Private Ownership - Complete Information

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Preliminaries

- the first variant of the basic model we study restricts to private values and complete information
- all preferences are known
- $X = \mathbb{R}^{Jn}$ - the set of feasible allocations
- an allocation $\{x_1, \dots, x_n\}$is a list of consumption vectors one for each of the traders. Consumption and production vectors are in $\mathbb{R}^k$ assuming there are k physical commodities
- feasiblity - there are m firms each firm can produce any vector $y_j \in Y_j$ its production set

- the set of feasible allocations is

$$Y =$$

$$\left\{x \in X : \sum_{i=1}^{n} x_i \leq \sum_{j=1}^{m} y_j + \sum_{i=1}^{n} \omega_i ; y_1 \in Y_1, \ldots, y_m \in Y_m\right\}$$

ω_i is consumer i's endowment.

- $\Theta = \{\overline{\theta}\}$ - everything is known which is why Y doesn't depend on the state.

- for consumers preferences are given by $u_i(x_i)$: - consumers care only about their own consumption vector, and not about other consumer's allocations, or production choices of firms.
- Firm production sets are constants, so there are no production externalities
- an allocation rule here just consists of a single outcome
- the physical description of the economy isn't enough to give an outcome, it just shows what is feasible

Institutions

- we study the Walrasian outcome which is generated the following way
- there is a price vector $p \in \mathbb{R}_+^k$
- each firm chooses a production vector to maximize profits
- each consumer i owns a share a strategy λ_{ij} of the profits of firm j.

- given a set of production choices $\{y_j\}_{j=1,\ldots m}$ each consumer has budget set

$$\left\{x_i : px_i \leq p\omega_i + \sum_{j=1}^{m} \lambda_{ij} p y_j \right\}$$

- an auctioneer announces price p
- consumers choose their favourite consumption bundle from their budget set
- firms maximize profits
- the auctioneer adjusts the price to minimize excess demand

- this institutional structure defines a model that now has predictive and normative content - notice that this content emerges from a combination of assumptions about the environment (eg, no externalities, complete information) and assumptions about the institutions through which agents interact
- the imaginary auctioneer isn't a shortcoming of the model - it is simply a conceptual device to keep track of forces that you might imagine involve some learning or gradual adjustment

- in a philosophical sense, a market emerges in this model once agents have a common belief about the price at which they can purchase a good - it is not a physical location like Granville market, or eBay. It is simply convenient to use Nash equilibrium to think about this expectation as a common expectation about a strategy that another agent will use in equilibrium.
- competitive models like this one assert that the consumer can believe things will happen off the equilibrium path that are actually infeasible - this makes it possible to effectively threaten consumers and firms when they don't do the right thing- this approach is now very unusual in most of micro theory which takes great pains to ensure that out of equilibrium behavior is actually sensible.

Walrasian Equilibrium

- a *Walrasian Equilibrium* is a list $\{x^*, y^*, p^*\}$ such that
- for each firm j

$$p^* y_j^* \geq p^* y' \forall y' \in Y_j$$

- for each consumer $u_i(x_i^*) \geq u_i(x_i)$ for every i and x_i such that

$$p^* x_i \leq p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j^*$$

- and

$$\sum_{i=1}^{n} x_i^* = \sum_{j=1}^{m} y_j^* + \sum_{i=1}^{m} \omega_i$$

First Welfare Theorem

- Preferences satisfy non-satiation if for any vector x and any open neighbourhood B containing x, there is a point $x' > x$ with $x' \in B$ such that $u(x') > u(x)$.
- Theorem: If preferences of every trader i satisfy non-satiation, then every Walrasian equilibrium allocation has the property that there is no alternative feasible allocation in which every consumer is at least as well off and some consumer is strictly better off.

- Proof: (by contradiction) Suppose (x^*, y^*, p^*) is a Walrasian allocation, but that it doesn't satisfy the property above. Then there is some feasible allocation (x', y', p') such that

$$u_i(x_i') \geq u_i(x_i^*)$$

for every i, with strict inequality holding for some i. Since preferences satisfy non-satiation, it must be that

$$p^* x_i' \geq p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j^*$$

- with strict inequality holding for the consumer who is made better off. Then since firms production choices are best replies to the prices they satisfy

$$p^* x_i' \geq p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j'$$

and so

$$\sum_{i=1}^{n} p^* x_i' > \sum_{i=1}^{n} p^* \omega_i + \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} p^* y_j' =$$

$$\sum_{i=1}^{n} p^* \omega_i + \sum_{j=1}^{m} p^* y_j'$$

taking into account the consumer who is strictly better off.

- This contradicts the fact that

$$\sum_{i=1}^{n} x_i' = \sum_{i=1}^{n} \omega_i + \sum_{j=1}^{m} y_j'$$

Second Welfare Theorem (First No-Trade Theorem)

- Suppose that there is some set of production vectors $\{y_1, \ldots, y_m\}$ for firms such that the allocation $\left\{\omega_1 + \sum_{j=1}^{m} s_{1j} y_j, \ldots, \omega_n + \sum_{j=1}^{m} s_{nj} y_j, y_1, \ldots, y_m\right\}$ is pareto optimal (it is obviously feasible). In words, given the production decisions of firms, the allocation where every consumer simply consumes his endowment of each of the goods, is already pareto optimal. Suppose preferences are monotonic in the sense describe before the first welfare theorem. Suppose further that this economy has some Walrasian equilibrium. Then there is an equilibrium in which firms produce outputs $\{y_1, \ldots, y_n\}$and every consumer consumes his or her endowment.

