

Arrow Debreu Equilibrium

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- ▶ $\Theta_0 = \{s_1, \dots, s_S\}$ - the set of (unknown) states of the world assuming there are S unknown states.
- ▶ information is complete but imperfect
- ▶ n - number of consumers
- ▶ K - number of physical commodities

- ▶ m number of firms
- ▶ this method proceeds by incorporating the state dependency directly in to the allocation - an *allocation* lies in

$$X = \mathbb{R}^{KnS}$$

a list of consumption plans for each individual in each state,

- ▶ AD is kind of Walrasian equilibrium
- ▶ each firm has a state contingent production set $Y_{js} \subset \mathbb{R}^K$

- ▶ an allocation $\{x_i\}_{i=1,\dots,n}$ if there is a collection of state contingent production plans $\{y_{js}\}_{j=1,\dots,m,s=1,\dots,S}$ satisfying $y_{js} \in Y_{js}$ for all j and s and

$$\sum_{i=1}^n x_{is} \leq \sum_{j=1}^m y_{js}$$

for each s

- ▶ Preferences subjective expected utility - π_s^i - the probability with which consumer i thinks that state s will occur
- ▶ consumer i 's utility for allocation x depends only on x_i and is given by $U_i(x)$ is $\sum_{s=1}^S \pi_s^i u_i^s(x_{is})$

- ▶ auctioneer announces a price vector $p \in R^{kS}$ (one price for each commodity in each state)
- ▶ each consumer owns the share λ_{ij} of firm j 's shares and a state contingent endowment ω_{is}
- ▶ firms maximize profits
- ▶ an *Arrow-Debreu* equilibrium is a Walrasian equilibrium for the problem above is a triple (p^*, x^*, y^*)

- ▶ (profit maximization) each firm's state contingent production plan y_j^* maximizes $p^* y_j$ subject to $y_j \in Y_j$
- ▶ (utility maximization) each consumer's consumption plan $x_i^* = \{x_{is}^*\}_{s=1, \dots, S}$ maximizes

$$\sum_s \pi_s^i u_i(x_{is})$$

subject to

$$\sum_{s=1}^S p_s^* x_{is} \leq \sum_{s=1}^S p_s^* \omega_{is} + \sum_{s=1}^S \sum_{j=1}^m \lambda_{ij} p_s^* y_{js}^*$$

- ▶ (feasibility)

$$\sum_{i=1}^n x_{is}^* \leq \sum_{i=1}^n \omega_{is} + \sum_{j=1}^m y_{js}^*$$

for each state s

- ▶ the preferences above satisfy the private value property, reasonable monotonicity restriction then ensure that the first Welfare Theorem holds so that an Arrow Debreu Equilibrium is Pareto Optimal which means that every Arrow Debreu equilibrium is interim efficient

- ▶ every Arrow-Debreu equilibrium allocation is *ex post efficient* provided $\pi_s^i > 0$ for each i and s .
- ▶ Proof: Let $\{x^*, y^*, p^*\}$ be the Arrow Debreu equilibrium allocation. By the first welfare theorem, there does not exist an alternative feasible allocation $\{x, y, p\}$ such that

$$\sum_{s=1}^S \pi_s^i u_i^s(x_{is}) \geq \sum_{s=1}^S \pi_s^i u_i^s(x_{is}^*)$$

with strict inequality holding for some i .

- ▶ Suppose that the allocation $\{x^*, y^*, p^*\}$ is not ex post efficient. Then there is an alternative feasible allocation $\{x', y', p'\}$ such that

$$u_i^s(x'_{is}) \geq u_i^s(x^*_{is})$$

with strict inequality holding for at least one i and s . But then it is immediate that

$$\sum_{s=1}^S \pi_s^i u_i^s(x'_{is}) \geq \sum_{s=1}^S \pi_s^i u_i^s(x^*_{is})$$

with strict inequality holding for at least one i . This contradicts the first welfare theorem.

