Auctions

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In an auction, a single seller tries to sell one unit of some commodity to one of n different buyers. Only one buyer can buy the unit. The seller's problem is to decide who to sell it to, and how much to charge them. We'll imagine that sellers are risk neutral expected revenue maximizers. They just want to sell the good in a way that will maximize their expected revenue. Buyers are interested in the difference between what an object is worth to them, and what they have to pay for it. They want to maximize the product of the probability that they win the auction times the difference between their value and what they expect to pay for the good when they win.

The thing that makes everyone's problem hard is that no one knows any of the buyers' values. It is a game of incomplete information. In everything that follows, we'll make the assumption that values are identically and independently distributed. To keep things simple, lets just suppose that this distribution has its support on the interval $[0,1]$, meaning that if $F(x)$ is the probability that a certain bidder's valuation is less than or equal to x, then $F(0) = 0$ and $F(1) = 1$. Otherwise, lets suppose this distribution has a density given by $f(x)$. This means, of course, that $F(x) = \int_0^x f(t)\,dt$.

The idea here is that each bidder believes that each of the other bidders has a value that is somewhere between 0 and 1, and that $F(x)$ is the probability that this value is less than or equal to x.

Second price auction

In a sealed bid second price auction, each bidder submits a bid to the seller. The seller then chooses the bidder who submits the highest bid, and offers him the good at a price which is equal to the second highest bid that was submitted.

To describe the payoff in the auction, let b be the vector consisting of the n bids submitted to the auction. A player who has value v_i has payoff

$$V(b_i, b_{-i}, v_i) = \begin{cases} \frac{v_i - \max_{j \neq i} b_j}{\left|\left\{j' \neq i; b_{j'} = b_i\right\}\right| + 1} & b_i \geq b_j \forall j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

There are two cases, if your bid isn't the highest bid, you get nothing. If your bid is highest, you have the same chance as each of the other bidders whose bid is also highest. So for example, if you are highest and tied with another bidder for highest bid, you will trade with probability $\frac{1}{2}$.

The notation $|\{j' \neq i; j' = \max_{j\neq i} b_j\}|$ means to count ($||$) the number of elements in the set consisting of all the others who are tied with you as highest bidder.

As you'll see, ties won't be much of an issue as long as the distribution function $F(\cdot)$ is continuously differentiable.

No bidder knows what the other bidders will bid. To handle this, we assume that each bidder has a plan, say $b(v)$ that describes what he or she plans to bid for each possible value that they might have. In that case, what the bidder expects to get from a bid b' is

$$\mathbb{E}_{v_{-i}} V\left(b', \{b(v_j)\}_{j\neq i}, v_i\right). \tag{2}$$

Notice that the expectation is taken of the *values* of the other players which the bidder doesn't know. What we assume the bidder does know is the plans of the other players. This is a fundamental part of Nash equilibrium - the players correctly guess what the other players plan to do.

A (symmetric) Bayesian Nash equilibrium of the second price auction is a bidding rule b^* such that

$$\mathbb{E}_{v_{-i}} V\left(b^*(v_i), \{b^*(v_j)\}_{j\neq i}, v_i\right) \geq \mathbb{E}_{v_{-i}} V\left(b', \{b^*(v_j)\}_{j\neq i}, v_i\right)$$

for each pair (i, v_i).

Lets solve the second price auction using pure math, assuming that all bidders are using the rule $b(v)$ and assuming there are only two bidders. If you think you know the rule $b(v)$ that the other player is using, then all you really have to do is to figure out for any bid b' whether or not the other player's value is high enough for him or her to outbid you. One way to think about this is to imagine that when your value is v instead of bidding $b(v)$, which is what you expect everyone else to do if they had the same value, you could submit b', which is the bid you would have submitted if your value were v' instead of v. Then you could write your payoff as if you were using the rule $b(\cdot)$, just like the other player, but pretending to have value v' instead of v. Then you know the other player would outbid you if their value were larger than v'.

