Cumulative Prospect Theory

October 13, 2011

- based on Advances in Prospect Theory, by Tversky and Kahneman, Journal of Risk and Uncertainty 5:297-323 (1992)
- S is the set of states, subsets $A \subset S$ are called events
- X is a set of outcomes,
- a *prospect* is a function $f : S \to X$

- supp f is measurable with respect to a partition $\{A_i\}_{i \in I}$ where I is some index set. (i.e $f(s) = x_i$ for each $s \in A_i$)
- *then a prospect f is equivalent to a* sequence $p = \{x_i, A_i\}_{i \in I}$. Write P to represent the set of all prospects. Use the notation $\{x_i, A_i\} \in p$ to indicate that the element $\{x_i, A\}$ is contained in the sequence p.

- assuming preferences over prospects are given by a transitive, complete and continuous preference relation $\succeq$ defined on P
- then from the existence theorem, there is a utility function V such that $p \succeq p'$ if and only if $V(p) \geq V(p')$.
- rather than deducing the properties of the function V from properties of choices, K&T choose a function that seems to capture the stuff they were seeing in their experiments, in particular, non-linear preferences over probabilities when they exist, simultaneous existence of risk seeking and risk aversion, loss aversion, and things like the Ellsberg problem.

- a *capacity* is a set function $F : \mathbb{P}(S) \to [0,1]$ satisfying $F(\emptyset) = 0$, $F(X) = 1$ and $F(A) \geq F(B)$ whenever $B \subset A$. (Here $\mathbb{P}(S)$ is the set of all subsets of S.
- cumulative prospect theory assumes that preferences over prospects can be represented by a function $v : X \to \mathbb{R}$ which satisfies $v(x_0) = 0$ for some element of X, and a pair of capacities F^+ and F^-.
- v also provides an implicit ranking of the elements of X
- the three objects v, F^+ and F^- together define a utlity function as follows

- for a prospect p and any $\{x_i, A_i\} \in p$, define

$$B(\{x_i, A_i\}) = \{\cup \{A_j : v(x_j) > v(x_i)\}\}$$

and

$$W(\{x_i, A_i\}) = \{\cup \{A_j : v(x_j) < v(x_i)\}\}.$$

From these define weights

$$\pi_i(\{x_i, A_i\}) = \begin{cases} F^+(\{A_i\} \cup B(x_i)) - F^+(B(x_i)) & v(x_i) \geq 0 \\ F^-(\{A_i\} \cup W(x_i)) - F^-(W(x_i)) & v(x_i) < 0 \end{cases} \quad (0.1)$$

- the utility function over prospects is assumed to be given by

$$V(p) = \sum_{\{x_i, A_i\} \in p} \pi_i(x_i, A_i) \, v(x_i)$$

- for example, a lottery with three outcomes (like the ones we studied doing expected utility) is a prospect that splits the state space into three partition elements and assigns monetary payoffs to each element. For example, the Allais gamble has payoffs \$1000, \$500 and \$0. The prospect representation of the payoff associated with the lottery that gives \$1000 with probability .1, \$500 with probability .89 and 0 with probability .01 is a prospect that pays \$1000 is a collection of states that happens to have probability .1, similarly for the other outcomes

- the utility value of this lottery as a prospect is

$$\pi\left(\{\$1000, A_1\}\right) v\left(\$1000\right) + \pi\left(\{\$500, A_2\}\right) v\left(\$500\right) + \pi\left(\{0, A_3\}\right)$$

- an aside - suppose we want to compare this prospect with a lottery that always pays \$500 as in Allais. There would seem to be two ways to represent such a lottery. We could describe it as a prospect $p^* = \{\$500, S\}$ (in other words a prospect describe by a single partition element - the set itself), or as $p^{**} = \{\{\$500, A_1\}, \{\$500, A_2\}, \{\$500, A_3\}\}$ where A_1, A_2 and A_3 are the partition elements described in the previous slide.