Radner Equilibrium

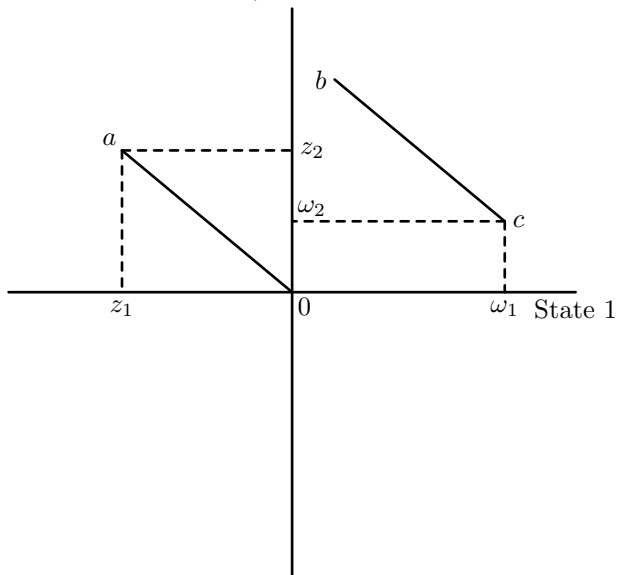
- ▶ now suppose that there are S *forward* markets for contingent delivery of good 1.
- ▶ z_s^i is the quantity of good 1 that consumer i buys on the forward market. Each such contract guarantees that the consumer will receive exactly 1 unit of commodity 1 if (and only if) state s occurs.
- ▶ this special kind of forward contract is referred to as an *Arrow security*
- ▶ let $q = \{q_1, \dots, q_S\}$ denote the vector of S *forward prices*

- ▶ once the state is realized deliveries occur as specified by the forward contracts (so consumer i get z_s^i units of good 1 in state s) - then trade in the other commodities occurs on a set of *spot markets*
- ▶ let $\{p_{1s}, \dots, p_{Ks}\}$ be the prices that each consumer i expects to prevail on the spot markets for each of the K commodities in state s .
- ▶ the budget constraint faced by consumer i on the forward markets is just

$$\sum_{s=1}^S q_s z_s^i \leq 0$$

- ▶ figure 1 gives a typical forward trade in the case where $S = 1$ and $K = 1$ in the figure, there are 2 states.

Consumption/Delivery in State 2



- ▶ The forward trade is given by the line segment $0a$ which has the trader selling z_1 units of good 1 forward (there is only one good) in state 1 and buying z_2 units of good 1 forward in state 2.
- ▶ the trader's state contingent consumption is given by adding the forward trades to the trader's endowment (ω_1, ω_2)
- ▶ in the first figure, the endowment is at point c while the forward trades take the consumer to point b
- ▶ the second figure shows some of the interesting things that can occur when there are multiple goods

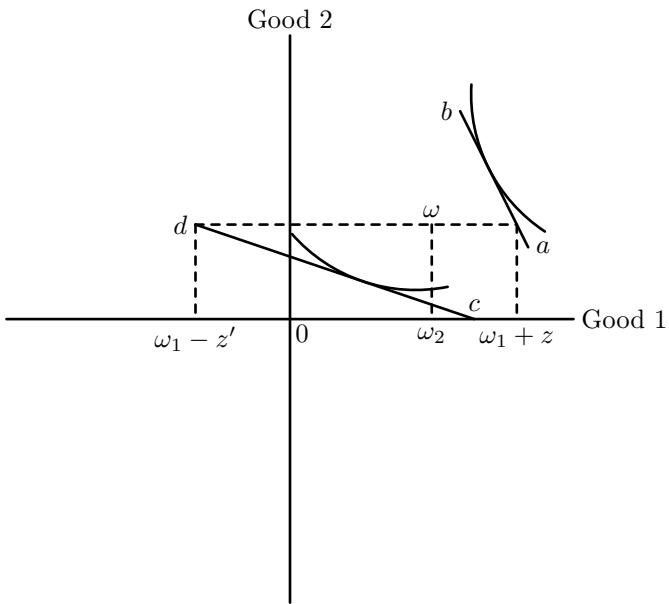


Figure: 2 Physical Goods, 2 States

- ▶ the trader's endowment is the same in both state 1 and state 2 and is given by the point ω
- ▶ the trader buys z units of good 1 forward in state 1. If the state is realized this takes the trade to the budget line ab and he trades along this to the tangency
- ▶ to pay for these forward deliveries, the trader sells z' units of good 1 forward in state 2 moving to the budget line cd when the state occurs
- ▶ notice that the forward trade in good 1 is larger than the consumer's total endowment of good 1 in state 2 - this is called *selling short*. It means that the consumer commits himself to buy good 1 in state 2 to meet his forward commitments.

