1 Demand Lecture

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- a demand correspondence is a special kind of choice correspondence where the set of alternatives is $\mathcal{X} = \left\{ x \in \mathbb{R}^n_+ \right\}$ and the family $\mathcal{B}$ consists of all sets of the form

$$B\left(p, w\right) = \left\{ x \in \mathbb{R}^n_+ : px \leq w; p \in \mathbb{R}^N_+; w \in \mathbb{R}_+ \right\}$$

- a choice correspondence for this problem takes each set $B\left(p, w\right)$ and converts it into some subset $C\left[B\left(p, w\right)\right] \subset B\left(p, w\right)$

- since each set in $\mathcal{B}$ is completely characterized by the price and wealth level that defines it we might as well write $C\left[B\left(p, w\right)\right]$ as $x\left(p, w\right)$ which is just the well known demand correspondence (or function if it is single valued).

- as before we want to know what conditions the demand correspondence has to satisfy in order for it to be consistent with 'rational utility maximizing behavior' (specifically there is a rational preference relation such that for each set $B(p, w)$, $x(p, w)$ coincides exactly with the alternatives in $B(p, w)$ that are at least as good as all alternatives in $B(p, w)$).

- a natural candidate for this condition is the weak axiom of revealed preference since we know that for *some* families $\mathcal{B}$, this is all that is required

- unfortunately, the family of budget sets $B(\cdot, \cdot)$ is not one of them

- to clarify the problem a bit, we have already shown that rational choice behavior will generate data that satisfies the weak axiom for every family of budget sets, so we could reject classical demand theory by coming up with data for which the weak axiom fails. We are looking for a more powerful test.

- equally unfortunately, the argument required to show that rational maximizing behavior implies more than the weak axiom is subtle and requires the introduction of new concepts

- we won't do this in full generality, in particular stick with demand functions.
- Thm: If the demand function satisfies $px(p,w) = w$ uniformly, then it satisfies the weak axiom if and only if for every pair (p,w) and every alternative price vector p'

  $$(p'-p)\left[x(p',p'x(p,w)) - x(p,w)\right] \le 0$$

  with strictly inequality whenever $x(p,w) \neq x(p',p'x(p,w))$.
- Proof: We show that the weak axiom implies the inequality. Multiplying out the left hand side gives

  $$p'\left[x(p',p'x(p,w)) - x(p,w)\right] - p\left[x(p',p'x(p,w)) - x(p,w)\right]$$

  Now $p'x(p',p'x(p,w)) = p'x(p,w)$. Suppose that $x(p',p'x(p,w))$ was inside the budget set $B(p,w)$ (obviously $x(p,w)$ is inside the budget set $B\left(p', p'x\left(p,w\right)\right)$), then by the weak axiom $x\left(p', p'x\left(p,w\right)\right)$ must be in the demand correspondence at price p and income w. So either $x\left(p', p'x\left(p,w\right)\right) = x\left(p,w\right)$ or $px\left(p', p'x\left(p,w\right)\right) > px\left(p,w\right)$ both of which give the inequality.

- the proof that the inequality implies the weak axiom is more tedious and is left out

- let dx be any change in a consumption vector that involves a movement along the budget hyperplane. By definition $p \cdot dx = 0$ which is sometimes formally expressed by saying that the price vector is orthogonal to the budget hyperplane.

- the weak axiom forces dp and dx to point in opposite directions (and conversely, if dp and dx always point in opposite directions, then the weak axiom must hold).

- our goal is to show that rational utility maximization implies strictly more than this

