1 Choice Theory with Uncertainty

- let $\mathcal{X}$ be a finite set of possible outcomes, or consequences
- for example, consumption bundles, payoffs to a stock portfolio, outcomes of a research project
- N is the number of elements in $\mathcal{X}$, $x \in \mathcal{X}$ is a typical element
- a *lottery* is a pair $(\mathcal{X},p)$ with $p \in \mathbf{R}_{+}^{N}$, where $p = \{p_1, \ldots p_N\}$ and

$$\sum_{i=1}^{N} p_i = 1$$

- interpret p_i as the probability with which outcome or consequence i happens
- $\mathcal{X}$ is very general - it could be simply money payoffs, or it could be a finite set of lotteries

$$\mathcal{X} = \{(\mathcal{Y}_1, p_1), \ldots (\mathcal{Y}_N, p_N)\}$$

- In this case $(\mathcal{X},q)$ is called a *compound lottery*
- if $\mathcal{Y}_i = \mathcal{Y}$ for each i, where $\mathcal{Y}$ is some finite set containing M elements, and

  $$\mathcal{X} = \{(\mathcal{Y}, p_1), \ldots (\mathcal{Y}, p_N)\}$$

  then the *reduced lottery associated with* $(\mathcal{X},q)$ is defined to be $\left(\mathcal{Y}, \sum_{i=1}^{N} q_i p_i\right)$
- generally, let $\mathcal{L}$ be the set of possible lotteries. When the set of outcomes is the same in every lottery, we can refer to lotteries by referring to the probabilities that they generate. So for example, p' and p'' are elements of $\mathcal{L}$
- the picture that follows depicts a lottery with 3 outcomes (this is not the same diagram as the one in the text). Outcome 1 occurs with probability 1/2 outcome 2 occurs with probability 1/4 while outcome 3 occurs with the residual probability