Radner Equilibrium with Arrow Securities

- ▶ describing firms' preferences in Radner equilibrium is complicated, so to start, fix the production plans of each firm in each state and assume there is no trade in firms shares
- ▶ consumers trade arrow securities - for each state s there is an arrow security that pays 1 unit of good 1 if and only if state s occurs. No one has any endowment of arrow securities
- ▶ an allocation rule is, as always $X = \mathbb{R}^{nKS}$
- ▶ consumer's preferences over any allocation rule x are $U_i(x_i) = \sum_{s=1}^S \pi_s^i u_i^s(x_{is})$ - consumers only care about their own consumption

- ▶ to describe the institutions, each consumer chooses a trade vector $z_i \in \mathbb{R}^S$ describing the quantity of arrow securities bought and sold
- ▶ there is a *common belief* $\{p_1, \dots, p_s\}$ describing the spot price vectors that every consumer believes will prevail in each state s
- ▶ there is a price vector $q \in \mathbb{R}_+^S$ describing the prices of the Arrow Securities.

► A *Radner Equilibrium* is a set (x^*, z^*, q^*, p^*) of consumption plans, forward trades, securities prices and spot prices such that

1. (feasibility) $\sum_{i=1}^m z_i^* = 0$; and for all $s \in S$
 $\sum_{i=1}^m x_{is}^* = \sum_{i=1}^m \omega_s^i + \sum_{j=1}^m \lambda_{ij} p^* y_{js}$
2. for each i , $\{x_{is}^*\}, z_i^*$ maximizes

$$\sum_{s=1}^S \pi_s^i u_i(x'_s)$$

subject to the constraints

$$p^* x' \leq p^* \omega_s^i + p_{1s}^* z'_s + \sum_{j=1}^m \lambda_{ij} p^* y_{js}$$

for each $s = 1, \dots, S$; and

$$q^* z' \leq 0$$

- ▶ Every Radner Equilibrium is interim efficient (or pareto optimal).
- ▶ the proof introduces one of the most important ideas in the theory of asset pricing - equilibrium in asset markets generates a set of implicit *state contingent prices*
- ▶ in this approach asset traders act as if they were trading along a budget set defined by a set of state contingent prices

- ▶ *Proof:* Technique - use the Radner equilibrium prices to construct a set of state contingent prices - then show that the feasible set of consumption plans is the same under both price systems. Radner equilibrium delivers asset prices q and spot prices p_s in state s . Define

$$\lambda_s = \frac{q_s}{p_{1s}}$$

and let $\lambda = \{\lambda_1, \dots, \lambda_S\}$. Define state contingent prices equal to $p_s \lambda_s$. Now show that a consumption plan $\{x'_s\}$ for $s = 1, \dots, S$ can be supported by some set of asset trades in the Radner equilibrium if and only if it can be supported by direct trade at state contingent prices $\{p_s \lambda_s\}$.

- ▶ step 1 - $\{x'_s\}$ is feasible in a Radner equilibrium with asset prices q and spot prices p_s then it must be feasible with direct trade at state contingent prices $p_s \lambda_s$. Radner feasibility implies that there is some vector z' of asset trades such that $qz' \leq 0$ and

$$p_s x'_s \leq p_s \omega_s^i + p_{1s} z'_s$$

Since the budget set is homogenous of degree zero this implies that

$$p_s \lambda_s x'_s \leq p_s \lambda_s \omega_s^i + p_{1s} \lambda_s z'_s$$

which implies by definition that

$$p_s \lambda_s x'_s \leq p_s \lambda_s \omega_s^i + q_s z'_s$$

- ▶ Sum both sides over s to get

$$\sum_{s=1}^S p_s \lambda_s x'_s \leq \sum_{s=1}^S p_s \lambda_s \omega_s^i + \sum_{s=1}^S q_s z'_s$$

which by the feasibility of the asset trade z'_s implies that

$$\sum_{s=1}^S p_s \lambda_s x'_s \leq \sum_{s=1}^S p_s \lambda_s \omega_s^i$$

which says that the consumption plan is feasible at state contingent prices $\{p_s \lambda_s\}$

