1 Abstract Choice Theory

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- Let $\mathcal{X}$ be a set of alternatives, $\mathcal{X} \times \mathcal{X}$ is the Cartesian product of $\mathcal{X}$ with itself. A binary relation on $\mathcal{X}$ is a subset $\mathcal{P} \subset \mathcal{X} \times \mathcal{X}$
- orderings of alternatives can be thought of as binary relations - e.g. if x and y both elements of $\mathcal{X}$ and $(x, y) \in \mathcal{P}$ then one might say that x is at least as good as y, or $x \succeq y$
- examples include

  1. a set of alternative consumption bundles
  2. a set of alternative policies with $x \succeq y$ meaning that x is 'socially preferred' to y
  3. a set of alternative policies with $x \succeq y$ meaning that x would defeat y in a referendum between the two
  4. a set of alternative business propositions with $x \succeq y$ meaning that opportunity x is riskier than y

5. a set of econometric tests with $x \succeq y$ meaning that test x is a more powerful test than y
6. a set of numbers with $x \succeq y$ meaning that x is bigger than y (an example where a binary relation is an ordering)

- some binary relations have strange properties - for example

	TC	BB	IS
C	1	2	3
F	2	3	1
M	3	1	2

rows are parties, numbers represent their preferences over policies TC,BB and IS. Every policy is defeated in a majority vote against some alternative (Condorcet paradox).

- a *preference* relation is a special binary relation intended to represent an individual choice process - one imagines that it has two properties

1. Completeness for any pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ either $x \succeq y$ or $y \succeq x$ or both.
2. Transitivity for any $x, y, z \in \mathcal{X}$ $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$

- a preference satisfying these two properties is sometimes called a *rational preference relation*
- rational preference relations are often used in elementary theory since an indifference curve through a point x is defined to be the set $\{y \in \mathcal{X} : y \succeq x \text{ and } x \succeq y\}$
- a function $u : \mathcal{X} \to \mathbb{R}$ is called a *utility function* representing preference relation $\succeq$ if for all $x, y \in \mathcal{X}$

$$x \succeq y \iff u(x) \geq u(y)$$

- intransitive preference relations typically can't be represented by utility functions - if a binary relation $\succeq$ is intransitive, then there are three options x, y,and z such that $x \succeq y$; $y \succeq z$ but not $x \succeq z$. Now suppose there is a utility function representing this relation.

Then $x \succeq y \Rightarrow u(x) \geq u(y)$ while $y \succeq z \Rightarrow u(y) \geq u(z)$ so that $u(x) \geq u(z)$ which by definition means that $x \succeq z$. Since we know this is false, the assertion that there is a utility function must also be false. This is an example of a proof *by contradiction.*

- binary relations are an elegant way to think about preferences (since you can use them to construct indifference curves and create utility functions) and underly much of modern economic theory.

- however, they are completely impractical since there is no way to observe directly what these relations are (of course, you could always ask people to tell you their preference relations. To the extent that they could understand what that meant, it is unlikely they would have any reason to tell you the truth).

- a critical question is whether there is some way to infer the existence of a preference relation from something that you can observe - this leads to a very nice theorem

- Let $\mathcal{B}$ be a family of subsets of $\mathcal{X}$ and $\mathcal{P}(\mathcal{X})$ the collection of all subsets of $\mathcal{X}$ (the power set of $\mathcal{X}$) - a correspondence $C : \mathcal{B} \rightarrow \mathcal{P}(\mathcal{X})$ is called a *choice correspondence* if $C(B) \subset B$ for all $B \in \mathcal{B}$

- it would be possible in principle to construct a number of experiments in which participants were offered repeated chances to choose outcomes in the set B. The outcomes that they actually choose would be the choice correspondence.

- the choice correspondence C satisfies the *weak axiom of revealed preference* if for any pair of sets B and B' and points $x \in B \cap B'$ and $y \in B \cap B'$, $x \in C(B)$ and $y \in C(B') \Rightarrow x \in C(B')$.