- from equation (0.1) and the fact that a capacity must assign 0 weight to the empty set and weight 1 to S, we have

$$V(p^*) = v(500).$$

- a similar calculation using (0.1) on the lottery p^{**} gives its utlity value as

$$\left(F^+(A_1) + F^+(A_2) + F^+(A_3)\right) v(\$500)$$

which could be less than $v(\$500)$ since F^+ is a capacity. So prospect theory captures the inability (or reluctance to) reduce compound lotteries.

- now we want a utility function that will 'explain' the Allais experiment. To do it, start with a probability distribution G on S. Define a capacity $F^{+}(A_i)$ on S as follows

$$F^{+}(A_i) = g(G(A_i))$$

where g is some convex function from $[0,1]$ into itself which satisfies $g(0) = 1$ and $g(1) = 1$.
- notice that for any partition $\{A_i\}_{i=1,\dots,n}$ of S,

$$\sum_{i=1}^{n} F^{+}(A_i) < 1,$$

so this is a 'real' capacity.

- Recall the Allais experiment with payoffs always fixed at $\{\$1000, \$500, \$0\}$. For many decision makers, it seems plausible that the lottery $p = \{0, 1, 0\}$ is preferred to $p' = \{.1, .89, .01\}$, while $q' = \{.1, 0, .9\}$ is preferred to$q = \{0, .11, .89\}$.
- To start, represent these choices as prospects. Fix $v(0) = 0$ to make things simple. Write

$$p = \{\{\$500, S\}\},$$

$$p' = \{\{\$1000, A_1\}, \{\$500, A_2\}, \{0, A_3\}\}$$

as above, where $G(A_1) = .1$, $G(A_2) = .89$ and $G(A_3) = .01$.

- Using (0.1) and the capacity F^+ described above, $p \succeq p'$ implies

$$v(500) > F^+(A_1)\, v(1000) + \big(F^+(A_1 \cup A_2) - F^+(A_1)\big)\, v(500)$$

or

$$v(\$500)\big(1 - (F^+(A_1 \cup A_2) - F^+(A_1))\big) > F^+(A_1)\, v(\$1000)$$

$$v(\$500)(1 - (g(.99) - g(.1))) > g(.1)\, v(\$1000)$$

- Using a similar approach for $q' \succeq q$ gives

$$g(.1)\, v(\$1000) > g(.11)\, v(\$500)$$

  which will work as long as $1 - (g(.99) - g(.1)) > g(.11)$ or $1 - g(.99) > g(.11) - g(.10)$ which will be true if g is an increasing convex function.

- ellsberg with the random and uncertain boxes - each containing white and black balls, events $A_1 = \{(b,b),(b,w)\}$,$A_2 = \{(w,b),(w,w)\}$, $A_3 = \{(b,b),(w,b)\}$, $A_4 = \{(b,w),(w,w)\}$ a bet on the white ball in the random urn gives the prospect $\{(A_2,10),(A_1,0)\}$, while a bet on the white ball in the uncertain urn gives prospect $\{(A_4,10),(A_3,0)\}$, bet on black in the random urn is $\{(A_1,10),(A_2,0)\}$ and black in the uncertain urn is $\{(A_3,10),(A_4,0)\}$

- $\Pr(A_1) = .51$, $\Pr(A_2) = .49$ so subjective expected utility says the choice of the random urn in the first bet gives

$$.51v(10) + .49v(0) > \pi_w v(10) + \pi_b v(0)$$

where π_w is the subjective probability weight assigned to the event A_1. This implies $.51 > \pi_w$ if we assume $v(0) = 0$

- the preference for the random urn on the second gamble (using $v(0) = 0$) gives

$$.49v(10) > \pi_b v(10)$$

since $\pi_w + \pi_b = 1$ in subjective expected utility, this means $\pi_w > .51$ which contradicts the outcome in the first experiment.

