General Equilibrium as a Fixed Point

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Preliminaries

- the first variant of the basic model we study restricts to private values and complete information
- all preferences are known - there are m firms and n consumers and k physical goods
- $X = \mathbb{R}^{mnk}$ - the set of feasible allocations listing production of firms and consumption by consumers
- an outcome $\{x_1, \ldots, x_n\}$is a list of consumption vectors one for each of the consumers. Consumption and production vectors are in $\mathbb{R}^k$ assuming there are k physical commodities
- feasiblity - there are m firms each firm can produce any vector $y_j \in Y_j$ its production set

- the set of feasible allocations is

$$Y = \left\{ x \in X : \sum_{i=1}^{n} x_i \leq \sum_{j=1}^{m} y_j + \sum_{i=1}^{n} \omega_i ; y_1 \in Y_1, \ldots, y_m \in Y_m \right\}$$

ω_i is consumer i's endowment.

- the set of feasible allocations could distinguish physical commodities depending on when they are consumed, or under what conditions they are consumed. We won't worry too much about this here.

- for consumers preferences are given by $u_i\left(x_i\right)$: - consumers care only about their own consumption vector, and not about other consumer's allocations, or production choices of firms.
- Firm production sets are constants, so there are no production externalities

Institutions

- we study the Walrasian outcome which is generated the following way
- there is a price vector $p \in \mathbb{R}_+^k$
- each firm chooses a production vector to maximize profits
- each consumer i owns a share a strategy λ_{ij} of the profits of firm j.

- given a set of production choices $\{y_j\}_{j=1,\ldots m}$ each consumer has budget set

$$\left\{x_i : px_i \leq p\omega_i + \sum_{j=1}^{m} \lambda_{ij} p y_j\right\}$$

- an auctioneer announces price p
- consumers choose their favourite consumption bundle from their budget set
- firms maximize profits
- the auctioneer adjusts the price to minimize excess demand

Walrasian Equilibrium

- a *Walrasian Equilibrium* is a list $\{x^*, y^*, p^*\}$ such that
- for each firm j

$$p^* y_j^* \geq p^* y' \forall y' \in Y_j$$

- for each consumer $u_i(x_i^*) \geq u_i(x_i)$ for every i and x_i such that

$$p^* x_i \leq p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j^*$$

- and

$$\sum_{i=1}^{n} x_i^* = \sum_{j=1}^{m} y_j^* + \sum_{i=1}^{m} \omega_i$$

Existence of Walrasian Equilibrium

- firms' supply correspondence $y_j(p)$ is defined as

$$\arg\max\left\{py_j : y_j \in Y_j\right\}$$

- consumer i's *demand correspondence*

$$x_i(p) = \arg\max\left\{u_i(x_i) : px_i \leq p\omega_i + \sum_{j=1}^{m} \lambda_{ij} p y_j(p)\right\}$$

- the *aggregate excess demand correspondence* is given by

$$z(p) \equiv \sum_{i=1}^{n} x_i(p) - \sum_{i=1}^{n} \omega_i - \sum_{j=1}^{m} y_j(p)$$

- assume that preferences are monotonic in the sense defined in the description of the first welfare theorem. Then

$$pz(p) \equiv 0$$

- this is called Walras Law - it follows trivially from the fact that consumers with monotonic preferences will always choose consumption vectors that completely exhaust their budgets

- *Brouwer's Fixed Point Theorem:* Let $f : S^{k-1} \rightarrow S^{k-1}$ be a continuous mapping. Then $\exists p \in S^{k-1}$ *such that* $f(p) = p$.
- *Existence of Walrasian Equilibrium:* Suppose that $z(p)$ is a continuous function and that preferences are monotonic so that Walras Law holds. Then there is a Walrasian equilibrium allocation.

- Proof: We show that there is a price the auctioneer can set such that excess demand is zero for every commodity. For each commodity i define

$$g_i(p) = \frac{p_i + \max(0, z_i(p))}{1 + \sum_{j=1}^{n} \max(0, z_j(p))}$$

The denominator is always strictly positive and continuous, the numerator is non-negative and continuous. This follows from a number of facts: first we assumed $z(p)$ is continuous. Since z is a vector valued function, that means that each of its components is continuous. Second, the maximum of two continuous functions is continuous. Third, sums of continuous functions are continuous.

- Next observe that

$$\sum_{i=1}^{k} g_i(p) =$$

$$\sum_{i=1}^{k} \frac{p_i + \max(0, z_i(p))}{1 + \sum_{j=1}^{n} \max(0, z_j(p))} = 1$$

So $g(p)$ is a continuous mapping from S^{k-1} into itself.
So by Brouwer's theorem, there is a fixed point p^* such that $p^* = g(p^*)$.
We want to show that $z(p^*) = 0$. Note that

$$p_i^* = \frac{p_i^* + \max(0, z_i(p^*))}{1 + \sum_{j=1}^{n} \max(0, z_j(p^*))}$$

- so that

$$p_i^* \left[1 + \sum_{j=1}^{n} \max(0, z_j(p^*)) \right] = p_i^* + \max(0, z_i(p^*))$$

for each i. Cancel the p_i^* on each side and multiply both sides by $z_i(p^*)$ to get

$$z_i(p^*) p_i^* \sum_{j=1}^{n} \max(0, z_j(p^*)) = z_i(p^*) \max(0, z_i(p^*))$$

- Sum over i and use Walras law on the left hand side to get

$$0 = \sum_{i=1}^{k} z_i(p^*) \max(0, z_i(p^*))$$

which ensures that $z_i(p^*) \leq 0$ for each i. If $z_i(p^*) < 0$ for any i, then Walras law requires $p_i^* = 0$. But if that is true $z_i(p^*) = \infty$ because of monotonicity of preferences. This contradiction proves that $z_i(p^*) = 0$ for every i.