_{i=1}^{n} q_i x_i \leq w$$

- *Diversification Theorem* Suppose that asset 1 generates a sure return $\bar{z}_1$ and normalize the price of asset 1 to equal 1. Suppose that there exists a security $j > 1$ such that $\int z_j dF(z_2, \ldots z_n) > q_j \bar{z}_1$. Then no matter how risky the assets are, the optimal portfolio will have $x_1 < w$ (that is, the optimal portfolio will always involve purchase of some risky assets).

- *Proof:* Suppose to the contrary that $x_1 = w$ at the optimal portfolio. Imagine investing a small amount y in asset j and reducing investment of asset 1 by the same amount. Then the payoff to the portfolio will be

$$\int u\left([w - y]\,\bar{z}_1 + \frac{y}{q_j} z_j\right) dF(z_2, \ldots z_n)$$

The derivative of this expression is

$$\int u'\left([w-y]\,\bar{z}_1+\frac{y}{q_j}z_j\right)\left(\frac{z_j}{q_j}-\bar{z}_1\right)dF\left(z_2,\ldots z_n\right)$$

Evaluated at $y=0$ this is

$$\int u'\left(w\bar{z}_1\right)\left(\frac{z_j}{q_j}-\bar{z}_1\right)dF\left(z_2,\ldots z_n\right)>0$$

So the expected utility is strictly increasing in y when y is 0.

- *Comparative Statics* - Suppose there are two assets each has price 1. Asset 1 is safe and has a sure return equal to 1. Asset 2 is risky and has a random return given by $1+\tilde{s}$ where s is a random variable whose distribution is given by F. Total wealth is w while a is the amount invested in the risky asset. The investors problem is to maximize

$$\int u\left[a\left(1+\tilde{s}\right)+w-a\right]dF\left(\tilde{s}\right)$$

$$= \int u [w + a\tilde{s}] \, dF(\tilde{s})$$

subject to the constraint that $0 \leq a \leq w$

- *changes in wealth* Suppose that the utility for wealth function is concave and that the arrow pratt measure of absolute risk aversion is decreasing in wealth. Furthermore, suppose that the optimal solution to the portfolio problem given above is bounded away from 0 and 1. Then an increase in wealth w will increase the optimal value for a.

- *Proof:* The first order necessary condition is

$$\int u' [w + a\tilde{s}] \, \tilde{s} dF(\tilde{s}) = 0$$

Differentiate this implicitly with respect to wealth to get

$$\int u'' [w + a\tilde{s}] \left[1 + \frac{da}{dw} \tilde{s} \right] \tilde{s} dF(\tilde{s}) = 0$$

which gives

$$\frac{da}{dw} = -\frac{\int u'' [w + a\tilde{s}] \tilde{s} dF(\tilde{s})}{\int u'' [w + a\tilde{s}] \tilde{s}^2 dF(s)}$$

since -$\frac{u''(\cdot)}{u'(\cdot)}$ is decreasing, then

$$-\frac{u'' [w + a\tilde{s}]}{u' [w + a\tilde{s}]} \leq -\frac{u''(w)}{u'(w)}$$

if $s > 0$ while the inequality is reversed whenever $\tilde{s} < 0$. But this implies

$$-\frac{u'' [w + a\tilde{s}]}{u' [w + a\tilde{s}]} \tilde{s} \leq -\frac{u''(w)}{u'(w)} \tilde{s}$$

for all $\tilde{s}$. Then integrating

$$-\int u''(w + a\tilde{s}) \tilde{s} dF(\tilde{s}) \leq$$

$$-\frac{u''(w)}{u'(w)} \int u' [w + a\tilde{s}] \tilde{s} dF(\tilde{s}) = 0$$

This gives $\frac{da}{dw} \geq 0$

1.2 Stochastic Dominance

- A lottery F is said to *first order stochastic dominate* G (or $F \succcurlyeq_{FSD} G$) if for *every* nondecreasing function u

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

- stochastic dominance is an alternative way of ranking lotteries but it is an incomplete ordering

- Let $\mathcal{X}$ be a compact subset of $\mathbf{R}$. Then $F \succcurlyeq_{FSD} G$ if and only if $F(x) \leq G(x)$ for all x

- *Proof:* $F \succcurlyeq_{FSD} G \Rightarrow F(x) \leq G(x)$ for all x :Stochastic dominance implies

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

for all non decreasing u. Let

$$u_y(x) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{otherwise} \end{cases}$$

This is a non-decreasing function, so for all y

$$\int u_y(x)\,dF(x) \geq \int u_y(x)\,dG(x)$$

which implies

$$1 - F(y) \geq 1 - G(y)$$

or

$$F(y) \leq G(y)$$

- *second part:* $F(x) \leq G(x)$ for all $x \Rightarrow \int u(x)\,dF(x) \geq \int u(x)\,dG(x)$ for all non-decreasing u : Since F is compact

$$\int u(x)\,dF(x) =$$

$$u(y)\,F(y)\big|_a^b - \int F(x)\,u'(x)\,dx$$

$$= u(b) - \int F(x)\,u'(x)\,dx$$

$$\geq u(b) - \int G(x)\, u'(x)\, dx$$

$$= \int u(x)\, dG(x)$$

- *Second Order Stochastic Dominance* $F \succcurlyeq_{SSD} G$ if

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

for every non-decreasing concave function u

- Let x be a random variable distributed according to F and let z be any other random variable with mean conditional on x is zero for all values of x- this is called a *mean preserving spread.* The distribution F second order stochastically dominates the distribution G of the random variable $x+z$. To see this observe that for u concave

$$\int u(x+z)\, dG(x+z)$$

$$= \int u(x+z)\, dG(z|x)\, dF(x)$$

$$\leq \int u\left(x + \int z dG(z|x)\right) dF(x)$$

$$= \int u(x)\, dF(x)$$

where the inequality follows by Jensen's inequality.