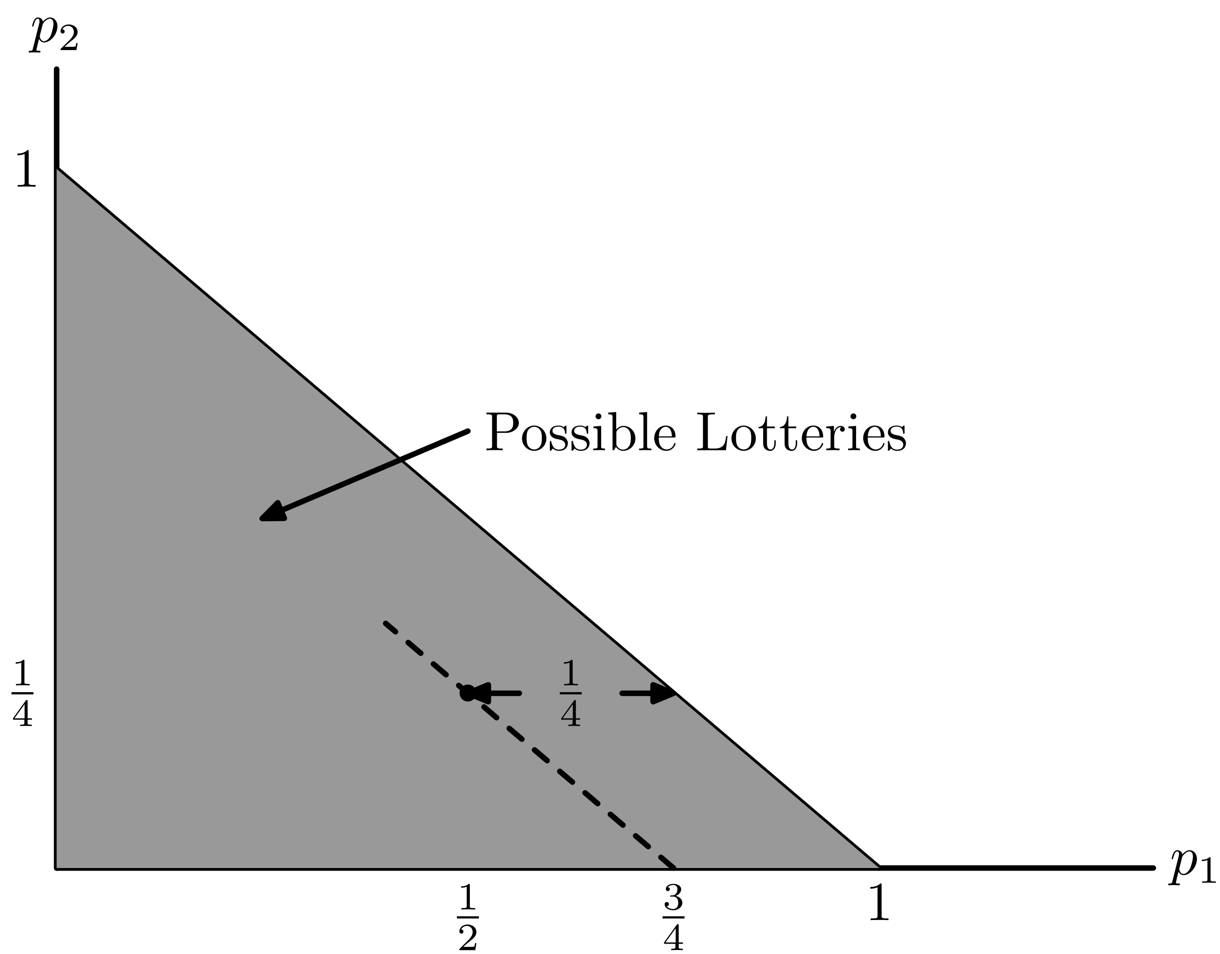

Figure 1: Feasible Lotteries

1.0.1 Preferences

- it is natural to suppose that lotteries can be ordered in the way that consumption bundles can
- interpret $p \succcurlyeq p'$ to mean that the lottery p is at least as good as the lottery p'
- the preference ordering $\succcurlyeq$ is said to be *complete* if for *any* pair $p, p' \in \mathcal{L}$, either $p \succcurlyeq p'$ or $p' \succcurlyeq p$
- $\succcurlyeq$ is said to *transitive* if for any $p, p', q \in \mathcal{L}$, $p \succcurlyeq p'$ and $p' \succcurlyeq q \Rightarrow p \succcurlyeq q$
- interpret $p \sim p'$ to mean indifference in the preference relation $\succcurlyeq$ (i.e., $p \succcurlyeq p'$ and $p' \succcurlyeq p$)
- $\succcurlyeq$ is said to satisfy *Reduction of Compound Lotteries* if for any $(\mathcal{X}, p), (\mathcal{X}, p') \in \mathcal{L}$

$$\lambda (\mathcal{X}, p) + (1 - \lambda) (\mathcal{X}, p') \sim (\mathcal{X}, \lambda p + (1 - \lambda) p')$$

- *Continuity* For any $p, p', p'' \in \mathcal{L}$ the sets

$$\{\lambda \in [0,1] : \lambda p + (1-\lambda)\, p' \succcurlyeq p''\}$$

and

$$\{\lambda \in [0,1] : p'' \succcurlyeq \lambda p + (1-\lambda)\, p'\}$$

are both closed.

- Recall that transitivity,completeness and continuity imply the existence of a utility function $U : \mathcal{L} \to \mathbf{R}$ such that $p' \succcurlyeq p$ if and only if $U\,(p') \geq U\,(p)$

- *Independence Axiom* $p \succcurlyeq p'$ if and only if for all $\lambda \in [0,1]$ and $p'' \in \mathcal{L}$

$$\lambda p + (1-\lambda)\, p'' \succcurlyeq \lambda p' + (1-\lambda)\, p''$$

- A utility function U has the expected utility property if $U\,(p) = \sum_{i=1}^{n} p_i u\,(x_i)$