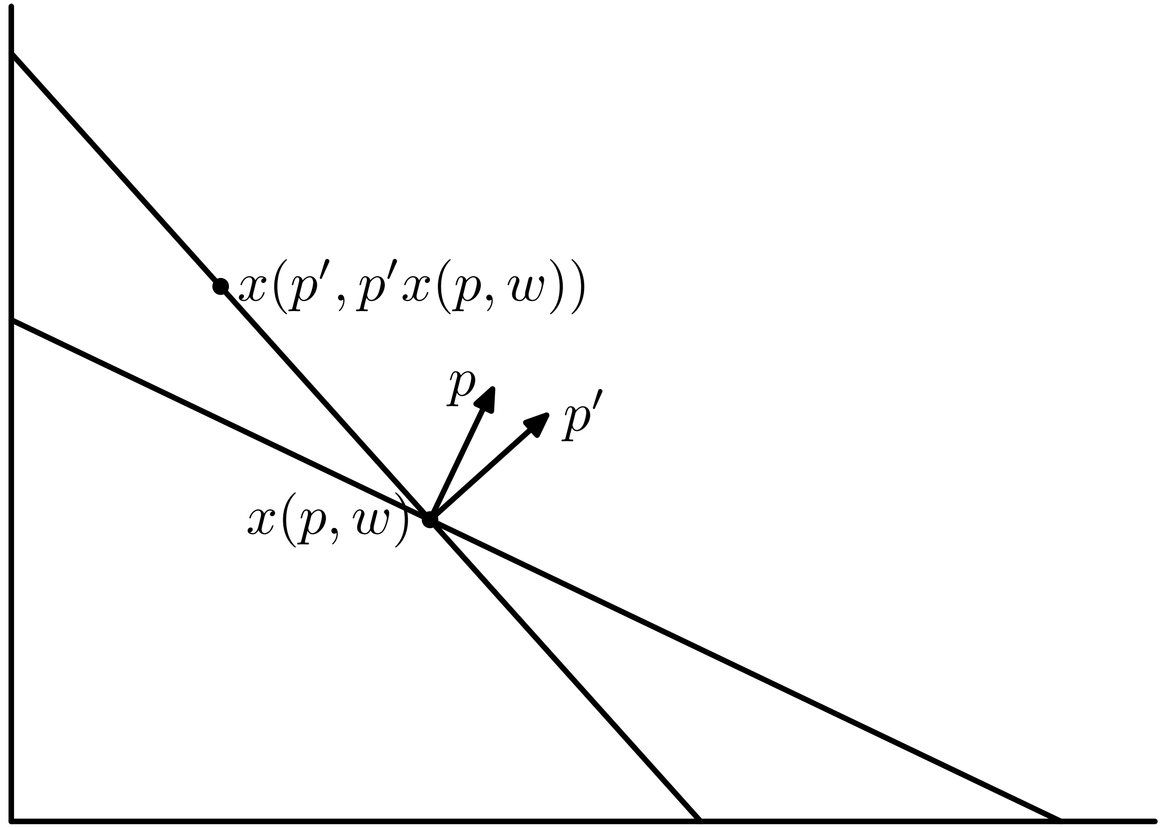

Figure 1: Weak Axiom

- at the same time, this exercise provides an opportunity to point out exactly what the implications of the weak axiom, and rational decision making are in the demand context. Since rational decision making implies the weak axiom, we can reject traditional demand theory by finding violations of the weak axiom. The objective is to show that if the weak axiom does hold, then consumers may or may not be acting as rational utility maximizers

- to draw the implication of the weak axiom in a slightly different form, write for the income compensated change in demand

$$dx = x_p(p,w)\,dp + x_w(p,w)\,dw =$$

$$x_p(p,w)\,dp + x_w(p,w)\,x(p,w)\,dp$$

  then since $dpdx \leq 0$

$$dp'[x_p(p,w) + x_w(p,w)\,x(p,w)]dp \leq 0$$

- since $[x_p(p,w) + x_w(p,w)\,x(p,w)]$ is an $N \times N$ matrix, the weak axiom implies that it is negative semi-definite (and that is all it implies).

- demand theory derived from rational preferences

- assuming that a rational preference relation $\succeq$ exists on the consumption set $\mathcal{X}$, we sometimes impose some restrictions on it that have nothing to do with rationality per se

- *local non-satiation* means that the preference ordering satisfies

$$\forall x \in \mathcal{X}; \forall \varepsilon > 0 \exists y : ||y - x|| < \varepsilon; y \succ x$$

- *convex* if for every pair $x, x' \in \mathcal{X}$ such that $x \succeq x'$ and $x' \succeq x$, (i.e. $x \sim x'$) $\lambda x + (1 - \lambda) x' \succeq x$ for all $\lambda \in [0, 1]$

- *homothetic* if for every pair $x, x' \in \mathcal{X}$ such that $x \sim x'$; $\alpha x \sim \alpha x'$ for all $\alpha \geq 0$.

- *quasi-linear* with respect to commodity k if for every $x \sim x'$, $x + \alpha e_k \sim x' + \alpha e_k$ for all $\alpha \geq 0$, where e_k is a vector whose coordinates are all zero except for the k^{th} which is equal to 1.

- *continuous* if whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of consumption bundles satisfying $x_n \succeq y_n$ for all n, then $\lim_{n\to\infty} x_n \succeq \lim_{n\to\infty} y_n$

- Existence of utility functions

- Thm: let $\succeq$ be a continuous rational preference ordering satisfying $x \geq x' \Rightarrow x \succeq x'$ (weak-monotonicity). Then there exists a utility function u that represents $\succeq$.