Specifically, in the two bidder case, your payoff would then be

$$\int_0^{v'} (v - b(\tilde{v})) f(\tilde{v}) \, d\tilde{v}.$$

Since the rule $b(\cdot)$ that each player is using should be a best reply to against the belief that the other player is also using that rule, this payoff should be maximized when v' is equal to v, your real value. What that means is that

$$\left.\frac{d}{dv'} \int_0^{v'} (v - b(\tilde{v})) f(\tilde{v}) \, d\tilde{v}\right|_{v'=v} = 0 \tag{3}$$

by the usual way we find the maximum of a function. Taking the derivative gives the condition $(v - b(v)) f(v) = 0$ which can only be true when $b(v) = v$. Now you know one way to show that bidders in a second price private value auction will bid their value.

Diversion: Fixed points

Let $x \in [0,1]$ and consider the function $\frac{1}{4} + \frac{1}{4}x$. This function takes and x and transforms it into a number that lies in the same set that x does. So we could write

$$T(x) = \frac{1}{4} + \frac{x}{4}.$$

A *fixed point* of the transformation T is a number x^* such that $T(x^*) = x^*$. Obviously there is only one fixed point for this T, $x^* = \frac{1}{3}$. Most equilibrium concepts in economics involve fixed points. For example, let $D(p)$ and $S(p)$ be demand an supply curves written with p on the vertical axis and q on the horizontal axis. Here is a transformation

$$p' = S(D^{-1}(p))$$

Make sure you draw a little picture of this transformation so you can see what it is doing. Convince yourself that any fixed point satisfying $p^* = S\left(D^{-1}(p^*)\right)$ is a market clearing price. If you choose some linear demand and supply curves, I am sure you can find the corresponding fixed point.

We are looking for an equilibrium for an auction, so you can use fixed points. The only leap you have to make is to realize that you can transform functions into new functions the same way you can transform numbers into new numbers. The approach we used for the second price auction is just like this.

Start with an arbitrary function $b(v)$ which is a non-decreasing continuous function that converts a number in $[0,1]$ into another number in $[0,1]$. Now instead of transforming numbers, lets transform the whole function b. We'll do it this way

$$\tilde{b}(v) = b\left(\arg\max_{v'} \int_0^{v'} (v - b(\tilde{v})) f(\tilde{v})\, d\tilde{v}\right)$$

Notice that whenever the argmax of $\int_0^{v'} (v - b(\tilde{v})) f(\tilde{v})\, d\tilde{v}$ is not equal to v then $\tilde{b}(v)$ and $b(v)$ will be different. So just like the two previous examples, we could look for a fixed point by finding a *function* $b^*(v)$ such that

$$b^*(v) = b^*\left(\arg\max_{v'} \int_0^{v'} (v - b^*(\tilde{v})) f(\tilde{v})\, d\tilde{v}\right).$$

where we say two function $\tilde{b}$ and b are equal if they are equal for all arguments. The argument in the previous section shows you how to find a fixed point for a functional transformation.

Boilerplate

If that seems mysterious, and you are wondering why it works out that way, here is another argument. If b_i is a bid for player i and b_{-i} are the bids of the other players, then there are a bunch of possibilities.

1. $b_i \geq b_j \forall j$ (i is the winning bidder) and $v_i > \max_{j \neq i} b_j$. Good news, the bidder gets a positive payoff. As long as his bid remains above $\max_{j \neq i} b_j$ the bidder continues to make this positive payoff. So the bidder would be just as happy to bid v_i.

2. $b_i \geq b_j \forall j$ (i is the winning bidder) and $v_i \leq \max_{j \neq i} b_j$. Bad news, the bidder is worse of buyer because he/she pays too much. If she changes her bid to v_i she either won't care, or will be strictly better off, depending on whether $b_i < \max_{j \neq i} b_j$.

3. $b_i \leq \max_{j \neq i} b_j$. Loser - gets nothing. Then if she switches her bid to v_i, one of two things will occur, either $v_i \leq \max_{j \neq i} b_j$, in which case i will still be a loser. Otherwise, i would become the high bidder, but pay less than her value, which is better than nothing.

