# 1 Rational Expectations Equilibrium

- $S$ - the (finite) set of states of the world - also use $S$ to denote the number
- $m$ - number of consumers
- $K$- number of physical commodities
- each trader has an endowment vector $\omega^i \in \mathbb{R}^{kS}$. Firms production plans are fixed in what follows, so this vector could be $\sum_{j=1}^{m} \lambda_{ij} y_j$ where $y_j \in \mathbb{R}^{kS}$ is firm $j$'s production plan and $\lambda_{ij}$ is $i$'s share of firm $j$.
- $R - S \times J$ matrix of asset returns .
- the assets are traded in period 0 with spot markets in period 1 that open after all assets have paid off.

- $\Theta_i$ the set of types for trader $i$ - this information is private to trader $i$. Suppose this set is finite with $T_i$ elements.

- $\Theta = \prod_{i=1}^{n} \Theta_i$ is private information, $\Theta \times S$ is the stuff that is unknown

- $i$'s type $\theta_i$ will be intereted as his private information or signal about the state.

- let $G(\theta_1, \ldots \theta_m, s)$ be the probability with which the state is $s$ and each of the traders has type $\theta_i$. It is important in this story types and the state are correlated.

- for convenience we will sometimes write $G(\theta_1, \ldots, \theta_m, s)$ as $G(\theta_i, \theta_{-i}, s)$

- trader $i$'s *posterior belief* conditional on his type $\theta_i$ about the probability with which state $s$ occurs is

$$\frac{\sum_{\theta_{-i}} G\left(\theta_i, \theta_{-i}, s\right)}{\sum_{\theta_{-i}, s'} G\left(\theta_i, \theta_{-i}, s'\right)} \tag{1}$$

- as you will see, traders beliefs about the probability with which different states occur will be endogenous - we introduce the idea here
- let $\rho(\theta)$ be an arbitrary function from $\Theta$ into $\mathbb{R}^J$ representing the way that traders believe private information is related to asset prices.
- Define

$$b_s^i\left(\theta_i, q' | \rho\right) = \frac{\sum_{\theta'_{-i}:\rho\left(\theta_i, \theta'_i\right)=q'} G\left(\theta_i, \theta'_{-i}, s\right)}{\sum_{\theta'_{-i}:\rho\left(\theta_i, \theta'_i\right)=q', s'} G\left(\theta_i, \theta_{-i}, s'\right)}$$

be the posterior probability with which trader $i$ believes that state $s$ will occur conditional on his own type, and on the asset price vector $q'$

- this function only describes beliefs when asset prices are in the range of $\rho$. To handle other cases, use the convention that if there is no $\theta_{-i}$ such that $\rho(\theta, \theta_{-i})$ is equal to the observed vector of asset prices $q'$, then beliefs are equal to interim beliefs as defined in (1).

- one example occurs when traders believe that asset prices are independent of prices, so $\rho_0(\theta) = \rho_0$ for all $\theta$. Then for every array $\theta$, traders beliefs are equal to their interim beliefs

$$b_s^i(\theta_i, q' | \rho_0) = \frac{\sum_{\theta_{-i}} G(\theta_i, \theta_{-i}, s)}{\sum_{\theta_{-i}, \theta_0'} G(\theta_i, \theta_{-i}, \theta_0')}$$

- a *Radner equilibrium relative to beliefs* $\rho$ is a price function $q : \Theta \to \mathbb{R}^J$, a set of spot price expectations $p : \mathbb{R}^J \times \Theta_0 \to \mathbb{R}^k$, a set of consumption plans $\left\{x_i : \Theta_i \times \mathbb{R}^J \times S \to \mathbb{R}^k\right\}$, and a set of portfolios $\left\{z_i : \Theta_i \times \mathbb{R}^J \to \mathbb{R}^J\right\}$ such that for every $\theta \in \Theta$:

  - each trader's consumption plan $x_{is}\left(\theta_i, q\left(\theta\right)\right)$ is affordable in every state using the portfolio $z_i\left(\theta_i, q\left(\theta\right)\right)$ and provides at least as much expected utility as any other affordable plan, i.e.,

$$\sum_{s=1}^{S} b_s^i\left(\theta_i, q\left(\theta\right)|\rho\right) u_i\left(x_{is}\left(\theta_i, q\left(\theta\right)\right)\right) \geq \sum_{s=1}^{S} b_s^i\left(\theta_i, q\left(\theta\right)|\rho\right) u_i\left(x'_s\right.$$

    for every alternative plan $x'$ that satisfies

$$p_s\left(q\left(\theta\right)\right) x'_s \leq p_s\left(q\left(\theta\right)\right) \omega_{is} + \sum_{j=1}^{J} p_{s1}\left(q\left(\theta\right)\right) r_{js} z'_{js}$$

for each $s$ and

$$q(\theta) z' \leq 0$$

– for each $\theta$

$$\sum_{i=1}^{n} x_i(\theta_i, q(\theta)) = \sum_{i=1}^{n} \omega_i$$

and

$$\sum_{i=1}^{n} z_i(\theta_i, q(\theta)) = 0$$

- observe that every part of the Radner equilibrium depends on the array of types $\theta$
- the environment is still extremely specialized - traders' payoffs don't depend on other traders' types, utility doesn't even depend directly on trader's own type

- **Example:** $m = 2, k = 1, \Theta_0 = \{1, 2\}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Trader 1 has either the type $\theta_1$ or the type $\theta_2$ while trader 2 always has the type $\overline{\theta}$. The joint distribution of types and outcomes is given by

| Event | Probability |
|---|---|
| $\theta_1, \overline{\theta}, 2$ | $3/8$ |
| $\theta_2, \overline{\theta}, 2$ | $1/8$ |
| $\theta_1, \overline{\theta}, 1$ | $1/8$ |
| $\theta_2, \overline{\theta}, 1$ | $3/8$ |

- the event records the type received by trader 1, the type of trader 2, and the state, either 1 or 2

- The traders have utility functions $\pi \ln(x) + (1 - \pi)\ln(y)$ where $\pi$ is the probability with which they believe state 1 will occur, $x$ is consumption in state 1 and $y$ is consumption in state 0. Each consumer has an endowment of one unit in each state

- interim and posterior beliefs for trader 1 are the same and are given by

$$b_1^1(\theta_1, q) = \frac{1/8}{(1/8) + (3/8)} = \frac{1}{4}$$

- trader 2 has interim belief given by $b_1^2(\theta_2, q) = \frac{1}{2}$

- now compute the usual Radner equilibrium (relative to interim beliefs) for each array of types $(\theta_1, \overline{\theta})$ and $(\theta_2, \overline{\theta})$ - in the first case trader 1 solves

$$\max b_1^1(\theta_1) \ln\left(1 + z_1^1\right) + \left(1 - b_1^1(\theta_1)\right) \ln\left(1 + z_2^1\right)$$

subject to

$$q_1 z_1^1 + q_2 z_2^1 \leq 0$$

- add $q_1+q_2$ to each side of the budget constraint to get a standard cobb douglas problem. The demands for consumption in state 1 are given by

$$\frac{b_1^1(\theta_1)(q_1+q_2)}{q_1}$$

for trader 1 and similarly for trader 2. Normalizing $q_2=1$ and setting the sum of demand equal to total endowment gives

$$\frac{b_1^1(\theta_1)(1+q)}{q}+\frac{b_1^2\left(\overline{\theta}\right)(1+q)}{q}=2$$

or

$$q=\frac{b_1^1(\theta_1)+b_1^2\left(\overline{\theta}\right)}{2-b_1^1(\theta_1)-b_1^2\left(\overline{\theta}\right)}$$

- the formula for the Radner equilibrium price given the array $\left(\theta_2,\overline{\theta}\right)$ is computed exactly the same way

- Now when trader 1 has type $\theta_1$ the price of asset 1 should be

$$q = \frac{\frac{1}{4} + \frac{1}{2}}{2 - \frac{1}{4} - \frac{1}{2}} = \frac{3}{5}$$

while if his type is $\theta_2$ the price of asset 1 is 1.

