1 Choice Theory with Uncertainty

- let $\mathcal{X}$ be a finite set of possible outcomes, or consequences
- for example, consumption bundles, payoffs to a stock portfolio, outcomes of a research project
- N is the number of elements in $\mathcal{X}$, $x \in \mathcal{X}$ is a typical element
- a *lottery* is a pair $(\mathcal{X}, p)$ with $p \in \mathbf{R}_+^N$, where $p = \{p_1, \ldots p_N\}$ and

$$\sum_{i=1}^{N} p_i = 1$$

- interpret p_i as the probability with which outcome or consequence i happens
- $\mathcal{X}$ is very general - it could be simply money payoffs, or it could be a finite set of lotteries

$$\mathcal{X} = \{(\mathcal{Y}_1, p_1)\,,$$

1.0.1 Preferences

- it is natural to suppose that lotteries can be ordered in the way that consumption bundles can
- interpret $p \succcurlyeq p'$ to mean that the lottery p is at least as good as the lottery p'
- the preference ordering $\succcurlyeq$ is said to be *complete* if for *any* pair $p, p' \in \mathcal{L}$, either $p \succcurlyeq p'$ or $p' \succcurlyeq p$
- $\succcurlyeq$ is said to *transitive* if for any $p, p', q \in \mathcal{L}$, $p \succcurlyeq p'$ and $p' \succcurlyeq q \Rightarrow p \succcurlyeq q$
- interpret $p \sim p'$ to mean indifference in the preference relation $\succcurlyeq$ (i.e., $p \succcurlyeq p'$ and $p' \succcurlyeq p$)
- $\succcurlyeq$ is said to satisfy *Reduction of Compound Lotteries* if for any $(\mathcal{X}, p), (\mathcal{X}, p') \in \mathcal{L}$

$$\lambda (\mathcal{X}, p) + (1-\lambda)(\mathcal{X}, p') \sim (\mathcal{X}, \lambda p + (1-\lambda) p')$$

- *Continuity* For any $p, p', p'' \in \mathcal{L}$ the sets

$$\{\lambda \in [0,1] : \lambda p + (1-\lambda)\, p' \succcurlyeq p''\}$$

  and

$$\{\lambda \in [0,1] : p'' \succcurlyeq \lambda p + (1-\lambda)\, p'\}$$

  are both closed.

- Recall that transitivity,completeness and continuity imply the existence of a utility function $U : \mathcal{L} \to \mathbf{R}$ such that $p' \succcurlyeq p$ if and only if $U(p') \geq U(p)$

- *Independence Axiom* $p \succcurlyeq p'$ if and only if for all $\lambda \in [0,1]$ and $p'' \in \mathcal{L}$

$$\lambda p + (1-\lambda)\, p'' \succcurlyeq \lambda p' + (1-\lambda)\, p''$$

- A utility function U has the expected utility property if $U(p) = \sum_{i=1}^{n} p_i u(x_i)$

- *Regularity Properties*

  1. $\exists b, w \in \mathcal{L} : b \succcurlyeq p \succcurlyeq w$ for all $p \in \mathcal{L}$

2. $\lambda > \lambda'$ if and only if $\lambda b + (1-\lambda) w \succ \lambda' b + (1-\lambda') w$
3. U satisfies the expected utility property if for every $\lambda \in [0,1], p, p' \in \mathcal{L}$

$$U(\lambda p + (1-\lambda) p') = \lambda U(p) + (1-\lambda) U(p')$$

– Let x_1 be the lottery in which the outcome x_1 is received with probability 1. Let x_{-1} be the lottery where x_1 occurs with probability 0 while each of the other outcomes x_j occurs with probability $p_j/(1-p_1)$. By the reduction of compound lotteries

$$p \sim p_1 x_1 + (1-p_1) x_{-1}$$

Then by the hypothesis in the theorem and the property of utility

$$U(p) = U(p_1 x_1 + (1-p_1) x_{-1})$$

$$= p_1 U(x_1) + (1-p_1) U(x_{-1})$$

Now expand $U\left(x_{-1}\right)$ in the same manner, and proceed recursively to get

$$U\left(p\right)=\sum_{i=1}^{n}p_{i}U\left(x_{i}\right)$$

– *Expected Utility Theorem*

– Theorem: Suppose $\succcurlyeq$ satisfies transitivity,completeness,continuity, reduction of compound lotteries, the technical restrictions and the independence axiom. Then there is a utility function U associated with $\succcurlyeq$ that has the expected utility property.

