Econ 600 - Peters
Fall Midterm - October 18,2011

Do each question. Marks are as indicated.

1. (30 Marks) State whether each of the following preference relations for lotteries on the outcome space $\mathcal{X} = \{x_1, x_2, x_3\}$ satisfies the independence axiom. If yes, give a short proof, if no, give a counterexample. Use $q = (q_1, q_1, q_3)$ to represent a lottery over outcomes $\mathcal{X}$. If there is some doubt, always assume that preferences satisfy the reduction of compound lotteries.

   (a) For any two distinct lotteries q and q', $q \succeq q'$ iff $q_1 > q_1'$ or if $q_1 = q_1'$ and $q_2 > q_2'$;
   Answer: Suppose $q \succeq q'$, the for any q'', either

   $$\lambda q_1 + ((1-\lambda) q_1'' > \lambda q_1' + (1-\lambda) q_1''$$

   or these are equal and

   $$\lambda q_2 + ((1-\lambda) q_2'' > \lambda q_2' + (1-\lambda) q_2''$$

   (depending on why $q \succeq q'$. So the independence axiom holds

   (b) For any two lotteries q and q', $q \succeq q'$ iff $\ln(aq_1 + bq_2 + cq_3) \geq \ln(aq_1' + bq_2' + cq_3')$;
   Answer: Since ln is monotonically increasings $\ln(aq_1 + bq_2 + cq_3) \geq \ln(aq_1' + bq_2' + cq_3')$ iff $aq_1 + bq_2 + cq_3 \geq aq_1' + bq_2' + cq_3'$. As a result there is also a linear (in probabilities) utility function that represents the preference ordering $\succeq$ and any linear (in probabilities) utility function satisfies the idependence axiom.

   (c) There is a strictly convex function $g : [0,1] \to [0,1]$ satisfying $g(0) = 0$ and $g(1) = 1$ such that $q \succeq q'$ iff

   $$g(q_1) a + g(q_2) b + g(q_3) c \geq g(q_1') a + g(q_2') b + g(q_3') c.$$

   Answer: Let q be a degenerate lottery that assigns probability 1 to outcome 1, q' a lottery that assigns probability 1 to outcome 2, and $q'' = q'$. Then if $q \succeq q'$, and $\lambda = \frac{1}{2}$

   $$g\left(\frac{1}{2}\right) a + g\left(\frac{1}{2}\right) b < g(1) b$$

   (by reduction of compound lotteries). So $\lambda q + (1-\lambda) q'$ is *not* preferred to $\lambda q' + (1-\lambda) q'$ which violates the independence axiom.

2. (30 Marks) The point of the trading problem we studied in class was to show how we might endogenize the part of preferences that seem to distinguish between outcomes that are 'better' or 'worse' than what the decision maker expected. The model suggested how firms might actually manipulate preferences by playing with prices. Another way to tell the same story is to imagine that consumers value a firms brand name in addition to the functionality of the good they buy (interdependent preferences). The value of a brand is given by the proportion of consumers who buy the brand. Assume there are two brands A and B with prices $p_b > p_a$. Suppose that half of all consumers like the functionality in brand A better, the other half prefer the functionality in brand B. A consumer's payoff is equal to some constant V less what he pays for the good he buys. If he/she buys a good that doesn't have the functionality they want they they lose surplus α. They also lose the brand benefits associated with the brand that they didn't buy. This loss is d times the proportion of consumers who they expect to buy the other brand. Let λ be the proportion of consumers that everyone expects to buy from firm A. A consumer who likes the functionality in product A best and buys it, receives payoff

$$V - p_a - (1 - \lambda) d$$

whereas if he/she decides to buy the less functional good B, their payoff is

$$V - p_b - \alpha - \lambda d$$

Find the value of λ that supports an equilibrium (the proportion of consumers expect to buy A is equal to the proportion of consumers who actually buy it) in which some of the consumers who find A more functional end up buying the higher priced brand anyway, while all the consumers who find B more functional buy it. What happens to this equilibrium if firm B raises its price? Can you provide a verbal explanation for why this happens?

Answer: Consumers who prefer the functionality of brand A are indifferent between the two brands when the proportion of consumers who are expected to buy brand A satisfies

$$V - p_a - (1 - \lambda) d = V - p_b - \alpha - \lambda d$$

or

$$\lambda = \frac{d - (p_b - p_a) - \alpha}{2d}$$

Then the proportion ρ of consumers who find the functionality of good A better who actually purchase good A must satisfy

$$\lambda = \frac{\rho}{2}$$

so $\rho = \frac{d-(p_b-p_a)-\alpha}{d}$. So raising the price of good B causes both the proportion of all consumers, and the proportion who prefer the functionality who actually buy good A to fall. The reason this happens is that in order to support the higher price a higher proportion of consumers must buy good B in order to increase the brand value of good B to make up.