The upshot is that *whatever* the bids submitted by the other players, i can do at least as well by bidding v_i and sometimes she will do strictly better. Since the expectations used in (2) just consider all the possible profiles of bids, it must be that the expected payoff a player receives by using the rule b^* are at least as high as the payoff she gets from using any alternative bidding rule no matter what the other players are doing. Now just continue - if there is an equilibrium b^* then replace b^* with $\hat{b}(v_1) = v_1$ for player 1. This may improve 1's expected payoff, but will never hurt it. If 1 adopts this new strategy, then the other players might be hurt, but even if they are, we can make the same argument for player 2, then 3, and so on, until we have replaced all the bidding rules with $\hat{b}$. At that point we won't be able to improve any player's payoff. So $\hat{b}(v_i) = v_i$ must be a Bayesian Nash equilibrium.

Maybe this will help you see the reasoning behind (3). If b' is less that v, then sometimes you are going to lose auctions you might have won at prices less than your value. You could win these auctions by raising the bid to v. The rest of the time you won't care one way or the other which of the bids b' or b you had submitted. So the derivative in (3) evaluated at $v' < v$ must be positive. Similarly if you choose $v' > v$, then you will sometimes be winning auctions and paying prices that are higher than what the good is worth to you. The rest of the time, it won't matter. So the derivative in (3) will be negative. The upshot, just bid v.

Lets check some of the implications of this. If a bidder wants to figure out how likely it is he or she will win in equilibrium, they just compute the probability that their value is highest. This is the probability that each of the other bidders has a lower value than they do, and this is just $F(v)^{n-1}$. So the probability they win the auction when their value is v is going to be $F(v)^{n-1}$.

What are they likely to pay if they win? This is the expectation of the highest value of the other bidders, conditional on the other bidders having values lower than v. This is

$$\frac{(n-1)\int_0^v \tilde{v}F^{n-2}(\tilde{v})\,f(\tilde{v})\,d\tilde{v}}{F^{n-1}(v)}. \tag{4}$$

Lets go over this calculation. Pick one of the other bidders, say bidder 1, and imagine that his value is $\tilde{v}$. This event occurs with probability $f(\tilde{v})$. In that case, the probability that all the others have lower values than his is $F(\tilde{v})^{n-2}$, because there are $n-2$ bidders other than you and bidder 1. Of course, it could also have been bidder 2 who had this value $\tilde{v}$. Summing this up over the $n-1$ bidders other than yourself, this says that the probability that the highest value bidder among the others has value $\tilde{v}$ is $(n-1)\,F(\tilde{v})^{n-2}\,f(\tilde{v})$.

Now we want to use Bayes rule. You know that your value is v. So the joint probability that your value is v and the highest among the others is $\tilde{v}$ is just $(n-1)\,F(\tilde{v})^{n-2}\,f(\tilde{v})$, as we just calculated as long as $\tilde{v} < v$. For Bayes rule, we then need to divide by the probability that your value v is highest, which is just $F^{n-1}(v)$. We then take the expectation using this conditional probability distribution.

In words, we just decided that the amount you should expect to pay if your value is v in a second price auction is given by (4).

Problems

1. Work out the expected payment when there are 2 other bidders and F is uniform (i.e. $F(x) = x$). Now do the same when there are three other bidders. How does the amount you expect to pay change between 2 and 3 other bidders?

2. Answer question 1 again, but assume that $F(x) = x^2$. What impact does this change in the distribution have.

Why worry about the expected payment?

If you bid in a second price auction, you will do okay as long as you don't bid more than your value. In a way, there isn't really much reason to do the calculation we did above. However, it is an important calculation for the seller. Lets do the calculation from the seller's point of view. Suppose that v is the highest value among the bidders in the auction. Then the revenue that the seller should expect to get from the bidder who wins the auction is exactly what that bidder expects to pay, i.e., the expression (4). Now integrate this over all the possible values bidder 1 could have, then multiply it by n because there are n bidders in all, and you get the revenue the seller expects to get from the second price auction

$$n\int_0^1 (n-1)\left\{\int_0^v \tilde{v}F^{n-2}(\tilde{v})\,f(\tilde{v})\,d\tilde{v}\right\} f(v)\,dv. \tag{5}$$

Problems

1. Calculate expected revenue when $F(v) = v$ and show that it is equal to $\frac{(n-1)}{n+1}$.

First Price Auctions.