– **Proof: Step 1** - Define a potential utility function. Using the technical restrictions, there is a best and worst lottery in $\mathcal{L}$ so define a function u such that $u\left(b\right)=1$ and $u\left(w\right)=0$. Now for each $p\in\mathcal{L}$ define

$$u\left(p\right)=\left\{\lambda:\lambda b+\left(1-\lambda\right)w\sim p\right\}$$

Notice that this means

$$p\sim u\left(p\right)b+\left(1-u\left(p\right)\right)w \tag{1}$$

– **Step 2** - we don't know whether u is actually a function. By the technical restrictions the sets

$$\{\lambda' : \lambda' b + (1 - \lambda') w \succcurlyeq p\}$$

and

$$\{\lambda' : p \succcurlyeq \lambda' b + (1 - \lambda') w\}$$

are both closed subsets of $[0, 1]$ by continuity. The union of these sets is $[0, 1]$ by completeness. Thus they have at least one point in common. Suppose they have more than one point in common, say λ' and λ''. Then without loss of generality $\lambda'' > \lambda'$. Then by the second technical restriction

$$\lambda'' b + (1 - \lambda'') w \succ \lambda' b + (1 - \lambda') w$$

which is a contradiction to the supposition that they are in the same set.

– **Step 3** Show that u satisfies the expected utility property, i.e

$$u (\lambda p + (1 - \lambda) p')$$

$$= \lambda u(p) + (1-\lambda) u(p')$$

– start with

$$\lambda p + (1-\lambda) p'$$

by 1 and the independence axiom this is indifferent to

$$\lambda [u(p) b + (1-u(p)) w] + (1-\lambda) p'$$

and consequently to

$$\lambda [u(p) b + (1-u(p)) w] + (1-\lambda) [u(p') b + (1-u(p') w)]$$

By the reduction in compound lotteries this is indifferent to

$$[\lambda u(p) + (1-\lambda) u(p')] b + [1 - [\lambda u(p) + (1-\lambda) u(p')]] w$$

Then by the fact that this is indifferent to $\lambda p + (1-\lambda) p'$ and using the definition of u this gives

$$u(\lambda p + (1-\lambda) p') =$$

$$\lambda u(p) + (1-\lambda) u(p')$$

so u has the expected utility property by the previous proposition.

– **Final step** - is u a utility function, i.e.

$$u(p) \geq u(p')$$

iff

$$p \succcurlyeq p'$$

This follows immediately from the second technical restriction

– **Problems with the Independence Axiom**

– Allais - monetary prizes are $\{1000, 500, 0\}$. You are given the choice between two lotteries

$$a = \{0, 1, 0\}$$

and

$$b = \{.10, .89, .01\}$$

which means you can have 500 for sure, or take a 10 percent chance of raising 500 to 1000 at the cost of a slight chance of losing everything. The second pair of choices looks as follows

$$a' = \{0, .11, .89\}$$

and

$$b' = \{.10, 0, .9\}$$

which means that you have an 11 percent chance of winning 500 which you can turn in to a 10 percent chance of winning 1000 at the cost of raising the chance of getting nothing slightly. Most people choose a over b and b' over a', which is inconsistent with the independence axiom

– **Subjective Expected Utility**

– start with S different *states of the world.* Associated with each state is some kind of state contingent outcome $x_s \in \mathcal{X}_s$

– for example, in a Bayesian Game.

	x	y
A	25,20	14,12
B	14,20	25,12
C	18,12	18,22

- a pair of mixed strategies $q = \{q_A, q_B, q_C\}$ and $q' = \left\{q'_x, q'_y\right\}$ (one for each player) generates a lottery over the outcome space $\mathcal{X} = \{(A, x), (A, y), (B, x), (B, y), (C, x), (C, y)\}$. The probabilities in this lottery are

$$\{p_{Ax}, p_{Ay}, p_{Bx}, p_{By}, p_{Cx}, p_{Cy}\} =$$

$$\left\{q_A q'_x, q_A q'_y, q_B q'_x, q_B q'_y, q_C q'_x, q_C q'_y\right\}$$

- let $\mathcal{X} = \{\mathcal{X}_1, \mathcal{X}_2, \ldots \mathcal{X}_S\}$ and $\mathcal{L} = \{(\mathcal{X}_1, \mathcal{P}_1), \ldots, (\mathcal{X}_S, \mathcal{P}_S)\}$ where $(\mathcal{X}_s, \mathcal{P}_s)$ is the set of all lotteries over the outcomes in $\mathcal{X}_s$
- objects in $\mathcal{L}$ are sequences with S elements. Suppose there is a complete transitive binary relation over the objects in $\mathcal{L}$

– If we assume the $\mathcal{X}_s$ are all finite and contain the same J elements, then we can represent the sequence of state contingent lotteries and a sequence of vectors, i.e.,

$$\mathcal{L} = \left\{ p; p = \{p_1, p_2, \ldots, p_S\}; p_s \in \mathbb{R}_+^J; \sum_{j=1}^{J} p_{sj} = 1 \right\}$$

– Suppose that a continuous rational preference relation $\succeq$ exists on the set $\mathcal{L}$. For any pair p, p' in $\mathcal{L}$ let

$$\lambda p + (1-\lambda)p' = \{\lambda p_1 + (1-\lambda)p'_1, \ldots, \lambda p_S + (1-\lambda)p'_S\}$$

– define $\pi_{jk}(p)$ to be the state contingent lottery where the j^{th} and k^{th} components of p are interchanged.