3. (40 Marks) Suppose there are two states s_1 and s_2. In each state, there is exactly 1 unit of wealth to be shared between two individuals. Each individual has expected utility preferences with utility for wealth given by $u(x) = \ln(x)$. Let x_s^i be the amount of wealth that individual i gets in state s. What is an outcome function for this problem? Describe (formally with notation) the set of ex post efficient outcome functions. Does this set depend on individuals beliefs about how probable the states are?
Now describe (formally) the set of interim efficient outcome functions when the individuals have beliefs that come from a common prior. Does this set of outcome function depend on individuals beliefs about how probable the states are?
Now describe (formally) the set of outcome functions that are interim efficient when individual 2 believes that state s_1 occurs with probablity q while individual 1 believes that state 1 occurs with probability αq with $\alpha < 1$.
Finally suppose that indidual 1 has two types, one type (θ_1) believes that state 1 occurs with probability q while the other believes that state 1 occurs with probability αq with $\alpha < 1$. Is there an interim efficient outcome function for this situation which is also incentive compatible? What conditions must it satisfy. Explain why such an outcome function must be interim incentive efficient.
Answer: An outcome function is a desription of the share of wealth of each individual in each of the two states and for each profile of their types. Assuming complete information and outcome function is a vector $(x_{11}, x_{21}, x_{12}, x_{22})$ interpreting x_{ij} as the share that i gets in state j. Since an outcome is ex post efficient if there isn't an alternative outcome function that makes them better off in each state, the set of ex post efficient outcome functions is

$$\{(x_{11}, x_{21}, x_{12}, x_{22}) : x_{11} + x_{21} = 1; x_{12} + x_{22} = 1\}.$$

Under the common prior assumption, each individual shares the same belief about the probability with which state 1 will occur - call this belief $b \in (0,1)$. An outcome function is interim efficient if there isn't an alternative outcome function that makes both better off given their interim beliefs, i.e., that state 1 will occur with probability b. Because interim efficient outcome functions must be ex post efficient, we can focus on outcome functios of the form $(x_{11}, x_{12}, 1 - x_{11}, 1 - x_{22})$. For them to be interim efficient, they must satisfy

$$b \ln(x_{11}) + (1-b)\ln(x_{12}) = \max b \ln(x'_{11}) + (1-b)\ln(x'_{12})$$

subject to the constraint that

$$b \ln(1 - x'_{11}) + (1-b)\ln(1 - x'_{12}) \geq b \ln(1 - x_{11}) + (1-b)\ln(1 - x_{12})$$

The Lagrangian is

$$b \ln \left(x'_{11}\right) + (1-b) \ln \left(x'_{12}\right) -$$
$$\lambda \left(b \ln \left(1 - x_{11}\right) + (1-b) \ln \left(1 - x_{12}\right) - b \ln \left(1 - x'_{11}\right) + (1-b) \ln \left(1 - x'_{12}\right)\right)$$

which, after some simple algebra gives necessary conditions $x_{11} = x_{12}$ (and $\lambda \leq 0$). So the set of interim efficient allocations is

$$\{x = (x_{11}, x_{12}, x_{21}, x_{22}) : x \text{ is ex post efficient}; x_{11} = x_{12}; x_{21} = x_{22}\}.$$

Without the common prior, the same approach gives necessary conditions

$$\frac{\alpha b x_{12}}{(1-\alpha b)\, x_{11}} = \frac{b\,(1-x_{12})}{(1-b)\,(1-x_{11})}$$

or

$$x_{12} = \frac{(1-\alpha b)\, x_{11}}{(1-\alpha)\, x_{11} + \alpha\,(1-b)}$$

The set of interim efficient outcome function is the set

$$\left\{x = (x_{11}, x_{12}, x_{21}, x_{22}) : x \text{ is ex post efficient}; x_{12} = \tfrac{(1-\alpha b)x_{11}}{(1-\alpha)x+\alpha(1-b)}; x_{21} = 1 - x_{11}; x_{22} = 1 - x_{12}\right\}.$$

Finally, when individual 1 has two types, an allocation rule is a pair $\left(x^{\theta_1}, x^{\theta_2}\right)$ satisfying

$$x_{11}^{\theta_1} = x_{12}^{\theta_1}; x_{21}^{\theta_1} = 1 - x_{11}^{\theta_1}; x_{22}^{\theta_1} = 1 - x_{12}^{\theta_1};$$

and

$$x_{12}^{\theta_2} = \frac{(1-\alpha b)\, x_{11}^{\theta_2}}{(1-\alpha)\, x + \alpha\,(1-b)}; x_{21}^{\theta_2} = 1 - x_{11}^{\theta_2}; x_{22}^{\theta_2} = 1 - x_{12}^{\theta_2};$$

and

$$\alpha b \ln \left(x_{11}^{\theta_2}\right) + (1-\alpha b) \ln \left(x_{12}^{\theta_2}\right) \geq$$
$$\alpha b \ln \left(x_{11}^{\theta_1}\right) + (1-\alpha b) \ln \left(x_{12}^{\theta_1}\right)$$

then the allocation rule is both interim efficient and interim incentive efficient.