- ▶ step 2 the consumption plan x'_s is feasible at state contingent prices $p_s \lambda_s$. Then

$$\sum_{s=1}^S p_s \lambda_s x'_s \leq \sum_{s=1}^S p_s \lambda_s \omega_s^i$$

Define the asset trade

$$z'_s = \frac{p_s x'_s - p_s \omega_s^i}{p_{s1}}$$

- ▶ This asset trade trivially makes the consumption x'_s affordable at the Radner equilibrium spot prices. Further

$$\sum_{s=1}^S q_s z'_s = \sum_{s=1}^S q_s \frac{p_s x'_s - p_s \omega_s^i}{p_{s1}} = \sum_{s=1}^S \lambda_s (p_s x'_s - p_s \omega_s^i) \leq 0$$

- ▶ since the set of feasible consumption plans at Radner prices is identical to the set of feasible plans at state contingent prices $p_s \lambda_s$ these plans must maximize consumers utility at the state contingent prices. Since they are feasible as required by Radner equilibrium, the state contingent prices $\lambda_s p_s$ along with the consumption plans $\{x_s^i\}$ constitute an Arrow Debreu equilibrium, and are therefore pareto optimal.

Incomplete Markets

- ▶ suppose there are J assets that pay off in units of good 1. Each asset $j = 1, \dots, J$ generates a vector of returns $r_j \in \mathbf{R}_{++}^S$
- ▶ r_{js} denotes the amount of good 1 delivered in state s
- ▶ Arrow security $r_s = \{0, 0, \dots, 1, \dots, 0\}$
- ▶ un-contingent delivery $r_j = \{1, 1, \dots, 1\}$

- ▶ option to purchase security $r_j = \{r_{j1}, r_{j2}, \dots, r_{jS}\}$ at a price c gives security $r_j^o = \{0, 0 \dots r_{jS} - c, r_{jS+1} - c, \dots, r_{jS} - c\}$
- ▶ define the matrix R to be equal to

$$\begin{array}{cccc}
 r_{11} & r_{21} & \dots & r_{J1} \\
 r_{12} & \vdots & \vdots & r_{J2} \\
 \vdots & \vdots & \dots & \vdots \\
 r_{1S} & r_{2S} & \dots & r_{JS}
 \end{array}$$

- ▶ the *rank* of R is the number of linearly independent columns in R (which is, of course, less than or equal to J)
- ▶ the return to a portfolio $z = \{z_1, \dots, z_J\}$ is given by Rz^T (in the rest of this section I will treat all vectors as if they were row vectors)
- ▶ an *arbitrage* is a trade z satisfying either $qz^T \leq 0, Rz^T > 0$, or $qz^T < 0, Rz^T \geq 0$
- ▶ The *separating hyperplane theorem* - Let A and B be disjoint convex subsets of \mathbf{R}^N . Then there is a non-zero $p \in \mathbf{R}^N$ such that

$$px^T \leq py^T$$

for all $x \in A$ and $y \in B$. Furthermore the inequality is strict if either x or y are interior points of A or B .

- ▶ if there is no arbitrage at asset price vector q , then the asset price vector is said to satisfy the *no arbitrage condition*
- ▶ if the asset price vector q satisfies the no-arbitrage condition, then there exists a strictly positive state price vector \tilde{p} such that $\tilde{p}R = q$ (which says that the price of each security is equal to the value of the returns it delivers in terms of the state prices).

► *Proof:* Let

$$A \equiv \{(x, y) : x \in \mathbf{R}; y \in \mathbf{R}^S; x = qz^T \text{ and } y = Rz^T\}$$

The set A is a *linear subspace* of the linear space \mathbf{R}^{S+1} .
Define

$$B \equiv \{(x, y) : x \leq 0; y \geq 0\}$$

- ▶ B is the set of vectors in \mathbf{R}^{S+1} that represent payoffs associated with arbitrages. Both these sets are convex and closed, and if the no-arbitrage condition holds, they are disjoint except for the point 0. Let

$$2\varepsilon = \inf \{ \|x - y\| : x \in A, y \in B, \|y\| = 1 \} > 0$$

where the strict inequality follows from the fact that A and B have only the point 0 in common, and if $\|y\| = 1$ then $y \neq 0$. Define the set

$$B^+ \equiv \{ (x, y) + \|(x, y)\| (\xi, \theta) : (x, y) \in B; \|(\xi, \theta)\| < \varepsilon \}$$