– State Uniformity: For any p and p' that differ only in their j^{th} component, if $p \succeq p'$ then $\pi_{jk}(p) \succeq \pi_{jk}(p')$

– Technical Conditions: (T1) there are best and worst elements b and w in $\mathcal{L}$; and (T2): if $\lambda > \lambda'$ then

$$\lambda b + (1-\lambda)w \succ \lambda' b + (1-\lambda')w$$

– Anscombe-Aumann theorem: Suppose preferences over $\mathcal{L}$ are rational, continuous, state uniform and satisfy the independence axiom. Then if the technical conditions T1 and T2 hold, there is a unique probability vector $\rho \in \mathbb{R}_{+}^{S}; \sum_{s=1}^{S} \rho_s = 1$ and utility function $u : \mathcal{L}_s \to \mathbb{R}$ for each s such that $p \succeq p'$ if and only if

$$\sum_{s=1}^{S} \rho_s u(p_s) \geq \sum_{s=1}^{S} \rho_s u(p'_s)$$

where $u\left(p_s\right) = \sum_{j=1}^{J} v_j p_{sj}$.

– The vector ρ is referred to as *subjective beliefs.*

– Proof: Follow the proof of the expected utility theorem to show that there is a linear utility function representing preferences: i.e. $p \succeq p'$ iff $u(p) \geq u(p')$, and for any $\lambda \in [0,1]$, $u(\lambda p + (1-\lambda)p') = \lambda u(p) + (1-\lambda)u(p')$. Since there are J elements in each $\mathcal{X}_s$, then each p has $J \times S$ components p_{ij}

and the linear utility function can be written as

$$\sum_{s=1}^{S}\sum_{j=1}^{J} v_{sj} p_{sj}$$

where the v_{sj} represent a $J \times S$ vector of constants describing the utility value of each of the physical outcomes in $\mathcal{X}_s$. If p and p' differ only in their j^{th}component, and $p \succeq p'$, then

$$\sum_{s=1}^{s}\sum_{j=1,J} v_{sj} p_{sj} \geq \sum_{s=1}^{s}\sum_{j=1,J} v_{sj} p'_{sj}$$

implies that

$$\sum_{k=1}^{J} v_{jk} p_{jk} \geq \sum_{k=1}^{J} v_{jk} p'_{jk}.$$

By state uniformity, $\pi_{j1}(p) \geq \pi_{j1}(p')$ implies, using the same reasoning that

$$\sum_{k=1}^{J} v_{1k} p_{jk} \geq \sum_{k=1}^{J} v_{1k} p'_{jk}.$$

Since this must be true for all p_{jk} and p'_{jk} pairs and for every every index j, this requires that $v_{nk} = \gamma_n v_{1k}$ for some non-negative vector of constants γ_n. Then we can write the utility function as

$$\sum_{s=1}^{S} \sum_{j=1}^{J} \gamma_s v_{1j} p_{sj}$$

By T2 at least one of the γ_i must be positive. Then define the utility function

$$u\left(p\right) = \sum_{s=1}^{S} \frac{\gamma_s}{\sum_{s'=1}^{S} \gamma_{s'}} \sum_{j=1}^{J} p_{sj} v_{1j}$$

where one can interpret $\frac{\gamma_s}{\sum_{s'=1}^{S} \gamma_{s'}}$ as the subjective probability with which the dm believes state s will occur and $u(p) = \sum_{j=1}^{J} p_{sj} v_{1j}$ with v_{1j} representing the 'value' of outcome j.

– **Anscombe-Aumann and Rationalizability**

– Suppose the players play the following game:

– (which is called matching pennies)

$1,-1$	$-1,1$
$-1,1$	$1,-1$

– We say the profile of strategies (top,left) is rationalizable because the row player expects the column player to play left because he thinks the row player will play down because he thinks the column player will play right because he thinks the row player will play top (which he does).

– Using the AA approach above, suppose the row player believes that the column player can have one of two types t_r or t_l. These types could use any strategy rule you like, but for the sake of argument, suppose the rule used by t_l is to play left while t_r uses the rule play right.

– Then a row player who subjectively assigns probability 1 to the state t_l will play top for sure - not very interesting. Also not Nash because the row player seems to believe the column player will do something that doesn't make sense.