## Rational Expectations Equilibrium

- a *rational expectations equilibrium* is a Radner equilibrium relative to some belief $\rho$ such that $q(\theta) = \rho(\theta)$.
- In words, traders understand the true relationship between hidden information and prices
- notice that this involves a fixed point - given beliefs $\rho$, Radner equilibrium relative to beliefs $\rho$ imposes a restriction on the equilibrium relationship between private

information and price given by $q(\theta)$. A rational expectations equilibrium is a fixed point for which $\rho(\theta) = q(\theta)$

## Full information

- for each array of signals $\theta$, define beliefs for each trader as

$$\pi_{is}(\theta) = \frac{G(\theta_i, \theta_{-i}, s)}{\sum_{s'} G(\theta_i, \theta_{-i}, s')}$$

  and let $q^f(\theta)$ be any selection from the ordinary Radner equilibrium asset price correspondence associated with these beliefs. Let $p_s^f(\theta)$ be spot prices in the full information Radner equilibrium when trader types are $\theta$, and let $x_i^f(\theta)$ be trader $i$'s equilibrium consumption plan in the full information equilibrium.

- traders don't make inferences from price here, they simply have specific beliefs which are different for each array

$\theta$

- if there are multiple Radner equilibrium for some array of types $\theta$, the function $q^f$ simply assigns one of them arbitrarily.
- the function $q^f$ is called the *full information equilibrium price function*
- the full information equilibrium price depends on the vector of types. For example, in the example given above the price of asset 1 when the array of types is $\left(\theta_1, \overline{\theta}\right)$ is

$$q = \frac{b_1^f(\theta_1) + b_1^f(\theta_1)}{2 - b_1^f(\theta_1) - b_1^f(\theta_1)}$$

and this is equal to $1/3$ when trader 1 has type $\theta_1$ and is 3 when he has type $\theta_2$

- the full information price function is *revealing* if for any $\theta \in \Theta$, the solution to $q' = q^f(\theta)$ is unique whenever it exists. In words each array of prices that occurs with positive probability with full information is consistent with only one array of possible type.

- *Theorem:* Suppose that $q^f$ is revealing. Then there is a rational expectations equilibrium with asset price function $q^f$.

- Proof: We need to show that we can construct a Radner equilibrium relative to the belief function $q^f$ in which the asset pricing function is $q^f$. To do this, let each trader $i$ use any consumption plan such that $x_i(\theta_i, q(\theta)) = x_i^f(\theta)$ and let spot prices be given by any function such that $p\left(q\left(\theta\right)\right)=p^{f}\left(\theta\right)$. Then for each $\theta$

$$\sum_{s=1}^{S}b_{s}^{i}\left(\theta_{i},q^{f}\left(\theta\right)|q^{f}\right)u_{i}\left(x_{is}\left(\theta_{i},q\left(\theta\right)\right)\right)=$$

$$\sum_{s=1}^{S}\frac{G\left(\theta_{i},\theta_{-i},s\right)}{\sum_{s'}G\left(\theta_{i},\theta_{-i},s'\right)}u_{i}\left(x_{is}^{f}\left(\theta\right)\right)\geq$$

$$\sum_{s=1}^{S}\frac{G\left(\theta_{i},\theta_{-i},s\right)}{\sum_{s'}G\left(\theta_{i},\theta_{-i},s'\right)}u_{i}\left(x_{s}'\right)$$

for any consumption plan satisfying the budget constraints

$$p_{s}^{f}\left(\theta\right)x_{s}'\leq p_{s}^{f}\left(\theta\right)\omega_{is}+\sum_{j=1}^{J}p_{s1}^{f}\left(\theta\right)r_{js}z_{js}'$$

for each $s$ and

$$q^f(\theta) z' \leq 0$$

All this follows from properties of the Radner equilibrium when types are $\theta$. Substituting back in the definitions $x_i(\theta_i, q(\theta)) = x_i^f(\theta)$ and $p(q(\theta)) = p^f(\theta)$ shows that the tentative Radner equilibrium relative to beliefs satisfies the optimality conditions of equilibrium. The market clearing conditions follow in a similar way.

- we could turn this reasoning around. Let $q(\theta)$ be a rational expectations equilibrium price function. Suppose that for every $q'$ in the range of $q(\cdot)$, there is a unique $\theta$ such that $q' = q(\theta)$. Then we could say that the rational expectations equilibrium is fully revealing. It is straightforward that if it is fully revealing, then it must coincide with a full information Radner equilibrium price function.