- ▶ By construction, the sets A and $B^+ - \{0\}$ are disjoint, convex sets. Then by the separating hyperplane theorem, there is a vector $(\alpha, p) \in \mathbf{R}^{S+1}$ such that

$$\alpha x + py^T \leq \alpha x' + py'^T$$

for all $(x, y) \in A$ and $(x', y') \in B^+ - \{0\}$ with strict inequality holding for any (x', y') in the interior of B^+ . Since $0 \in A$ this implies $\alpha x' + py'^T \geq 0$. Furthermore, since $(x, y) \in A$ implies $-(x, y) \in A$, $\alpha x + py^T = 0$ for all $(x, y) \in A$. Now for any $x < 0$, the point $(x, 0)$ is interior to the set B^+ , so $\alpha x > 0$ which implies that $\alpha < 0$. Alternately, the point $(0, 0, \dots, 1, \dots, 0)$ is interior to B^+ so $p_s > 0$ for each s .

- ▶ Then for any $(x, y) \in A$

$$-\alpha x = py^T$$

Now evidently, $(q_j, r_j) \in A$ for each j so

$$-\alpha q_j = pr_j$$

or

$$q_j = -\frac{p}{\alpha} r_j$$

since $\alpha < 0$. Thus $-\frac{p}{\alpha} = \hat{p}$ is the set of strictly positive state prices.

- ▶ what is the use of state prices? - recall the theorem with Arrow Securities - we started with a budget set given by

$$\left\{ x_i : \exists z_i; qz_i \leq 0; p_s x_{is} \leq p_s \omega_{is} + \sum_{j=1}^m \lambda_{ij} p_s y_{js} + \sum_{j=1}^J p_{1s} z_{ij} r_{js} \forall s \in S \right.$$

then derived a set of arrow debreu prices \hat{p}_s such that everything that was feasible in the set above is also feasible in the set

$$\left\{ x_i : \sum_{s=1}^S \hat{p}_s x_{is} \leq \sum_{s=1}^S \left\{ \hat{p}_s \omega_{is} + \sum_{j=1}^m \lambda_{ij} \hat{p}_s y_{js} \right\} \right\}$$

- ▶ the state price theorem says that there is always a vector of arrow debreu prices like this provided the no-arbitrage condition holds
- ▶ the arrow debreu prices are given by

$$\hat{p}_{sj} = \hat{p}_s \frac{p_{sj}}{p_{1j}}$$

where \hat{p}_s are the state prices from the theorem above and p_{sj} are spot prices

- ▶ value firms production plans using arrow debreu prices
- ▶ prices might no be unique

Radner Equilibrium with a Stock Market

- ▶ generally there will be many asset price vectors that support no arbitrage, and for each of them there may be many state price vectors.
- ▶ Radner equilibrium can be used to narrow the set of admissible asset prices, because Radner equilibrium implies the existence of state prices (and as a consequence Radner equilibrium security prices will satisfy the no-arbitrage condition).
- ▶ Radner equilibrium with arbitrary assets has the same definition as it does with Arrow securities (see above) except for an alteration in the consumers' budget constraint in the spot market for each state s
- ▶ as before assume there are J securities that pay state contingent returns expressed as units of physical good 1. Security j trades at price q_j .

- ▶ assume there are m firms whose production plans are fixed in the sense that firm j produces output vector y_{js} in state s . This time however, we treat the firms' shares as assets that can be traded along with or instead of the existing Radner securities
- ▶ the return matrix R is now augmented with firms share, and looks like the following

$$\begin{bmatrix} r_{11} & \cdots & r_{J1} & \frac{p_{1Y11}}{p_{11}} & \cdots & \frac{p_{1Ym1}}{p_{11}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ r_{1S} & \cdots & r_{JS} & \frac{p_{SY1S}}{p_{1S}} & \cdots & \frac{p_{SYmS}}{p_{1S}} \end{bmatrix} = [R_J | R_m]$$

- ▶ each consumer i has an endowment λ_{ij} of firm j 's shares. These shares trade in the securities market at price q_j
- ▶ a portfolio now consists of a trade vector $z_i \in \mathbb{R}^{J+m}$, as before z_{ij} is interpreted as the amount of asset j (either a firm's shares or a radner security) that i buys or sells. In particular if i decides to sell his own shares we just subtract this from his initial endowment
- ▶ allocation $\mathbb{R}^{KmS} \times \mathbb{R}^{(J+m)m} \times \mathbb{R}^{KS} \times \mathbb{R}^{J+n}$ (consumption plans-portfolios-spot prices-securities prices)