- Proof: Let $e \in \mathbb{R}_+^N$ be such that $e = [1, 1, \ldots 1]$. Let

$$Z = \left\{ x' \in \mathbb{R}_+^N : x = \alpha e \, \text{for some } \alpha \geq 0 \right\}$$

For any $x \in \mathbb{R}_{++}^N$ there is a $z \in Z$ such that $z \geq x$ (one such would be $z \equiv [\max_j x_j] \cdot e$), and so by weak-monotonicity, $z \succeq x$. Similarly, there is a $z' \in Z$ (i.e. 0) such that $x \geq z'$, and therefore $x \succeq z'$. So the sets $P^+(x) = \{z \in Z : z \succeq x\}$ and $P^-(x) = \{z' \in Z : x \succeq z'\}$ are both non-empty. Furthermore both sets are closed by the continuity of preferences. By completeness of preferences, $z \succeq x$ or $x \succeq z$ for all $z \in Z$, so $Z \subset P^{+}(x) \cup P^{-}(x)$ and so $Z = P^{+}(x) \cup P^{-}(x)$.

Now $P^{+}(x)$ and $P^{-}(x)$ must have a point in common (if they don't then $P^{-}(x)$ is the complement of $P^{+}(x)$ in Z, which could not be closed). Furthermore they can have only one point in common by monotonicity. Let $\alpha(x)e$ be this point. The claim is that $\alpha(x)$ is the desired utility function.

To see it, suppose that $\alpha(x) \geq \alpha(y)$. Then by monotonicity $\alpha(x)e \succeq \alpha(y)(e)$. By transitivity, $x \sim \alpha(x)e \succeq \alpha(y)e \sim y$ implies $x \succeq y$. The reverse implication is proved in a similar way.

- a preference ordering that is not continuous may not be representable by a utility function - an example is lexicographic $(x, y) \succ (x', y')$ if $x > x'$ or if $x = x'$ and $y > y'$.

- a preference ordering that is continuous may be representable by a discontinuous utility function

- once the utility function has been defined, there are three functions of interest
- *the Walrasian demand function* (we assume enough to make these correspondences differentiable functions in this section) is given by

$$x\left(p,w\right)\equiv\arg\max\left\{u\left(x\right):px\leq w\right\}$$

- the *indirect utility function* is given by

$$v\left(p,w\right)=u\left(x\left(p,w\right)\right)$$

- the *Hicksian demand function* is given by

$$h\left(p,\overline{u}\right)=\arg\min\left\{px:u\left(x\geq\overline{u}\right)\right\}$$

- the *(minimum) expenditure function* is given by

$$e\left(p,\overline{u}\right)=ph\left(p,\overline{u}\right)$$

- only the Walrasian demands are directly observable, nonetheless these functions can be used to draw out the testable implications of demand theory.

- First Property: The expenditure function is concave in prices

- Proof:

  $$e\left(\lambda p+(1-\lambda)\, p', \overline{u}\right)=\min \left\{\left(\lambda p+(1-\lambda)\, p'\right) x : u\,(x) \geq \overline{u}\right\}$$

  Let x^* be a solution to this problem. Then

  $$\lambda p x^*+(1-\lambda)\, p' x^* \geq$$

  $$\min \{\lambda p x : u\,(x) \geq \overline{u}\}+\min \{(1-\lambda)\, p' x : u\,(x) \geq \overline{u}\}=$$

  $$\lambda \min \{p x : u\,(x) \geq \overline{u}\}+(1-\lambda) \min \{p' x : u\,(x) \geq \overline{u}\}=$$

  $$\lambda e\,(p, \overline{u})+(1-\lambda)\, e\,(p', \overline{u})$$

- from standard properties of concave functions, note that this implies that if the expenditure function is twice differentiable, then the matrix of second derivatives of the function is *symmetric* and negative semi-definite.

- Second Property $h(p, \overline{u}) = e_p(p, \overline{u})$

- Proof: follows immediately from the envelope theorem.

- notice that this implies that the matrix of first derivatives of the Hicksian demand function with respect to prices is equal to the matrix of second derivatives of the expenditure function with respect to prices. Since the expenditure function is concave, this matrix is *symmetric and negative semi-definite.*

- Primary Result: $h_p(p, \overline{u}) = x_p(p, w) + x_w(p, w)\, x(p, w)$ where $w = e(p, \overline{u})$

- Proof:

$$h_p(p, \overline{u}) = x_p(p, e(p, \overline{u})) + x_w(p, e(p, \overline{u}))\, e_p(p, \overline{u})$$

$$x_p\left(p, e\left(p, \overline{u}\right)\right) + x_w\left(p, e\left(p, \overline{u}\right)\right) h\left(p, \overline{u}\right) =$$
$$x_p\left(p, e\left(p, \overline{u}\right)\right) + x_w\left(p, e\left(p, \overline{u}\right)\right) x\left(p, e\left(p, \overline{u}\right)\right)$$

- this is the same matrix we saw previously when we derived the implications of the weak axiom in the differentiable case. Notice that the concavity of the expenditure function forces this matrix to be symmetric as well as negative definite

- so rationality implies more than the weak axiom, the two are not equivalent.