- the column player is doing the same thing - lets assign types t_u and t_d to the row player.
- then all the t_i's will be interpreted as *belief types.*
- for example, the belief type t_u is a row player who believes that the column player has type t_l with probablity 1. The column player of type t_l has belief type that assigns probability 1 to the row player having belief type t_d.
- the row player of type t_d believes the column player has type t_r with probability 1 while the column player of type t_r believes the row player has type t_u with probability 1.
- Now, each player belief type chooses a best reply to a strategy rule which is itself a best reply. No one believes anyone is doing anything stupid, nor do they believe anyone believes that anyone else is doing anything stupid, etc.
- **The state space**

- the state is a pair of belief types (t_R, t_C). Each player learns something about the state space because they see their own type
-

(t_u, t_l)	(t_u, t_r)
(t_d, t_l)	(t_d, t_r)

- $(up, left)$ - Bayesian equilibrium without a common prior
- **Problems with Subjective Expected Utility:** Ellsberg - there are two urns consisting of red and black balls. The first urn has 51 red balls and 49 black balls. The second urn has 100 red and black balls (the composition is unknown). The first experiment proposes to give you 10 dollars if a red ball is drawn - you choose which urn you want the ball to be drawn from, the second experiment is the same except that you are paid 10 dollars if a black ball is drawn - again choose a urn. Most people choose the first urn in both cases
- Anscombe Aumann and the uncertain urn - the state is evidently the number of red balls in the second urn - $S = \{0, 1, \ldots, 100\}$.

– Betting on red in the second urn provides a state contingent lottery that looks like this

$$p = \left\{ \{0,1\}, \left\{ \frac{1}{100}, \frac{99}{100} \right\}, \ldots, \{1,0\} \right\}$$

where $\mathcal{X}_s = \{\$10, \$0\}$ for each s.

– Betting on black in the second urn also gives a state continent lottery

$$p' = \left\{ \{1,0\}, \left\{ \frac{99}{100}, \frac{1}{100} \right\}, \ldots, \{0,1\} \right\}$$

– Now use the theorem and the payoff associated with p is

$$\sum_{s=0}^{100} \rho_s \left(\frac{s}{100} v(10) + \left(\frac{100-s}{100} \right) v(0) \right).$$

– Now take $v(0) = 0$ so that this equals

$$v(10) \sum_{s=0}^{100} \rho_s \frac{s}{100} \equiv qv(10).$$

– Do the same for the state contingent lottery associated with betting on black in the second urn to get $(1-q)\,v\,(10)$

– From expected utility betting on red in the first urn has value (again normalizing) $\frac{51}{100}u\,(10)$ while betting on black in the first urn has value $\frac{49}{100}u\,(10)$.

– So the choices first urn in both cases require

$$\frac{51}{100}u\,(10) > qv\,(10)$$

and

$$\frac{49}{100}u\,(10) > (1-q)\,v\,(10) \iff qv\,(10) > \frac{51}{100}u\,(10)$$

– this is inconsistent with Anscombe Aumann

– **Alternatives to expected utility: Uncertainty Aversion**

– (sometimes called the multiple priors model): Define Π as the set of all probability distributions over the outcome space $\mathcal{X}$.

If the outcome space is finite with J elements, this is the set $\left\{\pi \in \mathbb{R}_+^J : \sum_{i=1}^J \pi_i = 1\right\}$.

– the bet is called a 'prospect'. A prospect where the probabilities are known is called a risky prospect. So betting on urn 1 in ellsberg is a risky prospect whether you bet on black or red.

– a prospect where you don't know the probabiities is called an *uncertain* prospects.

– each prospect is characterized by a set of outcomes, and a *set* of probability distributions $\mathcal{P} \subset \Pi$ - $(\mathcal{X}, \mathcal{P})$

– a prospect $(\mathcal{X}, \mathcal{P})$ is preferred to prospect $(\mathcal{X}, \mathcal{P}')$ if and only if there is a set of constants v_j (values of each of the outcomes) such that

$$\inf_{\pi \in \mathcal{P}} \sum_{j=1}^J \pi_j v_j \geq \inf_{\pi \in \mathcal{P}'} \sum_{j=1}^J \pi_s v_j$$

– in words, the decision maker has multiple prior beliefs and evaluates every plan using the prior in the set they think is possible which gives the plan the lowest expected utility.

– in the Ellsberg urn example, this is straightforward. The collection $\mathcal{P}$ can be thought of as a collection of prior probabilities π with which the ball drawn from the unknown urn B is red, for example $\pi \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Then the value of the unknown urn when betting on red is

$$\inf_{\pi \in \left[\frac{1}{4}, \frac{3}{4}\right]} \pi u(10) + (1-\pi) u(0)$$

$$= \frac{1}{4} u(10) + \frac{3}{4} u(0) < \frac{51}{100} u(10) + \frac{49}{100} u(0)$$

so it is better to choose the known urn when betting on red.

– for a bet on black, the value of the unknown urn B is

$$\inf_{\pi \in \left[\frac{1}{4}, \frac{3}{4}\right]} (1-\pi) u(10) + \pi u(0) = \frac{1}{4} u(10) + \frac{3}{4} u(0)$$

so it is still better to take the known urn.