- *Regularity Properties*

1. $\exists b, w \in \mathcal{L} : b \succcurlyeq p \succcurlyeq w$ for all $p \in \mathcal{L}$
2. $\lambda > \lambda'$ if and only if $\lambda b + (1-\lambda) w \succ \lambda' b + (1-\lambda') w$
3. U satisfies the expected utility property if for every $\lambda \in [0,1], p, p' \in \mathcal{L}$

$$U(\lambda p + (1-\lambda) p') = \lambda U(p) + (1-\lambda) U(p')$$

- Let x_1 be the lottery in which the outcome x_1 is received with probability 1. Let x_{-1} be the lottery where x_1 occurs with probability 0 while each of the other outcomes x_j occurs with probability $p_j/(1-p_1)$. By the reduction of compound lotteries

$$p \sim p_1 x_1 + (1-p_1) x_{-1}$$

Then by the hypothesis in the theorem and the property of utility

$$U(p) = U(p_1 x_1 + (1-p_1) x_{-1})$$

$$= p_1 U(x_1) + (1-p_1) U(x_{-1})$$

Now expand $U(x_{-1})$ in the same manner, and proceed recursively to get

$$U(p) = \sum_{i=1}^{n} p_i U(x_i)$$

- *Expected Utility Theorem*

- Theorem: Suppose $\succcurlyeq$ satisfies transitivity,completeness,continuity, reduction of compound lotteries, the technical restrictions and the independence axiom. Then there is a utility function U associated with $\succcurlyeq$ that has the expected utility property.

- **Proof: Step 1** - Define a potential utility function. Using the technical restrictions, there is a best and worst lottery in $\mathcal{L}$ so define a function u such that $u(b) = 1$ and $u(w) = 0$. Now for each $p \in \mathcal{L}$ define

$$u(p) = \{\lambda : \lambda b + (1-\lambda) w \sim p\}$$

Notice that this means

$$p \sim u(p) b + (1 - u(p)) w \tag{1}$$

- **Step 2** - we don't know whether u is actually a function. By the technical restrictions the sets

$$\{\lambda' : \lambda' b + (1 - \lambda') w \succcurlyeq p\}$$

and

$$\{\lambda' : p \succcurlyeq \lambda' b + (1 - \lambda') w\}$$

are both closed subsets of $[0, 1]$ by continuity. The union of these sets is $[0, 1]$ by completeness. Thus they have at least one point in common. Suppose they have more than one point in common, say λ' and λ''. Then without loss of generality $\lambda'' > \lambda'$. Then by the second technical restriction

$$\lambda'' b + (1 - \lambda'') w \succ \lambda' b + (1 - \lambda') w$$

which is a contradiction to the supposition that they are in the same set.

- **Step 3** Show that u satisfies the expected utility property, i.e

$$u(\lambda p + (1 - \lambda) p')$$

$$= \lambda u(p) + (1-\lambda) u(p')$$

- start with

$$\lambda p + (1-\lambda) p'$$

by 1 and the independence axiom this is indifferent to

$$\lambda [u(p) b + (1 - u(p)) w] + (1-\lambda) p'$$

and consequently to

$$\lambda [u(p) b + (1 - u(p)) w] + (1-\lambda) [u(p') b + (1 - u(p') w)]$$

By the reduction in compound lotteries this is indifferent to

$$[\lambda u(p) + (1-\lambda) u(p')] b + [1 - [\lambda u(p) + (1-\lambda) u(p')]] w$$

Then by the fact that this is indifferent to $\lambda p + (1-\lambda) p'$ and using the definition of u this gives

$$u(\lambda p + (1-\lambda) p') =$$

$$\lambda u(p) + (1-\lambda) u(p')$$

so u has the expected utility property by the previous proposition.