It is interesting that second price auctions have an equilibrium where bidders bid their true values. Yet one might wonder whether there might not be better ways to sell something. For example, imagine that you are trying to sell some public land to make money for taxpayers. You decide to hold a second price auction. Some big company gives you a bid of \$1 million. By what we have just said, that is the amount the company thinks the land is worth. Why not just charge them \$1 million - that seems better for taxpayers. After all, why deliberately charge the company something less than what you know they are willing to pay.

The answer is that if they know you are going to do this, they won't bid \$1 million, they will bid something considerably less. If you want to figure out if it would be better to charge them what they bid, you need to figure out exactly what they will bid.

To do this we can use the same approach we used above. Start with the payoff function. Since we changed the rules of the auction, the payoff function for a bidder becomes

$$V(b_i, b_{-i}, v_i) = \begin{cases} v_i - b_i & b_i \geq b_j \forall j \\ 0 & \text{otherwise.} \end{cases}$$

As before, we'll ignore ties. This payoff function itself doesn't allow you to figure out what to bid because you aren't sure what the b_{-i} are going to be. Bayesian equilibrium says that you should use your prior beliefs about the distribution of valuations to figure out what to do. You believe that each bidder has a value that is independently drawn from the same distribution F. So we proceed by assuming that each bidder will be expected to make a bid that depends on his or her valuation. Lets suppose that if two bidders have the same valuation, we expect them to have the same bid. Then what you are expecting to happen in the auction is that a bidder who have a valuation v will bid $b(v)$. We're still not sure what $b(v)$ is, but it is a start.

Now suppose we guess that $b(v)$ is an increasing function of v, which means bidders with higher values probably bid more. Then we can evaluate the expected payoff associated with a bid b' as follows

$$\mathbb{E}_{b_{-i}}\left\{V(b', b_{-i}, v_i)\right\} = (v_i - b') \Pr\left\{\text{every } b_j \text{ is less than } b'\right\} =$$

$$(v_i - b') \Pr\left(\text{every } v_j \text{ is such that } b(v_i) < b'\right) =$$

$$(v_i - b') \Pr\left(\text{every } v_i \text{ is less that } v' \text{ where } b(v') = b'\right) =$$

$$(v_i - b') F^{n-1}\left(b^{-1}(b')\right) =$$

$$\left(v_i - b\left(v'\right)\right) F^{n-1}\left(v'\right)$$

As we pointed out last time, bidding b' is the same as acting as if your value is v' (even though it isn't) and using the rule b.

If the function b is a Bayesian Nash equilibrium, then bidding $b(v)$ when your value is v should be a best reply to what you think others are doing, no matter what your actual value v. This is the same as saying that

$$\left(v_i - b\left(v_i\right)\right) F^{n-1}\left(v_i\right) \geq \left(v_i - b\left(v'\right)\right) F^{n-1}\left(v'\right).$$

In particular, that means that the derivative of the function

$$\left(v - b\left(v'\right)\right) F^{n-1}\left(v'\right)$$

with respect to v' should be zero when $v' = v$. In other words

$$\left(v - b\left(v\right)\right)\left(n-1\right) F^{n-2}\left(v\right) f\left(v\right) = b'\left(v\right) F^{n-1}\left(v\right). \tag{6}$$

One way we could approach this is to solve for $b'(v)$

$$b'\left(v\right) = \frac{\left(v - b\left(v\right)\right)\left(n-1\right) f\left(v\right)}{F\left(v\right)}. \tag{7}$$

If you observe that must hold for every value of v, it becomes a differential equation that we could try to solve.