- **Alternatives: Recursive Expected Utility**
- similar to multiple priors, this focuses on uncertain prospects.
- Instead of assuming the worst, it instead assume a probability distribution $\mathcal{F}$ over the set of probability distributions π over the outcomes in $\mathcal{X}$.
- For simplicity, suppose that $\mathcal{F}$ consists of a finite set $\mathcal{P}$ of probability vectors, i.e , $\mathcal{P} = \left\{\pi^1, \ldots, \pi^T\right\}$, where each π^t is a vector of probabilities with J elements. Let p be a vector of T probabilities, p^t representing the probability with which the probability distribution π^t is the correct one.
- An uncertain prospect is a pair $\{p, \mathcal{P}\}$. Notice that a prospect is just a compound lottery.
- Recursive Expect Utility assumes that decision makers can't reduce compound lotteries. Instead, it assumes that $\{p, \mathcal{P}\}$ is preferred to a prospect $\{p', \mathcal{P}'\}$ if there exists a concave function u and a set of constants $v \in \mathbb{R}^J$ such that $\{p, \mathcal{P}\} \succeq \{p', \mathcal{P}'\}$ iff

–

$$\sum_{t=1}^{T} u\left(\sum_{j=1}^{J} \pi_j^t v_j\right) p^t \geq$$

$$\sum_{t=1}^{T} u\left(\sum_{j=1}^{J} \pi_j v_j\right) p^{t\prime} \qquad (2)$$

– Notice that if p and p' are degenerate, in the sense that they assign probability 1 to some π^t, then p and p' will be evaluated using expected utility.

– Notice also that the values v_j assigned to each outcome are independent of t. So recursive expected utility decision makers rank risky lotteries the same way that expected utility decision makers do.

– For lotteries with uncertainty, this is no longer true. For example, in the Ellsberg problem, the 'first' box had 51 Red balls and 49 Black balls. The second box has 100 red and black balls.

– when betting on black, there are 2 outcomes in $\mathcal{X}$, $\{\text{win } 10, \text{win } 0\}$ while there are 101 outcomes in $\mathcal{P}$, these are

$$\left\{(0,1)\,,\left(\frac{1}{100},\frac{99}{100}\right),\ldots,(1,0)\right\}$$

where the first element in each pair is the probability of drawing a black ball when there are t black balls in the box.

– The first box represents a prospect where $p^{49} = 1$ (i.e the probability of drawing a black ball is $\frac{49}{100}$, while all the other p^t are zero. That is the definition of a risky prospect. The second box represents an uncertain prospect because a lot of the p^t are non-zero.

– The subjective expected utility decision maker reduces compound lotteries so when betting on black in the unknown box their payoff is

$$\sum_{t=0}^{100} p^t \left\{\frac{t}{100}v_{10} + \frac{100-t}{100}v_0\right\}$$

$$\geq \sum_{t=0}^{100} p^{t\prime} \left\{ \frac{t}{100} v_{10} + \frac{100-t}{100} v_0 \right\} =$$

$$v_{10} \sum_{t=0}^{100} p^{t\prime} \frac{t}{100} + v_0 \left(1 - \sum_{t=0}^{100} p^{t\prime} \frac{t}{100} \right).$$

As a result they act as if they had assigned a subjective probability to the event draw a black ball equal to

$$q = \sum_{t=0}^{100} p^{t\prime} \frac{t}{100}$$

which should be larger than $\frac{51}{100}$ as long as they choose the first box when they bet on red.

– The recursive expected utility decision maker who is betting on black evaluates the first box the same way a subjective expected utility decision maker does. They assign it value $u\left(\frac{49}{100} u_{10} + \frac{51}{100} u_0\right)$. However, he or she evaluates the second box as

$$\sum_{t=0}^{100} p^{t\prime} u\left(\left\{\frac{t}{100} v_{10} + \frac{100-t}{100} v_0\right\}\right)$$
$$\leq u\left(q v_{10} + (1-q)\, v_0\right)$$

provided u is concave.

– If they choose box 1 when betting on red, they reveal

$$u\left(\frac{51}{100} u_{10} + \frac{49}{100} u_0\right) \geq u\left(q v_0 + (1-q)\, v_{10}\right)$$

so since u is an increasing function we have

$$u\left(q v_{10} + (1-q)\, v_0\right) \geq u\left(\frac{51}{100} u_0 + \frac{49}{100} u_{10}\right)$$

however, their actual payoff is now lower than the left hand side, so if u is strictly concave and the p^t are spread out enough, they may still choose the first box when betting on black.