- Proof: Let $\{x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*, p^*\}$be one of the Walrasian equilibrium for this economy. Then

$$x_i^* = \arg\max \left(u_i\left(x'\right) : p^* x' \leq p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j^* \right)$$

In particular, since y_j^* maximizes profits for each firm j,

$$p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j \leq p^* \omega_i + \sum_{j=1}^{m} \lambda_{ij} p^* y_j^*$$

- So, in particular, $u_i\left(x_i^*\right) \geq u_i\left(\omega_i + \sum_{j=1}^m s_{ij} y_j\right)$ for each trader i since the latter bundle is affordable. If this inequality were strict for any i, this would violate the pareto optimality of the endowment vector. We conclude that $u_i\left(x_i^*\right) = u_i\left(\omega_i + \sum_{j=1}^m \lambda_{ij} y_j\right)$ for each trader i.
- Suppose $p^* y_j < p^* y_j^*$ for some j. Then

$$p^*\left(\omega_i + \sum_{j=1}^{m} \lambda_{ij} y_j\right) < p^*\left(\omega_i + \lambda_{ij} y_j^*\right)$$

- By monotonicity of preferences there must then be a consumption bundle in some consumer's budget set when production choices are y_j^* which is strictly better than $\omega_i + \sum_{j=1}^m \lambda_{ij} y_j$. This contradicts the conclusion that $u_i\left(\omega_i + \sum_{j=1}^m \lambda_{ij} y_j\right) = u_i\left(x_i^*\right)$.
- We conclude that $p^* y_j = p^* y_j^*$. Since the endowment is feasible, the allocation $\left\{\omega_1 + \sum_{j=1}^m \lambda_{1j} y_j, \ldots, \omega_n + \sum_{j=1}^m \lambda_{nj} y_j, y_1, \ldots, y_m, p^*\right\}$ is another Walrasian equilibrium.

Existence of Walrasian Equilibrium

- firms' supply correspondence $y_j(p)$ is defined as

$$\arg\max\{py_j : y_j \in Y_j\}$$

- consumer i's *demand correspondence*

$$x_i(p) = \arg\max\left\{u_i(x_i) : px_i \leq p\omega_i + \sum_{j=1}^{m} \lambda spy_j(p)\right\}$$

- the *aggregate excess demand correspondence* is given by

$$z(p) \equiv \sum_{i=1}^{n} x_i(p) - \sum_{i=1}^{n} \omega_i - \sum_{j=1}^{m} y_j(p)$$

- assume that preferences are monotonic in the sense defined in the description of the first welfare theorem. Then

$$pz(p) \equiv 0$$

- this is called Walras Law - it follows trivially from the fact that consumers with monotonic preferences will always choose consumption vectors that completely exhaust their budgets

- *Brouwer's Fixed Point Theorem:* Let $f : S^{k-1} \to S^{k-1}$ be a continuous mapping. Then $\exists p \in S^{k-1}$ *such that* $f(p) = p$.
- *Existence of Walrasian Equilibrium:* Suppose that $z(p)$ is a continuous function and that preferences are monotonic so that Walras Law holds. Then there is a Walrasian equilibrium allocation.

- Proof: We show that there is a price the auctioneer can set such that excess demand is zero for every commodity. For each commodity i define

$$g_i(p) = \frac{p_i + \max(0, z_i(p))}{1 + \sum_{j=1}^{n} \max(0, z_j(p))}$$

The denominator is always strictly positive and continuous, the numerator is non-negative and continuous. This follows from a number of facts: first we assumed $z(p)$ is continuous. Since z is a vector valued function, that means that each of its components is continuous. Second, the maximum of two continuous functions is continuous. Third, sums of continuous functions are continuous.

- Next observe that

$$\sum_{i=1}^{k} g_i(p) =$$

$$\sum_{i=1}^{k} \frac{p_i + \max(0, z_i(p))}{1 + \sum_{j=1}^{n} \max(0, z_j(p))} = 1$$

So $g(p)$ is a continuous mapping from S^{k-1} into itself. So by Brouwer's theorem, there is a fixed point p^* such that $p^* = g(p^*)$.
We want to show that $z(p^*) = 0$. Note that

$$p_i^* = \frac{p_i^* + \max(0, z_i(p^*))}{1 + \sum_{j=1}^{n} \max(0, z_j(p^*))}$$

- so that

$$p_i^* \left[1 + \sum_{j=1}^{n} \max(0, z_j(p^*))\right] = p_i^* + \max(0, z_i(p^*))$$

for each i. Cancel the p_i^* on each side and multiply both sides by $z_i(p^*)$ to get

$$z_i(p^*) p_i^* \sum_{j=1}^{n} \max(0, z_j(p^*)) = z_i(p^*) \max(0, z_i(p^*))$$

- Sum over i and use Walras law on the left hand side to get

$$0 = \sum_{i=1}^{k} z_i(p^*) \max(0, z_i(p^*))$$

which ensures that $z_i(p^*) \leq 0$ for each i. If $z_i(p^*) < 0$ for any i, then Walras law requires $p_i^* = 0$. But if that is true $z_i(p^*) = \infty$ because of monotonicity of preferences. This contradiction proves that $z_i(p^*) = 0$ for every i.