- now illustrate the theorem with the example given above. The full information price function is

$$q^f(\theta) = \begin{cases} (1/3, 1) & \text{if } \theta = (\theta_1, \overline{\theta}) \\ (3, 1) & \text{if } \theta = (\theta_2, \overline{\theta}) \end{cases}$$

  and this is one to one

- given these beliefs, the price vector $(1/3, 1)$ prevails when the type received by 1 is $1/4$ and each trader believes that state 1 occurs with probability $1/4$. The conditional probability of state 1 given this price ratio is exactly $1/4$ as required in a rational expectations equilibrium

- this example is special in that the state that occurs has no effect on preferences or endowments. In examples like this, the states are often referred to as sunspots.

- notice that in the fully revealing REE for this example there is no trade

- in the simple Walrasian equilibrium where traders ignore the information in prices, there will be trade because traders have different posterior beliefs about the sunspots - thus they are willing to bet with one another.

- a more interesting example can be constructed in the case where trader 1's endowment is *correlated* with the state.

- here is a new probability matrix

| Event | Probability |
|---|---|
| $\theta_1, \overline{\theta}, s_1, (1,1)$ | $5/16$ |
| $\theta_2, \overline{\theta}, s_2, (\omega,1)$ | $4/16$ |
| $\theta_2, \overline{\theta}, s_1, (1,1)$ | $3/16$ |
| $\theta_1, \overline{\theta}, s_3, (1,1)$ | $4/16$ |

- in this example, $s_1$ and $s_3$ are sunspot states and endowments are exactly as before, 1 unit for each consumer.

However state $s_2$ is such that consumer 1 has an endowment of $\omega \neq 1$.

- suppose the Radner matrix is

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

- now trader 1 maximizes

$$b_{11}(\theta)\, ln\,(1 + z_{11}) + b_{12}(\theta')\, ln\,(\omega + z_{12}) + b_{13}(\theta')\ln(1 + z_{12})$$

subject to the contraint that $q_1 z_{11} + z_{12} \leq 0$, where $\theta'$takes the values $\theta_1$ and $\theta_2$.

- Observe that when $\theta' = \theta_1$, trader 1 thinks state 2 is impossible, when $\theta' = \theta_2$, he thinks state 3 is impossible.

- In particular, when $\theta' = \theta_1$, trader 1 knows his endowment will always be 1. So his maximization problem is

$$\max b_{11}(\theta_1)\, ln\,(1 + z_{11}) + (1 - b_{11}(\theta_1))\, ln\,(1 + z_{12})$$

subject to $qz_{11} + z_{12} \leq 0$ (or $q(1 + z_{11}) + (\omega + z_{12}) \leq q + 1$) when 1 has type $\theta_1$, which is a Cobb-Douglas problem

- if $\theta' = \theta_2$, then 1 believes that state 3 is impossible but is no longer sure of his endowment. His problem is

$$\max b_{11}(\theta_2)\, ln\,(1 + z_{11}) + (1 - b_{11}(\theta_2))\, ln\,(\omega + z_{12})$$

subject to $qz_{11} + z_{12} \leq 0$ (or $q(1 + z_{11}) + (\omega + z_2) \leq q + \omega$).

- Player 2 always has an endowment of 1, and because the Radner securities don't permit her to trade consumption between state 2 and 3, she always has the same consumption in states 2 and 3. As a consequence she maximizes

$$b_{21} \ln(1 + z_{21}) + (1 - b_{21}) \ln(1 + z_{22})$$

subject to $qz_{21} + z_{22} \leq 0$, which is the same as $q(1 + z_{21}) + (1 + z_{22}) \leq q + 1$.

- Using the trick above, 1's demand for consumption is state 1 is

$$b_{11}(\theta_1) \frac{q+1}{q}$$

when his type is $\theta_1$ and

$$b_{11}(\theta_2) \frac{q+\omega}{q}$$

when his type is $\theta_2$.

- similarly, trader 2's demand for consumption in state is $b_{21}\frac{q+1}{q}$

- the solution when no one believes price conveys information

$$b_{11}\left(\theta_1\right)\frac{q+1}{q}+b_{21}\frac{q+1}{q}=2$$

when one has type $\theta_1$ and

$$b_{11}\left(\theta_2\right)\frac{q+\omega}{q}+b_{21}\frac{q+1}{q}=2$$

when 1 has type $\theta_2$

- substituting posterior beliefs for type

$$\frac{5}{9}\frac{q+1}{q}+\frac{1}{2}\frac{q+1}{q}=2$$

which has solution $q=\frac{19}{17}$

- if $\theta' = \theta_2$ then the market clearing condition is

$$\frac{3}{7}\frac{q+\omega}{q} + \frac{1}{2}\frac{q+1}{q} = 2$$

which has solution $q = \frac{6\omega+7}{15}$

- notice that there is a value of $\omega = 83/51$ at which the prices are the same, this supports a ree where no information is revealed.