- **Alternatives: Prospect Theory**
- as before, imagine a finite set of outcomes $\mathcal{X}$. As before, these might be lotteries, the logic is the same
- suppose the objects are ordered from worst to best, i.e., x_1 is the worst outcome, x_J is the best. Choose a status quo outcome, say x_k sometimes called a reference point. Let $\{p, \mathcal{X}\}$ be lottery. Then prospect theory (roughly) says that

  $$p \succeq p'$$

  if and only if there are J constants $\{u_j\}_{j=1}^{J}$ representing cardinal payoff, a reference point x_k^* and a constant λ such that

  $$\sum_j p_j x_j + \sum_{j<k} \lambda p_j \left(u_j - u_k\right)$$

- in Ellsberg $\mathcal{X}$ is the set of objective lotteries we described before, $J = 101$ and x_j is a lottery where there are $j-1$ red balls in the unknown urn. So if you are betting on red, the constant to be used in prospect theory is $u_j = \frac{j-1}{100} u(10) + \left(1 - \frac{j-1}{100}\right) u(0)$. Notice, these lotteries are ordered from lowest to highest. When you are betting on black, the risky urn gives only a single lottery, so it has payoff $u_{49} = \frac{49}{100} u(10) + \frac{51}{100} u(0)$

– When betting on black, that seems a natural reference point when you evaluate the uncertain urn (the one with 100 read and black balls), so prospect theory would say the payoff to the uncertain urn is

$$\sum_{j=1}^{101} p_{j-1} \left[\frac{j-1}{100} u(0) + \left(1 - \frac{j-1}{100}\right) u(10)\right]$$

$$+ \sum_{j-1>51} p_{j-1} \lambda \left(u_{49} - u_{j-1}\right)$$

which is generally going to be much smaller than $\frac{49}{100} u(10) + \frac{51}{100} u(0)$ even if $\sum_{j=1}^{101} p_{j-1} \frac{j-1}{100}$ is equal to $\frac{49}{100}$

– three different ways of explaining the same behavior.

- **Personal Equilibrium - to define the reference point**
- there are two firms A and B producing products with different characteristics and offering them to a continuum of consumers
- at the first stage of the game, two firms advertize their prices to consumers and provide descriptions that reveal to consumers that the products are differentiated in such a way that each consumer will perceive a quality difference of value d between the products. However no consumer knows which product is better for them.
- Each consumer forms an expectation λ of the probability with which he or she will buy from firm A. This is the reference point the consumers take to the second stage of the game.
- At the second stage of the game, consumers learn which of the two products A or B is better for them and make a purchase decision.
- a consumer who learns ex post that product A is best suited to him, and who proceeds to buy from firm A receives payoff

that depends on his expectation

$$V - p_a - (1 - \lambda)\, d$$

– the first part of this $V - p_a$ is the intrinsic payoff associated with buying what turns out to be the best product for them, the second part is a perceived loss associated with the fact that he believed that with probability $(1 - \lambda)$ he was going to buy from firm B and from this perspective he is disappointed at how firm A's product compares to the one he thought he would buy

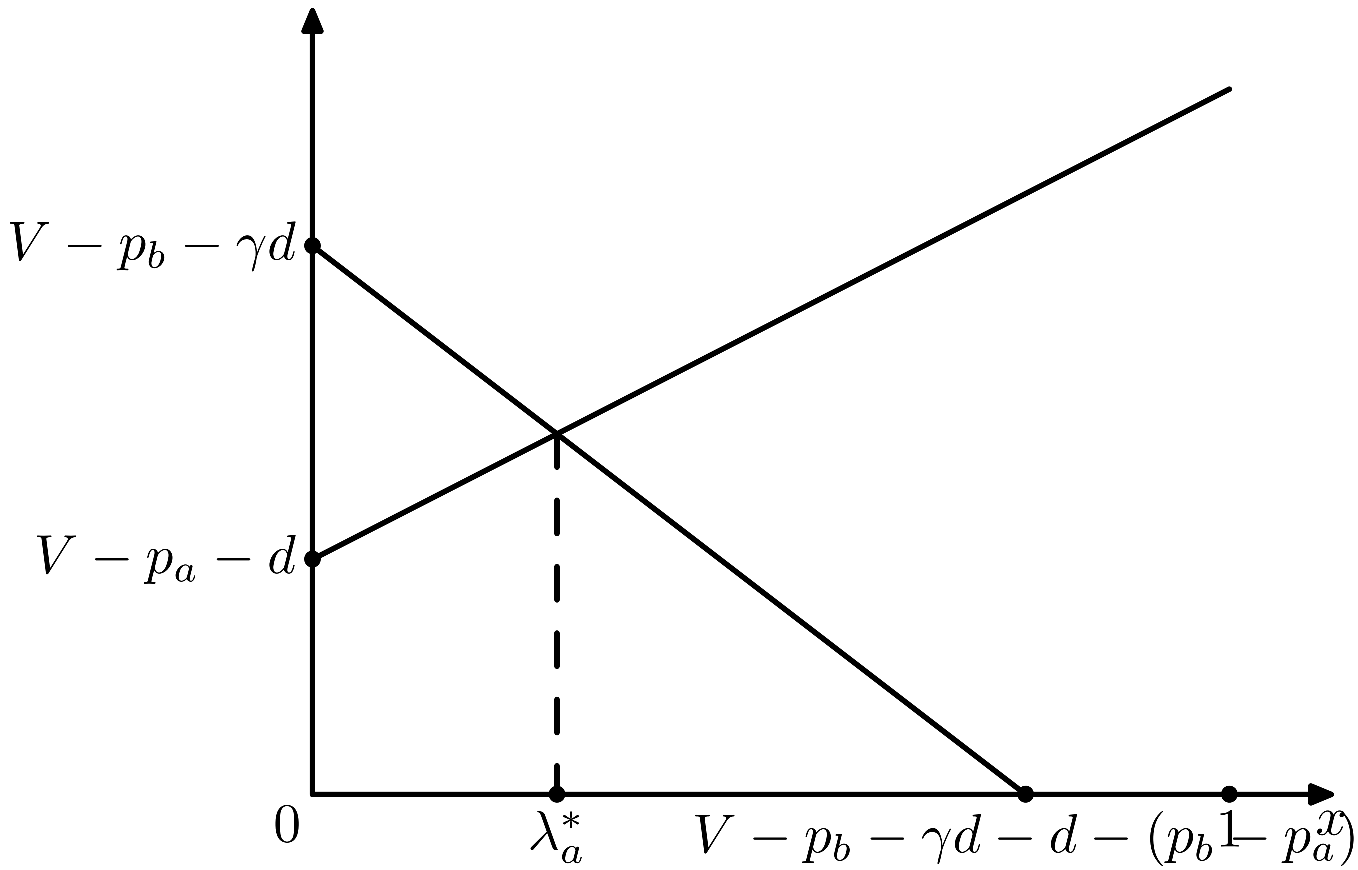
– he would also be pleased that he ended up buying at a lower price than he expected in this case, but we ignore this and focus on losses to make things simple.