- in prospect theory, choosing the random urn when betting on white gives

$$F^+(A_2)\, v(10) > F^+(A_4)\, v(10)$$

while choosing the random urn while betting on black implies

$$F^+(A_1)\, v(10) > F^+(A_3)\, v(10)$$

- Since F^+ is a capacity, but not necessarily a probability measure, $F^+(A_3) + F^+(A_4)$ can be less than 1 (which can't be true if F^+ is a probability measure)
- so we can explain Ellsberg by assuming $F^+(A_3)$ and $F^+(A_4)$ are small, reflecting a dislike of uncertainty.

- to illustrate the endowment effect, here is an example from a paper by Martin Pietz called Competing for loss averse consumers.
- it illustrates two things, how to model loss aversion, and how to think about the reference point.
- there are two firms A and B producing products with different characteristics and offering them to a continuum of consumers
- at the first stage of the game, two firms advertize their prices to consumers and provide descriptions that reveal to consumers that the products are differentiated in such a way that each consumer will perceive a quality difference of value d between the products.
- neither consumers nor firms know at this stage which product they will prefer. Each consumer forms an expectation λ of the probability with which he or she will buy from firm A. This is the reference point the consumers take to the second stage of the game.

- At the second stage of the game, consumers learn which of the two products A or B is better for them and make a purchase decision.
- a consumer who learns ex post that product A is best suited to him, and who proceeds to buy from firm A receives payoff that depends on his expectation

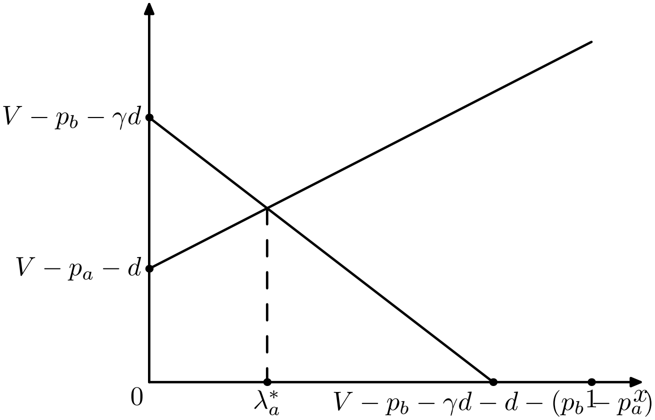
$$V - p_a - (1 - \lambda)\, d$$

- the first part of this $V - p_a$ is the intrinsic payoff associated with buying his favourite product, the second part is a perceived loss associated with the fact that he believed that with probability $(1 - \lambda)$ he was going to buy from firm B and from this perspective he is disappointed at how firm A's product compares to the one he thought he would buy
- he would also be pleased that he ended up buying at a lower price than he expected in this case, but we ignore this and focus on losses to make things simple.

- if he instead buys from firm B his payoff is

$$V - p_b - \gamma d - \lambda d - \lambda \left(p_b - p_a \right)$$

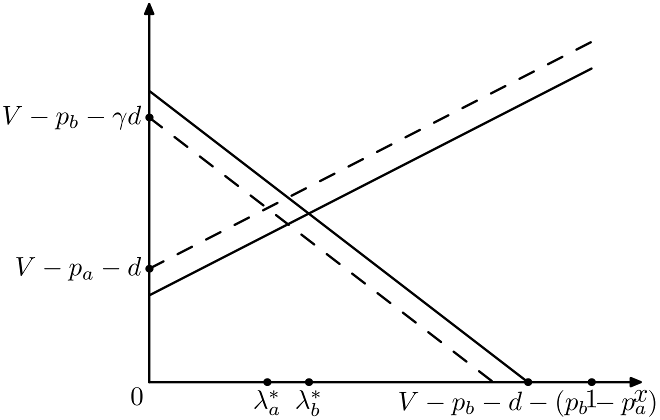
- here the term γd represents the intrinsic loss associated with buying something other than his ideal product. The reference point determined the rest - with probability λ the consumer expected to buy from firm a and his chosen product B is dissapointing different from what he expected. Furthermore, if he expected to buy from firm A, then the price p_b is disappointingly high, which is why we subtract the other term.
- his reference point λ will now determine which of the two products he buys - depending on which of these two payoffs is higher - the figure shows how the reference point affects the decision