- example - $\mathcal{X} = \{x, y, z\}$, $\mathcal{B} = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

- then $C(\{x, y\}) = \{x\}; C(\{x, y, z\}) = \{x, y\}; C(\{y, z\}) = \{y\}; C(\{x, z\}) = x$ fails the weak axiom because y is chosen given choice set $\{x, y, z\}$ and x is also in $\{x, y, z\}$. x is chosen in $\{x, y\}$ but y isn't

- every rational preference relation supports a choice correspondence in the obvious way

$$C_{\succeq}(B) = \{x \in B : x \succeq y \forall y \in B\}$$

provided that this set is always non-empty

- first the less interesting theorem - every choice correspondence supported by a rational preference relation satisfies the weak axiom

  Proof: Suppose not. Then there are sets $B, B'$and points $x \in B \cap B'$ and $y \in B \cap B'$ such that (i)$x \in C_{\succeq}(B)$; (ii) $y \in C_{\succeq}(B')$ while (iii) $x \notin C_{\succeq}(B')$. Since $C_{\succeq}$ is supported by a preference relation $x \succeq y$ by (i). By (iii) there is a point z in B' such that $z \succeq x$ but not $x \succeq z$ ($z \succ x$). By (ii) $y \succeq z \succ x$. Then $x \succeq y \succeq z$ but not $x \succeq z$, so the preference relation isn't transitive.

- the reason this isn't interesting is because we construct the choice correspondence from the unobservable preference ordering.

- we want the other way around - if we run a series of experiments and find that some agents' choices obey the weak axiom, can we conclude that the trader will behave *as if* he has a rational preference ordering?

- in other words, is there a rational preference ordering that supports the choice correspondence as above?

- Not generally - Example - $\mathcal{X}$ as above with $\mathcal{B} = \{\{x,y\},\{y,z\},\{x,z\}\}$ and (i) $C(\{x,y\}) = x$, (ii) $C(\{y,z\}) = y$ and (iii) $C(x,z) = z$. Note that this set of choices implies intransivity because if the rationalizing preference relation exists, then $x \succ y$ by (i), $y \succ z$ by (ii) and $z \succ x$ by (iii). The weak axiom holds because the sets in $\mathcal{B}$ simply don't give the decision maker an opportunity to violate the weak axiom.

- Thm: let C be a choice correspondence satisfying the weak axiom. Suppose that for any three distinct points x, y, and z in $\mathcal{X}$ there exist sets B and B' in $\mathcal{B}$ such that $B = \{x,y\}$ and $B' = \{x,y,z\}$. Then there is a rational preference relation supporting C.

- Proof: Define the binary relation $\succeq_C$as follows $x \succeq_C y$ iff $\exists B : x \in B; y \in B$ and $x \in C(B)$. Since C is defined on all sets in $\mathcal{B}$ and $\mathcal{B}$ contains all two element sets, then for any pair of points $\{x,y\}$ either $x \in C(\{x,y\})$ or $y \in C(\{x,y\})$ or both, which is equivalent to $x \succeq_C y$ or $y \succeq_C x$ or both. Hence the binary relation $\succeq_C$ is complete. Suppose that $x \succeq_C y$ and $y \succeq_C z$. $C(\{x,y,z\})$ must contain at least one point. If that point in x then $x \succeq_C z$ by definition, and the relation is transitive. If the point is y then since $x \succeq_C y$ there is some set B'' such that $y \in B''$, and $x \in C(B'')$, so by the weak axiom $x \in C(\{x, y, z\})$ which gives $x \succeq_C z$. If the point is z, use the same reasoning to show that $y \in C(\{x, y, z\})$, from which the same logic gives $x \in C(x, y, z)$ or $x \succeq_C z$. This proves that $\succeq_C$ is transitive. So $\succeq_C$ is a rational preference relation (note how the assumptions were used in this argument - what would go wrong if $\mathcal{B}$ did not contain all sets of the form $\{x, y, z\}$?).

At this point we have a rational preference relation $\succeq_C$, which must support *some* choice correspondence satisfying the weak axiom (we proved that just before) say C^*. It is isn't yet clear whether this choice correspondence is the same as C. To see that it is observe first that if $x \in C(B)$ for some B then $x \succeq_C y$ for all $y \in B$ so $x \in C^*(B)$, ie, $C(B) \subset C^*(B)$ for all B.

Alternatively if $x \in C^*(B)$ then $x \succeq_C y$ for all $y \in B$. Now $C(B)$ must contain some point. It could be x,of course, then $x \in C(B)$ implies $C^*(B) \subset C(B)$ and we are finished. On the other hand if the point were y then because $x \succeq_C y$ there is *some* set B_y such that $x \in B_y$ and $y \in B_y$ and $x \in C(B_y)$. Then by the weak axiom $x \in C(B)$, so again we get $C^*(B) \subset C(B)$ which completes the proof.