– if he instead buys from firm B his payoff is

$$V - p_b - \gamma d - \lambda d - \lambda\,(p_b - p_a)$$

– here the term γd represents the intrinsic loss associated with buying something other than his ideal product. The reference point determined the rest - with probability λ the consumer expected to buy from firm a and his chosen product B is disappointingly different from what he expected. Furthermore, if he expected to buy from firm A, then the price p_b is disappointingly high, which is why we subtract the other term.

– his reference point λ will now determine which of the two products he buys - depending on which of these two payoffs is higher - the figure shows how the reference point affects the decision



- if λ is zero (the consumer doesn't expect at all to buy from firm A, then provided the intrisic quality difference is small (i.e γd is close to zero), product B will be preferred because the loss associated with unexpectedly buying from A dominates
- one significant point is λ_a^* which is the point at which the con-

sumer is just indifferent between the two products - suppose the parameters are such that this is less than $\frac{1}{2}$.

– the reference point then affects the decision of a type A consumer in the following way: he buys

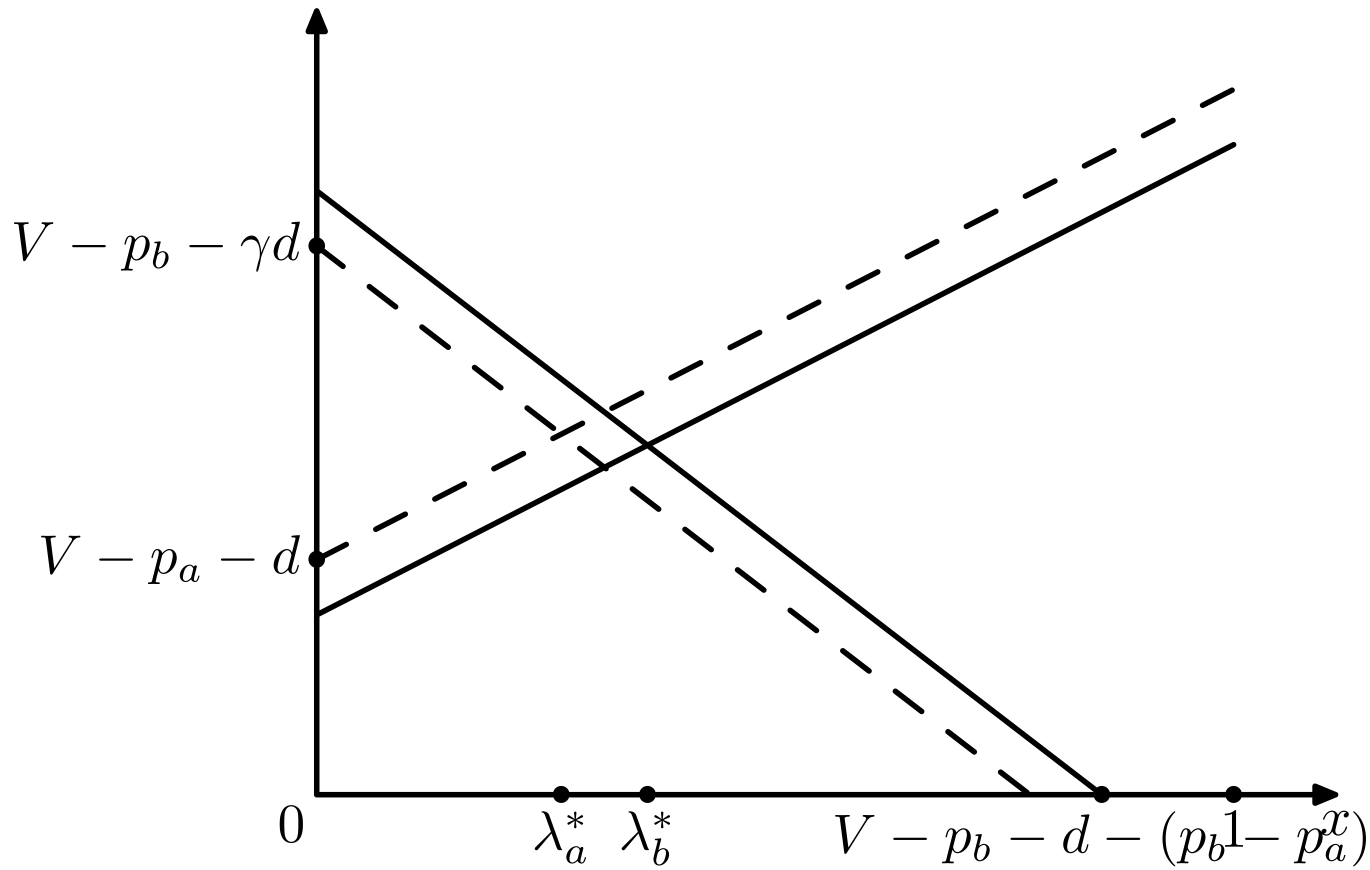
$$\begin{cases} B & \text{if } \lambda < \lambda_a^* \\ A \text{ or } B & \lambda = \lambda_a^* \\ A & \text{otherwise.} \end{cases}$$

– a similar argument applies when the consumer is type B

– the curve for product A is shifted down by the difference γd, the curve for B is shifted up

– the indifference point λ_b^* lies to the right of the point λ_a^*

– the final restriction is that the consumers expectation λ should be 'rational' or equal to the true expectation - 'personal equilibrium'



- there are then a number of equilibria depending on the values of λ_a^* and λ_b^*
- from the figures observe that if the consumer expects to buy from firm B for sure, then he will buy from firm B whether he is type B or A, similiarly if he expects to buy from firm A

for sure. In these two cases his expectations will be realized

- if $\lambda_a^* < \frac{1}{2}$, there is an equilibrium in which all the type B consumers buy from firm B and each of the type A consumers buys from firm A with probability ρ. If it happens that