An aside:

I did say that equilibrium in economics was a fixed point. Notice that if I integrate both sides of (7) I get:

$$\int_0^v b'\left(\tilde{v}\right) d\tilde{v} = \int_0^v \frac{\left(\tilde{v} - b\left(\tilde{v}\right)\right)\left(n-1\right) f\left(\tilde{v}\right)}{F\left(\tilde{v}\right)} d\tilde{v} = b\left(v\right).$$

I know the constant I should add must be zero because a bidder with zero value should bid 0. Not every function b will solve the differential equation, so I can just recast this problem by defining a transformation

$$T\left(b\left(v\right)\right) = \int_0^v \frac{\left(\tilde{v} - b\left(\tilde{v}\right)\right)\left(n-1\right) f\left(\tilde{v}\right)}{F\left(\tilde{v}\right)} d\tilde{v}$$

So finding the equilibrium bidding rule is the same as finding a fixed point to the transformation T, ie. find b^*such that

$$T\left(b^*\right) = b^*.$$

Back to the solution.

There is another way to get the solution that will help in our comparison to the second price auction. Lets just rewrite (6) as

$$v\,(n-1)\,F^{n-2}\,(v)\,f\,(v) = b\,(v)\,(n-1)\,F^{n-2}\,(v)\,f\,(v) + b'\,(v)\,F^{n-1}\,(v)\,.$$

Now observe that the right hand side of this expression is just the derivative of $b\,(v)\,F^{n-1}\,(v)$ with respect to v.

What that means is that uniformly in b

$$\frac{d\left\{b\,(v)\,F^{n-1}\,(v)\right\}}{dv} = v\,(n-1)\,F^{n-2}\,(v)\,f\,(v)\,.$$

Then we just use the fundamental theorem of calculus, and integrate the derivative to get the function itself, i.e.

$$b\,(v)\,F^{n-1}\,(v) = \int_0^v \tilde{v}\,(n-1)\,F^{n-2}\,(\tilde{v})\,f\,(\tilde{v})\,d\tilde{v},$$

or

$$b\,(v) = \frac{\int_0^v \tilde{v}\,(n-1)\,F^{n-2}\,(\tilde{v})\,f\,(\tilde{v})\,d\tilde{v}}{F^{n-1}\,(v)}.$$

Now you can look back at the expression we got in (4) describing the amount that a bidder in the second price auction expects to pay conditional on winning - you will see it is exactly the same. The stunning conclusion is that the amount that the seller should expect to receive from the winning bidder is exactly the same in both the first and second price auctions. They produce exactly the same revenue.

All Pay Auctions.

If you don't find the relationship between the first and second price auction surprising, here is an even more surprising result. Many auctions (or at least things that act like auctions) have the property that the high bidder wins the auction and pays whatever she bid. Yet everyone else in the auction has to pay what they bid as well. If you think that sounds unreasonable, that is in many ways what happens in education. To get a job you spend a lot of money on education - the most educated person gets the most desirable job. If you don't get the most desirable job, you still have to pay for the education you received.

Many kinds of litigation are like this. One party sues the other, then both lawyer up. The side that spends the most on lawyers wins the case, but both sides have to pay their lawyers.[1] Lobbying is like this. If you are a corporation

[1] An interesting example of this kind of thing that pertains to another part of this course is the companies that act as 'patent trolls'. The way patent trolls work is to apply for, or buy very vague patents, then suing a company for patent violation. Even if the patent doesn't apply, the company who is being sued has to defend itself in court, which requires them to lawyer up in the manner described above. The patent troll then offers to settle out of court for an upfront payment, which the company will normally pay. This is type of extortion which is perfectly legal under US *intellectual property* law. If you are getting bored with auctions, here is a story about patent trolls - https://www.youtube.com/watch?v=3bxcc3SM_KA

and you need a regulation relaxed or a pipeline approved, you contribute to party of the ruling candidate. The corporation that contributes the most gets the rule it wants, but no one gets their money back.

We can find an equilibrium for this sort of thing using the approach above. Assuming your are starting to get the idea behind Bayesian equilibrium, we can do it a bit more quickly. Lets suppose the bidders use a monotonic rule $b(v)$ to decide how much to bid. Once again, if $b(v)$ is a Bayesian equilibrium bidding rule, then the function

$$vF^{n-1}(v') - b(v')$$

should be maximized when $v' = v$.