- **Fully Revealing Equilibrium:** If the full information price function is one to one, there is a fully revealing equilibrium

- the market clearing condition under interim beliefs when 1 has type $\theta_1$ is

$$\frac{5}{9}\frac{q+1}{q} + \frac{5}{9}\frac{q+1}{q} = 2$$

and the market clearing price is $\frac{5}{4}$

- if 1 has types $\theta_2$ the market clearing price is

$$\frac{3}{7}\frac{q+\omega}{q} + \frac{3}{7}\frac{q+1}{q} = 2$$

  the market clearing price in this case is $\frac{3\omega+3}{8}$

- notice this full information price function is one to one as long as $\omega \neq \frac{7}{3}$.

- **Partially Revealing Equilibrium:** this example extends the rational expectations idea beyond the Radner security framework, it also illustrates how to use the auctioneer player to think about non-revealing equilibria.

- there are three traders, a buyer who observes the state $S$, and two sellers who don't. They believe the state is uniformly distributed on the interval $[0, 1]$. The buyer's payoff if he owns the good is $1 - s$ ($s$ is the state), seller 1 has cost $s$ the other has cost $\frac{1}{4} + \frac{3}{8}s$

- what is the rational expectations equilibrium?
- there are multiple equilibrium outcomes, we focus on one - we need a strategy for the auctioneer, beliefs, and demands and supplies for the buyers and sellers.
- beliefs of both sellers are just functions of price - to find them

## No Trade Theorem (the generalized second welfare theorem)

- Traders ex ante beliefs about the state are the beliefs they have before they see their signals.

- the ex ante payoff associated with an outcome function $\omega : S \times \Theta \rightarrow X^{Kn}$ is given for trader $i$ by

$$\sum_{\theta,s} G(\theta, s)\, u_{is}(\omega_{is}(\theta))$$

- if the outcome function is independent of $\theta$ then this expression becomes

$$\sum_{s} G(s)\, u_{is}(\omega_{is})$$

  where $G(s)$ is the marginal probability of $s$

- a trader is weakly risk averse if for an $\lambda$, $s$ and any pair $\omega_s$ and $\omega'_s$, $\lambda u_{is}(\omega_s) + (1-\lambda)\, u_{is}(\omega'_s) \leq u_{is}(\lambda \omega_{is} + (1-\lambda)\, \omega'_{is})$.

- an outcome function $\{\omega_{is}\}$ is *ex ante efficient* if there is no alternative outcome function $\{\omega'_{is}\}$ for which every traders's ex ante payoff is at least as high, and some player's ex ante payoff is strictly higher

- (note this only checks alternative outcomes that don't depend on $\theta$)
- there are no mutual gains to trade across states when an outcome is ex ante efficient
- Thm: Suppose the state contingent endowment is ex ante efficient, and every trader is weakly risk averse. Then in every rational expectations equilibrium, each trader's payoff is the same as the payoff he would get by not trading at all.
- Proof: Every rational expectations equilibrium consists of a price function $q : \Theta \to \mathbb{R}^J$, and a set of beliefs that

satisfy

$$b_i^s\left(\theta_i, q_0 | q\left(\theta\right)\right) = \frac{\sum_{\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i, \theta_{-i}, s\right)}{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i, \theta_{-i}, s'\right)}$$

for every price vector $q_0$ that prevails with positive probability in this equilibrium.