$$\frac{1}{2}\rho = \lambda_a^*$$

then the consumer's belief that he will buy from firm A with probability λ_a^* is actually right (he will be an A consumer half the time and buy in that case with probability ρ, while if he is a B consumer he won't buy from A at all)

- notice that ρ cannot exceed 1 which is why this will only work if $\lambda_a^* < \frac{1}{2}$.
- there is a similar equilibrium when the consumer believes he will buy from A with probability λ_b^*. This happens if

$$\frac{1}{2} + \frac{1}{2}\rho = \lambda_b^*$$

- from the figure above, when the consumer's reference point is λ_b^* and it turns out that A is better suited to him, then he will buy for sure. If he is better suited to B, he is indifferent between the two, so if he buys A with probability ρ, his belief is again justified.
- notice that this can only work if λ_b^* happens to be larger than $\frac{1}{2}$.
- some simple comparative statics - consumers expect to buy from A for sure (from the figure, if that is their reference point, they will always buy from A even when B turns out to be the product that is better suited to them)
- if firm A raises its price and consumers reference point doesn't change, then consumers will continue to buy from A for sure. Heuristically, firm A will have a pretty high price in equilibrium (the i-(pod,pad,book,phone) story). So (some) firms will do very well when selling to 'behavioral' consumers.
- start instead in the equilibrium where the reference point is λ_a^* and consumers buy from A with probability ρ. If firm A raises

it price, then it will take a higher reference point and a higher value of ρ to make consumers indifferent. Counterintuitively raising price will increase sales. In this kind of environment you might expect both firms to have very high prices and close market shares (Canadian cell phone service is like this - very high prices despite the fact there are many firms).

– there is also a very competitive outcome in which firms set low prices, references points are interior, but if any firm raises its price, consumers revert to an equilibrium in which they expect to buy for sure from the firm who didn't raise price.