$V - p_b - \gamma d$
$V - p_a - d$
0
λ_a^*
$V - p_b - \gamma d - d - (p_b - p_a)$
1
x

- if λ is zero (the consumer doesn't expect at all to buy from firm A, then provided the intrisic quality difference is small (i.e γd is close to zero), product B will be preferred because the loss associated with unexpectedly buying from A dominates
- one significant point is λ_a^* which is the point at which the consumer is just indifferent between the two products - suppose the parameters are such that this is less than $\frac{1}{2}$.
- the reference point then affects the decision of a type A consumer in the following way: he buys

$$\begin{cases} B & \text{if } \lambda < \lambda_a^* \\ A \text{ or } B & \lambda = \lambda_a^* \\ A & \text{otherwise.} \end{cases}$$

- a similar argument applies when the consumer is type B
- the curve for product A is shifted down by the difference γd, the curve for B is shifted up
- the indifference point λ_b^* lies to the right of the point λ_a^*
- the final restriction is that the consumers expectation λ should be 'rational' or equal to the true expectation - 'personal equilibrium'


$V - p_b - \gamma d$
$V - p_a - d$
0
λ_a^*
λ_b^*
$V - p_b - d - (p_b - p_a)$
1
x

- there are then a number of equilibria depending on the values of λ_a^* and λ_b^*
- from the figures observe that if the consumer expects to buy from firm B for sure, then he will buy from firm B whether he is type B or A, similiarly if he expects to buy from firm A for sure. In these two cases his expectations will be realized

- if $\lambda_a^* < \frac{1}{2}$, there is an equilibrium in which all the type B consumers buy from firm B and each of the type A consumers buys from firm A with probability ρ. If it happens that

$$\frac{1}{2}\rho = \lambda_a^*$$

  then the consumer's belief that he will buy from firm A with probability λ_a^* is actually right (he will be an A consumer half the time and buy in that case with probability ρ, while if he is a B consumer he won't buy from A at all)
- notice that ρ cannot exceed 1 which is why this will only work if $\lambda_a^* < \frac{1}{2}$.

- there is a similar equilibrium when the consumer believes he will buy from A with probability λ_b^*. This happens if

$$\frac{1}{2} + \frac{1}{2}\rho = \lambda_b^*$$

- from the figure above, when the consumer's reference point is λ_b^* and it turns out that A is better suited to him, then he will buy for sure. If he is better suited to B, he is indifferent between the two, so if he buys A with probability ρ, his belief is again justified.
- notice that this can only work if λ_b^* happens to be larger than $\frac{1}{2}$.

- some simple comparative statics - consumers expect to buy from A for sure (from the figure, if that is their reference point, they will always buy from A even when B turns out to be the product that is better suited to them)
- if firm A raises its price and consumers reference point doesn't change, then consumers will continue to buy from A for sure. Heuristically, firm A will have a pretty high price in equilibrium (the i-(pod,pad,book,phone) story). So (some) firms will do very well when selling to 'behavioral' consumers.

- start instead in the equilibrium where the reference point is λ_a^* and consumers buy from A with probability ρ. If firm A raises it price, then it will take a higher reference point and a higher value of ρ to make consumers indifferent. Counterintuitively raising price will increase sales. In this kind of environment you might expect both firms to have very high prices and close market shares (Canadian cell phone service is like this - very high prices despite the fact there are many firms).
- there is also a very competitive outcome in which firms set low prices, references points are interior, but if any firm raises its price, consumers revert to an equilibrium in which they expect to buy for sure from the firm who didn't raise price.