When players have these beliefs, there is, for each array of types $\theta$, a collection of portfolios $\left\{z_i\left(\theta\right)\right\}_{i=1,\ldots n}$ in $\mathbb{R}^J$, and consumption plans $\left\{x_i\left(\theta\right)\right\}_{i=1,\ldots,m}$ in $\mathbb{R}^{KS}$, along with spot price vectors $\left\{p_s\left(\theta\right)\right\}_{s=1,\ldots S}$ in $\mathbb{R}^K$, such that for each $\theta$, each trader's consumption plan is affordable at prices $p_s\left(\theta\right)$ in each state given his portfolio trade $z_i\left(\theta\right)$ and maximizes his expected utility subject to the securites market budget constraint, and every ex post budget constraint. Since not trading is always a feasible option, the outcome function $x_i(\theta)$ for player $i$ satisfies

$$\sum_{s=1}^{S} \frac{\sum_{\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i,\theta'_{-i},s\right)}{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i,\theta'_{-i},s'\right)} u_i\left(x_{is}(\theta)\right) \geq$$

$$\sum_{s=1}^{S} \frac{\sum_{\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i,\theta'_{-i},s\right)}{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i,\theta'_{-i},s'\right)} u_i\left(\omega_{is}\right) \qquad (2)$$

for every securities price vector $q_0$ that occurs with positive probability.
Suppose, contrary to the theorem, that this inequality is strict for some consumer when he has some type $\theta_i$ and some price $q_0$ occurs.

- Write $G(\theta_i)$ to be the marginal probability of $\theta_i$. The probability that the price $q_0$ occurs conditional on $\theta_i$ is just

$$\Pr\left(q_0|\theta_i\right) = \frac{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0} G\left(\theta_i,\theta'_{-i},s'\right)}{G\left(\theta_i\right)} \tag{3}$$

- you are summing up the probability of all the signals for other traders that would result in the observed price $q_0$ anf $i$'s signal $\theta_i$.

- Now multiply both sides of (2) $\Pr\left(q_0|\theta_i\right) G\left(\theta_i\right)$ (both strictly positive), then sum over all the $q_0$ that $i$ believes possible and all the $\theta_i$ that are possible, and use the fact that the inequality above is strict for some $\theta$, we get

$$\sum_{\theta_i} G\left(\theta_i\right) \sum_{q'_0} \Pr\left(q'_0|\theta_i\right) \frac{\sum_{\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q'_0} G\left(\theta_i,\theta'_{-i},s\right)}{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q'_0} G\left(\theta_i,\theta'_{-i},s'\right)} u_i\left(x_{is}\left(\theta\right)\right)$$

$$\sum_{\theta_i} G(\theta_i) \sum_{q_0'} \Pr(q_0'|\theta_i) \frac{\sum_{\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0'} G\left(\theta_i, \theta'_{-i}, s\right)}{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0'} G\left(\theta_i, \theta'_{-i}, s'\right)} u_i(\omega_{is})$$

Since the numerator in (3) is equal to the denominator of the term

$$\frac{\sum_{\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0'} G\left(\theta_i, \theta'_{-i}, s\right)}{\sum_{s',\theta'_{-i}:q\left(\theta_i,\theta'_{-i}\right)=q_0'} G\left(\theta_i, \theta'_{-i}, s'\right)}$$

and the $G(\theta_i)$ terms cancel, the inequality becomes

$$\sum_{s=1}^{S} \sum_{\theta} G(\theta, s) u_i(x_{is}(\theta)) >$$

$$\sum_{s=1}^{S} \sum_{\theta} G(\theta, s) u_i(\omega_{is})$$

However,

$$\sum_{s=1}^{S}\sum_{\theta}G\left(\theta,s\right)u_i\left(x_{is}\left(\theta\right)\right)=$$

$$\sum_{s=1}^{S}\sum_{\theta}G\left(s\right)G\left(\theta|s\right)u_i\left(x_{is}\left(\theta\right)\right)\leq$$

$$\sum_{s=1}^{S}G\left(s\right)u_i\left(\sum_{\theta}G\left(\theta|s\right)x_{is}\left(\theta\right)\right)$$

by the fact that trader $i$ is weakly risk averse. So

$$\sum_{s=1}^{S}G\left(s\right)u_i\left(\sum_{\theta}G\left(\theta|s\right)x_{is}\left(\theta\right)\right)\geq$$

$$\sum_{s=1}^{S}\sum_{\theta}G\left(\theta,s\right)u_i\left(\omega_{is}\right)$$

with strict inequality holding for at least one $i$. Since it is straightforward to show that the outcome function $\sum_{\theta} G(\theta|s) x_{is}(\theta)$ is feasible, this contradicts the assumption that the endowment is ex ante efficient.