The corresponding first order condition is

$$v(n-1)F^{n-2}(v)f(v) = b'(v).$$

This is actually really easy because we can use the fundamental rule of calculus right away to get

$$b(v) = \int_0^v \tilde{v}(n-1)F^{n-2}(\tilde{v})f(\tilde{v})\,d\tilde{v}.$$

If you compare this to the bid in the first price auction, it is much smaller.

However, the total expected payments to the seller are

$$n\int_0^1 b(v)f(v)\,dv =$$

$$n\int_0^1 \int_0^v \tilde{v}(n-1)F^{n-2}(\tilde{v})f(\tilde{v})\,d\tilde{v}f(v)\,dv.$$

If you compare this to our original formula for the revenue in the second price auction, given by (5), you will see that it is exactly the same.

Optimal Reserve price.

Each buyer in the auction might pay something to the seller. As we reasoned above, a buyer with valuation v will pay the seller his or her bid $b(v)$ if they happen to win the first price auction. We now know enough to write down this payment. The buyer will win the auction with probability $F^{n-1}(v)$ then pay $b(v)$. We showed above that

$$b(v) = \frac{\int_0^v \tilde{v}(n-1)F^{n-2}(\tilde{v})f(\tilde{v})\,d\tilde{v}}{F^{n-1}(v)}.$$

To find the payment the seller expects to receive from a buyer with valuation v we could then just take the expectation of $F^{n-1}(v)b(v)$ across all the values v the buyer might have. We have n buyers in all so the total expected revenue for the seller is going to be

$$n\int_0^1 F^{n-1}(v)b(v)f(v)\,dv.$$

Substituting in the equilibrium bidding rule gives

$$n\int_0^1\int_0^v \tilde{v}\,(n-1)\,F^{n-2}\left(\tilde{v}\right)f\left(\tilde{v}\right)d\tilde{v}f\left(v\right)dv$$

since $F^{n-1}(v)$ cancels itself out. You'll notice this is the same expected revenue we calculated for the second price auction. So what we'll do next is independent of which particular auction we are interested in. To keep things simple, lets just keep focus on the first price auction.

To understand why the seller might want to choose a starting bid greater than 0,it might help to start with a case in which there is only one bidder. If the seller runs a second price auction, this single bidder will bid his value. The second highest bid is 0 which is what it is worth to the seller. If the seller chooses a starting bid of p on the other hand, the high bidder will pay p instead of 0 provided his value is above p. This represents a benefit of having a starting bid. The downside is that the buyer may not have a value above v in which case the seller loses.

It is actually pretty easy to figure out what the seller should do. His expected revenue would be

$$p\left(1-F\left(p\right)\right).$$

Obviously any positive starting bid would be better than 0. By the same token, it wouldn't be wise to set the starting bid to 1 because no one would be willing to pay it.

The best starting bid is the one that maximizes this expectation. You can finding it as you always do using a first order condition

$$\frac{d}{dp}p\left(1-F\left(p\right)\right)=$$

$$\left(1-F\left(p\right)\right)-pf\left(p\right)=0.$$

In other words, you would choose p to satisfy

$$p-\frac{1-F\left(p\right)}{f\left(p\right)}=0.$$

The basic point, as a seller you are always better off if you can commit yourself not to trade with very low value buyers. Since having an auction with one buyer is a special case, this suggests that seller will want to do the same thing even if there are many buyers.

The most complicated part of the auction would seem to be to figure out how the change in the starting bid or reserve price will affect the bidding rule. You know from the logic of the second price auction that it won't - buyers will still bid their values. So choosing a reserve price in the second price auction directly involves choosing the lowest value you are willing to sell to.

It is just slightly more complicated in the first price auction. Go back to the basic payoff function for a bidder in the first price auction, given by

$$\left(v-b\left(v\right)\right)F^{n-1}\left(v\right).$$

If we put the first order condition together with the fixed point restriction as we did above we get

$$v\left(n-1\right)F^{n-2}\left(v\right)f\left(v\right)=\frac{d}{dv}F^{n-1}\left(v\right)b\left(v\right).$$

Recall what we did with this condition was to integrate both side and use the fundamental theorem of calculus. However, remember when you do this, there is always a constant left over (the integral of a derivative only gives you the function itself if you know the level where it started).