- **Final step** - is u a utility function, i.e.

$$u(p) \geq u(p')$$

iff

$$p \succsim p'$$

This follows immediately from the second technical restriction

- **Problems with the Independence Axiom**

- Allais - monetary prizes are $\{1000, 500, 0\}$. You are given the choice between two lotteries

$$a = \{0, 1, 0\}$$

and

$$b = \{.10, .89, .01\}$$

which means you can have 500 for sure, or take a 10 percent chance of raising 500 to 1000 at the cost of a slight chance of losing everything. The second pair of choices looks as follows

$$a' = \{0, .11, .89\}$$

and

$$b' = \{.10, 0, .9\}$$

which means that you have an 11 percent chance of winning 500 which you can turn in to a 10 percent chance of winning 1000 at the cost of raising the chance of getting nothing slightly. Most people choose a over b and b' over a', which is inconsistent with the independence axiom

- **Subjective Expected Utility**
- start with S different *states of the world.* Associated with each state is some kind of state contingent outcome $x_s \in \mathcal{X}_s$
- define the objects of choice $\mathcal{L}$ to be arbitrary lists $\{x_1, \ldots, x_S\}$of these state contingent outcomes. Suppose a rational continuous

preference relation exists over these lists. Then a utility function exists - if x and x' are two distinct contingent plans (i.e. lists of outcomes in each of the possible states), then $x \succeq x'$ if and only if $u(x) \geq u(x')$. In this argument the lists x and x' could be filled with complicated things.

- recall from the proof of the expected utility theorem that the independence axiom gives linearity.

- Does it make sense to assume that the independence axiom holds for lists?

- It depends on what $\mathcal{X}$ is. The most obvious problem is that the *mixture operation* $\lambda x + (1 - \lambda) x'$ might not make sense. For example, suppose that $\mathcal{X}$ is a finite list of dollar amounts, as in the examples above. Let x be a constant plan that involves the same dollar amount for every state, and x' another constant plan. The $\lambda x + (1 - \lambda) x'$ will generally be a dollar amount, but not a dollar amount which is in $\mathcal{X}$. More generally, the objects in $\mathcal{X}$ might not be algebraic, so something like $\lambda * \mathit{apple} + (1 - \lambda) * \mathit{orange}$ isn't a helpful idea.

- This is not the only problem. A more difficult problem occurs if $\mathcal{X}$ is the set of a consumption bundles in some budget set contained in $\mathbb{R}^n$. If x, x', and x'' are consumption *plans* then the independence axiom requires that $x \succeq x'$ and $\lambda \in [0,1]$, then $\lambda x + (1-\lambda)x'' \succeq \lambda x' + (1-\lambda)x''$, these last two things are both well defined consumption plans

- to see why it doesn't make sense to impose the independence axiom here notice that if the independence axiom holds for all of $\mathcal{L}$, then it must hold for any subset of $\mathcal{L}$ on which the the mixture operation $\lambda x + (1-\lambda)x$ is well defined.

- this would include the set $\overline{\mathcal{L}}$ of plans that involve the same consumption bundle in every state. Imposing the independence axiom on these plans would be almost the same as imposing the independence axiom on preference over consumption bundles. This wouldn't make sense since it would required indifference curves to be linear (or all goods are perfect substitutes).

- Anscombe-Aumann instead of ranking state contingent lists of consumption plans, imagine ranking state contingent lists of *lotteries* over consumption plans - then writing $\lambda x + (1-\lambda)x''$ in the independence axiom involves writing down a list of compound lotteries. If we assume that the independence axiom holds over the entire set of plans of this kind, then that will certainly imply that it must hold over the set of consumption lotteries, but this is something we are prepared to accept.

- Fix outcomes $\mathcal{X}$ and states $\mathcal{S}$ - imagine that both are finite. For each s define $\mathcal{L}_{\mathcal{X}}$ to be the set of lotteries over $\mathcal{X}$. Let

$$\mathcal{L} = \{p; p = \{p_1, p_2, \ldots, p_S\}; p_j \in \mathcal{L}_{\mathcal{X}}\}$$

- Suppose that a continuous rational preference relation $\succeq$ exists on the set $\mathcal{L}$. For any pair p, p' in $\mathcal{L}$ let

$$\lambda p + (1-\lambda)p' = \{\lambda p_1 + (1-\lambda)p'_1, \ldots, \lambda p_S + (1-\lambda)p'_S\}$$

- State Uniformity: write the lottery $\{p_1, \ldots, p_j, \ldots, p_S\}$ as $\{p_j, p_{-j}\}$. If $\{p^0, p_{-j}\} \succeq \{p^1, p_{-j}\}$ where p^0 and p^1 are simple lotteries over the outcomes $\mathcal{X}$, then $\{p^0, p_{-k}\} \succeq \{p^1, p_{-k}\}$ for any k and any p_{-k}.