- ▶ feasibility (x, z, p, q) is feasible if $\sum_{i=1}^m z^i = 0$ - net asset trades must be zero; and for all $s \in S$

$$\sum_{i=1}^m x_s^i = \sum_{j=1}^n y_{js}; p \geq 0; p1 = 1; q \geq 0; q1 = 1$$
 where $q1$ means the inner product of q and a vector of the same dimension consisting solely of 1s.
- ▶ the auctioneer announces spot prices p and securities prices q
- ▶ each consumer responds with a consumption plan x_i and portfolio z_i satisfying

$$p_s x_{is} \leq p_s \omega_{is} + \sum_{j=1}^n \lambda_{ij} p_s y_{js} + p_{1s} \sum_{j=1}^{J+m} z_j^i r_{js}$$

for each $s = 1, \dots, S$ and

$$\sum_{j=1}^{J+n} z_j^i q_j \leq 0$$

(notice that this would look just like the previous constraint if firms shares were untraded)

- ▶ *Theorem:* Suppose that the matrix of returns R associated with some Radner equilibrium allocation has rank S . Then there is a unique (up to normalization) set of Arrow Debreu prices associated with asset prices. Furthermore, the corresponding Radner equilibrium allocation is pareto optimal.

- ▶ *Proof:* The proof mimics the technique outlined above - we explicitly calculate the Arrow Debreu equilibrium prices that support the same allocation. Begin with the Radner equilibrium spot prices p and asset prices q . These define the returns matrix R . Since it has rank S , choose a submatrix \hat{R} consisting of S linearly independent columns of R . Define

$$\lambda_s = \frac{\left(\hat{q}\hat{R}^{-1}\right)_s}{p_{1s}}$$

where \hat{q} means the S dimensional subvector of q consisting of the prices of assets chosen in the submatrix \hat{R} . $\left(\hat{q}\hat{R}^{-1}\right)_s$ means the s^{th} component of the vector $\hat{q}\hat{R}^{-1}$. Let $\lambda = \{\lambda_1, \dots, \lambda_S\}$ and define arrow debreu prices equal to $\hat{p}_s \equiv p_s \lambda_s$.

- ▶ The non-negativity of λ_s is proved using something called Farkas Lemma. $\hat{q}\hat{R}^{-1}\hat{R} = \hat{q} \geq 0$ or to put it the way Farkas Lemma is usually expressed $\hat{R}^T (\hat{q}\hat{R}^{-1})^T = \hat{q}^T$.
By Farkas Lemma, if any of the components of $(\hat{q}\hat{R}^{-1})^T$ are negative, then the system

$$\hat{R}z \geq 0$$

and

$$\hat{q}^T z < 0$$

must have a solution - i.e., there must be an arbitrage, which is inconsistent with Radner equilibrium.

- ▶ Second, note that since a Radner equilibrium satisfies the no-arbitrage condition, then by the state price theorem, there is a set of state prices $\hat{p} \in \mathbb{R}_{++}^S$ such that $q_j = \hat{p}r_j$. Let z and z' be any two portfolios that generate the same state contingent returns $\omega \in \mathbb{R}^S$. Then

$$qz = \hat{p}Rz = \hat{p}\omega = \hat{p}Rz' = qz'$$

Every portfolio that generates the same state contingent return has the same value at Radner equilibrium security prices.

- ▶ Now show that a consumption plan $\{x'_s\}$ for $s = 1, \dots, S$ can be supported by some set of asset trades in the Radner equilibrium if and only if it can be supported by direct trade at state contingent prices $\{p_s \lambda_s\}$.