When there is no reserve price the constant is zero just because we know that a bidder with value 0 should bid 0. This means the bidding rule must pass through the point $(0,0)$.

When there is a reserve price, the lowest acceptable bid will be r. Whichever type is supposed to bid r, we know that this type should be the lowest typ who submits an acceptable bid. If this type earns strictly positive payoff by doing so, lower types will also want to bid, which can't be true. So whatever type is supposed to bid r should get exactly zero surplus whether they win or not. Evidently the type who submits a bid equal to r should have type eactly r.

So the equilibrium bidding rule has to satisfy $b\left(r\right)=r$. Putting that together with the first order condition gives us the rule

$$b\left(v\right)=r+\frac{\int_{r}^{v}\tilde{v}\left(n-1\right)F^{n-2}\left(\tilde{v}\right)f\left(\tilde{v}\right)d\tilde{v}}{F^{n-1}\left(v\right)}.$$

So lets use that rule to calculate the seller's revenue from a first price auction.

$$n\int_{r}^{1}\left\{rF^{n-1}\left(r\right)+\int_{r}^{v}\left(\tilde{v}\left(n-1\right)F^{n-2}\left(\tilde{v}\right)f\left(\tilde{v}\right)d\tilde{v}\right)\right\}f\left(v\right)dv.$$

The inner integral, as we have seen is

$$rF^{n-1}\left(r\right)+\int_{r}^{v}\left(\tilde{v}\left(n-1\right)F^{n-2}\left(\tilde{v}\right)f\left(\tilde{v}\right)\right)d\tilde{v}=$$

$$rF^{n-1}\left(r\right)+\int\tilde{v}dF^{n-1}\left(\tilde{v}\right)$$

which can be integrated by parts to get

$$rF^{n-1}\left(r\right)+vF^{n-1}\left(v\right)-rF^{n-1}\left(r\right)-\int_{r}^{v}F^{n-1}\left(\tilde{v}\right)d\tilde{v}.$$

Then our expression for seller revenues becomes

$$n\int_{r}^{1}\left\{vF^{n-1}\left(v\right)-\int_{r}^{v}F^{n-1}\left(\tilde{v}\right)d\tilde{v}\right\}f\left(v\right)dv=$$

$$n\int_{r}^{1}\left(vF^{n-1}\left(v\right)\right)f\left(v\right)dv-n\int_{r}^{1}\int_{r}^{v}F^{n-1}\left(\tilde{v}\right)d\tilde{v}f\left(v\right)dv.$$

Now we can pull out the second double integral

$$\int_r^1 \int_r^v F^{n-1}(\tilde{v})\, d\tilde{v} f(v)\, dv.$$

Integrating by parts this is equal to

$$\int_r^v F^{n-1}(\tilde{v})\, d\tilde{v} F(v) \Big|_r^1 - \int_r^1 F^n(v)\, dv =$$

$$\int_r^1 F^{n-1}(v) \frac{1-F(v)}{f(v)} f(v)\, dv.$$

Finally we can substitute this back into the revenue expression to get

$$n \int_r^1 F^{n-1}(v) \left(v - \frac{1-F(v)}{f(v)} \right) f(v)\, dv,$$

which is the expression we want to work with.

All that is left is to choose the lowest value buyer that the seller wants to sell to by choosing r to maximize the following expression

$$n \int_r^1 F^{n-1}(v) \left(v - \frac{1-F(v)}{f(v)} \right) f(v)\, dv.$$

The first order condition is

$$\left(v - \frac{1-F(v)}{f(v)} \right) = 0.$$

Now we are done. We choose the lowest valuation the seller wants to deal with by solving this first order condition. In the second price auction we'd set the starting value to this lowest valuation. In a first price auction we set the reserve price equal to the equilibrium bid of a bidder with this valuation.