- Technical Conditions: (T1) there are best and worst elements b and w in $\mathcal{L}$; and (T2): if $\lambda > \lambda'$ then

$$\lambda b + (1-\lambda)w \succ \lambda' b + (1-\lambda')w$$

- Anscombe-Aumann theorem: Suppose preferences over $\mathcal{L}$ are rational, continuous, state uniform and satisfy the independence axiom. Then if the technical conditions T1 and T2 hold, there is a probability vector $\pi \in \mathbb{R}^S$ and utility function $u : \mathcal{L}_{\mathcal{X}} \to \mathbb{R}$ such that $p \succeq p'$ if and only if

$$\sum_{s \in S} \pi_s u(p_s) \geq \sum_{s \in S} \pi_s u(p'_s)$$

- Proof: Follow the proof of the expected utility theorem to show that there is a linear utility function representing preferences: i.e. $p \succeq p'$ iff $u(p) \geq u(p')$, and for any $\lambda \in [0,1]$, $u(\lambda p + (1-\lambda)p') =$

$\lambda u(p) + (1-\lambda)u(p')$. If there are N elements in $\mathcal{X}$, then each p has $N \times S$ components p_{ij} and the linear utility function can be written as

$$\sum_{i=1,S;j=1,N} b_{ij} p_{ij}$$

By state uniformity $\{p^0, p_{-j}\} \succeq \{p^1, p_{-j}\}$ implies that $\{p^0, p_{-k}\} \succeq \{p^1, p_{-k}\}$ for all k so $b_{ij} = \gamma_i b_{1j}$ for some non-negative vector of constants γ_i. Then we can write the utility function as

$$\sum_{i=1,S;j=1,N} \gamma_i b_{1j} p_{ij}$$

By T2 at least one of the γ_i must be positive. If the utility function $\sum_{i=1,S} \gamma_i u(p_i)$ represents the preference ordering, then

$$\frac{1}{\sum_{i=1,S} \gamma_i} \cdot \sum_{i=1,S} \gamma_i u(p_i)$$

also represents the ordering so we let $\pi_s = \frac{\gamma_s}{\sum \gamma_{s'}}$.

- in the result above, we interpret π_s as the subjective probability with which state s occurs.

- this idea is used everywhere in modern economic theory, probably because of the following application:

- you are the row player in the famous prisoner's dilemma game, describe in the following table

	c	d
c	2,2	-1,4
d	4,-1	0,0

- when you both play c, you both get 2, but the column player can improve his payoff to 4 if you play c, etc

- think of the state of the world S as consisting of the two possible actions $\{c, d\}$ by your opponent. The choice set $\mathcal{X}$ consists of all possible lotteries over your own action, or if you know game theory, $\mathcal{X}$ is all your mixed strategies. You have to choose a mixed strategy without knowing what your opponent will do - so a mixed strategy for this game is just a special state contingent plan - we assume you have preferences over all plans, but we only need to know the preferences over uncontingent plans for this to work. You make a choice - assuming your preferences satisfy the assumptions in the Anscombe Aumann theorem, you will look like you have chosen a mixed strategy that maximizes your expected utility relative to some prior belief about what your opponent will do. (is state uniformity plausible here? in what sense?).

- **Problems with Subjective Expected Utility:** Ellsberg - there are two urns consisting of red and black balls. The first urn has 51 red balls and 49 black balls. The second urn has 100 red and black balls (the composition is unknown). The first experiment proposes to give you 10 dollars if a red ball is drawn - you simply choose which urn you want the ball to be drawn from, the second experiment is the same except that you are paid 10 dollars if a black ball is drawn - again choose a urn. Most people choose the first urn in both cases