- ▶ step 1 - if $\{x'_s\}$ is feasible in a Radner equilibrium with asset prices q and spot prices p_s then it must be feasible with direct trade at state contingent prices $\hat{p}_s = p_s \lambda_s$. Radner feasibility implies that there is some vector z' of asset trades such that $qz' \leq 0$ and

$$p_s x'_s \leq \sum_{j=1}^m s_{ij} p_s y_{js} + p_{1s} \sum_{j=1}^{J+n} z'_j r_{js}$$

Since the matrix of asset returns \hat{R} spans a subspace of \mathbb{R}^S of dimension S , there must be some portfolio \hat{z} consisting of trades only of assets in the set of securities that compose \hat{R} such that

$$p_s x'_s \leq \sum_{j=1}^m s_{ij} p_s y_{js} + p_{1s} \sum_{j=1}^S \hat{z}'_j r_{js}$$

- ▶ Since the budget set is homogenous of degree zero and $\lambda_s \geq 0$ as described above, this implies that

$$p_s \lambda_s x'_s \leq \lambda_s \sum_{j=1}^m s_{ij} p_s y_{js} + p_{1s} \lambda_s \sum_{j=1}^S \hat{z}'_s r_{js}$$

Sum both sides over s to get

$$\sum_{s=1}^S p_s \lambda_s x'_s \leq \sum_{s=1}^S \lambda_s p_s \sum_{j=1}^m s_{ij} y_{js} + \sum_{s=1}^S p_{1s} \lambda_s \sum_{j=1}^S \hat{z}'_s r_{js}$$



$$\sum_{s=1}^S p_s \lambda_s x'_s \leq \sum_{s=1}^S \lambda_s p_s \sum_{j=1}^m s_{ij} y_{js} + \hat{q} \hat{R}^{-1} \hat{R} \hat{z}' =$$

$$\sum_{s=1}^S \lambda_s p_s \sum_{j=1}^m s_{ij} y_{js} + q \hat{z}' \leq \sum_{s=1}^S \lambda_s p_s \sum_{j=1}^m s_{ij} y_{js}$$

\hat{q} can be replaced with q since \hat{z} is zero for all assets other than those used in the construction of \hat{R} . since $q \hat{z}' = q z' \leq 0$ as we argued above, this says that the consumption plan is feasible at state contingent prices $\{p_s \lambda_s\}$

- step 2 the consumption plan x'_s is feasible at state contingent prices $p_s \lambda_s$. Then

$$\sum_{s=1}^S p_s \lambda_s x'_s \leq \sum_{s=1}^S p_s \lambda_s \sum_{j=1}^m s_{ij} y_{js}$$

Define the return vector

$$y'_s = \frac{p_s x'_s - p_s \sum_{j=1}^m s_{ij} y_{js}}{p_{s1}}$$

- ▶ Any portfolio that generates this vector of state contingent deliveries of good 1 will make the consumption x'_s affordable at the Radner equilibrium spot prices. Since the returns matrix \hat{R} has full rank, there is a unique portfolio such that

$$y' = \hat{R}z$$

given by $\hat{R}^{-1}y'$. Then by definition

$$qz = q\hat{R}^{-1}y' =$$



$$\begin{aligned} & \sum_{s=1}^J \lambda_s p_{1s} \left(\frac{p_s x'_s - p_s \sum_{j=1}^m s_{ij} y_{js}}{p_{1s}} \right) \\ &= \sum_{s=1}^S \lambda_s \left(p_s x'_s - p_s \sum_{j=1}^m s_{ij} y_{js} \right) \leq 0 \end{aligned}$$

- ▶ since the set of feasible consumption plans at Radner prices is identical to the set of feasible plans at state contingent prices $p_s \lambda_s$ these plans must maximize consumers utility at the state contingent prices. Since they are feasible as required by Radner equilibrium, the state contingent prices $\lambda_s p_s$ along with the consumption plans $\{x_s^i\}$ constitute an Arrow Debreu equilibrium, and are therefore pareto optimal.
- ▶ The case where the matrix of returns has rank S is sometimes referred to as the *complete markets case*. That the matrix of returns has rank equal to the number of states is sometimes called the *spanning condition*.

- ▶ if there are fewer securities than states, Radner equilibrium will not be equivalent to Arrow Debreu and will generally fail to be pareto optimal.
- ▶ note that in the proof above, the submatrix \hat{R} can be chosen arbitrarily. The argument then goes on to replace a portfolio of securities consisting of all assets with a portfolio consisting only of securities involved in \hat{R} and show that these portfolios will have the same values and support exactly the same allocations
- ▶ if there are more than S firms, then we could have chosen only firms shares and supported equilibrium in which there is not trade in the Radner securities and conversely.