A Unified Approach to Mechanism Design with Multiple Principals and Agents

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July 6, 2005

Abstract

We show a general revelation principle, which we call the *perfect revelation principle*, in mechanism design with multiple principals and agents. The type-revelation principle fails in multi-principal games because the asymmetric information between each principal and agent includes the mechanisms offered by the other principals. In this paper, we introduce the *perfect revelation mechanism*, in which each agent is asked to report not only his type but also which (economic) outcome will be realized. We show that each equilibrium outcome of a mechanism design game with any strategy space of the principals is achieved at a truthful equilibrium of a game in which each principal's strategy space is restricted to a set of perfect revelation mechanisms.

Key words: Mechanism Design, Multiple Principals, Revelation Principle.

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1 Introduction

In this paper, we consider mechanism design games in which multiple principals simultaneously and independently offer mechanisms to agents. In mechanism design with only one principal, the standard revelation principle, which we call the *type-revelation principle*, holds true. This states that each equilibrium outcome of a game with any strategy space of the principal is achieved at a truthful equilibrium of the game in which the strategy space of the principal is the set of type-revelation mechanisms.[1] Thanks to this principle, we can restrict the principal's feasible mechanisms to type-revelation mechanisms when we search optimal mechanisms.

Unfortunately, the type-revelation principle does not generally hold in multi-principal situations. This is mainly because asymmetric information between each principal and agent includes the mechanisms offered by the other principals. This difficulty was pointed out by McAfee (1993), Peck (1997), Epstein and Peters (1999), Peters (2001) and Martimort and Stole (2002). Peters (2001) and Martimort and Stole (2002) gave examples in which the type-revelation principle does not hold in a simple two-principal model. In their examples, by asking an agent to report some messages in addition to the agent's type, each principal could detect and punish the other principal's deviation off the equilibrium path. Greater punishment for the deviation lowers the principal's payoff off the equilibrium path, and thus, the set of equilibrium outcomes becomes strictly larger than the set of equilibrium outcomes of the game in which each principal's strategy space is restricted to the set of type-revelation mechanisms.

If it were possible to construct the "everything-revelation mechanism" in which each agent was asked to report a type and the mechanisms that the other principals had actually offered, then we might simply prove a generalized revelation principle. As McAfee (1993) and Katz (2004) showed,

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[1] Several versions of the type-revelation principle have been established by Gibbard (1973), Green and Laffont (1977), Dasgupta, Hammond, and Maskin (1979), Myerson (1979), Myerson (1982) and Harris and Townsend (1981).

however, we cannot define the principals' strategy spaces so that each principal asks each agent to report which mechanisms the other principals have offered.[2]

In this paper, we introduce a new mechanism, which we call the *perfect revelation mechanism.* An agent who participates in the perfect revelation mechanism is asked to report not only a type but also an outcome. The *perfect revelation game* is the game in which each principal's strategy space is the set of perfect revelation mechanisms. An equilibrium is said to be *truthful* if the agents' true types and the outcome that will be realized are reported at least to the principals who do not deviate from the equilibrium strategy. Thus, on the truthful equilibrium path, each principal will obtain true information of the agents' types and of the allocations that the other principals will assign (i.e., whether and how the other principal has deviated). In this sense, each principal at a truthful equilibrium will "perfectly" obtain the information that the principal wishes to know. We prove that each equilibrium outcome of a mechanism design game with any strategy space of principals is achieved at a truthful equilibrium of the perfect revelation game. This result may be called the *perfect revelation principle.*

We also show some characteristics of the perfect revelation game. First, we study the dimension of incentive constraints in perfect revelation mechanisms. Because each agent's report consists of a type and an outcome, one may be concerned that the incentive constraints for a perfect revelation mechanism are multi- or infinite-dimensional, even in the cases in which each agent has a finite type space. However, if multiple agents participate in the perfect revelation mechanism offered by a principal, and if there is a "punishment" allocation for each agent, then with no information rent the principal will know the outcome that will be realized. Accordingly, only the type-incentive constraints have to be satisfied. Therefore, the dimension of the incentive constraints for each agent is at most the dimension of the agent's

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[2] This problem is called a "self-reference problem" or an "infinite regress problem." We rigorously argue this problem in Section 2.

type space. Second, we examine whether each equilibrium outcome of the perfect revelation game is robust to any change of the principals' strategy spaces. We consider the perfect revelation game which is the "largest" in the sense that each principal's strategy space is the set of all perfect revelation mechanisms. We show that if the agents have perfectly correlated types, then each equilibrium outcome of the perfect revelation game is achieved at some equilibrium of any mechanism design game in which each principal's strategy space is sufficiently large. The results apply, for example, to the complete information model and the single agent model.

Several papers provided variants of the revelation principle under some additional conditions. Epstein and Peters (1999) considered mechanism design games with multiple principals and agents in which each agent can participate in at most one principal's mechanism. They called this class of games the *competing mechanisms*, and introduced the universal type of an agent, which consists of the agent's type and the series of the agent's payoffs under each mechanism. The latter information to some extent conveys information about which mechanisms are actually offered. They derived sufficient conditions on the agents' utility functions for essentially different mechanisms to attain different series of payoffs to the agents, in which case the universal types contain sufficiently rich information about the mechanisms. We believe that our approach improves theirs in two respects. First, our model covers not only the competing mechanisms, but also other mechanism design games such as common agency and exclusive dealing. Second, we impose only mild assumptions on the players' utility functions.

Peters (2001) and Martimort and Stole (2002) introduced the menu mechanism. In this mechanism, each principal asks an agent to report an allocation that the principal should assign. They showed that each equilibrium outcome of a game with only one common agent is attained at an equilibrium of the game in which each principal's strategy space is restricted to the set of menu mechanisms. Unfortunately, their results do not generally hold in multi-agent games, as Han (2004) pointed out.[3] Our model covers not only the common agency but also multi-agent games, and hence can be applied to economic problems such as competing auctions, optimal taxation for international enterprises and sales competitions.

The rest of this paper is organized as follows. In Section 2, we introduce the model of mechanism design games with multiple principals and agents. In Section 3, in order to give intuition of the results of this paper under incomplete information, we analyze the complete information model. In Section 4, we analyze the general model under incomplete information. In Section 5, we examine whether the equilibrium outcomes in the perfect revelation games are "robust" in the sense that it will be achieved at some equilibrium of the game in which the principals can offer more complicated mechanisms.

We present an economic example in Section 6, in which two retailers (principals) compete in a Cournot market. In this example, there are two manufacturers (agents) who independently have private information about the demand curve. We show that the retailers can achieve the cartel outcome in the perfect revelation game, while they cannot achieve the cartel outcome if their strategy spaces are restricted to the set of type-revelation mechanisms or the set of menu mechanisms.

Section 7 is the conclusion.

2 Basic Definitions and Notation

First, we informally describe the game we consider. Assume there are multiple principals and agents. Each agent has private information ex ante, which we call the "type" of the agent. At date 1, each principal simultaneously and independently offers a "mechanism". A mechanism of each principal is an allocation rule that assigns a feasible allocation for each profile of the agents'

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[3] Han (2004) offered sufficient conditions in which each equilibrium outcome of a game will be attained at an equilibrium of the game in which each principal's strategy space is restricted to the set of menu mechanisms.

"messages" reported to the principal and for each profile of the agents' "participation actions." A participation action of each agent represents which principals' mechanisms the agent participates in. At date 2, each agent observes the profile of the mechanisms offered by the principals and chooses a participation action. By restricting the set of feasible participation actions for each agent, we can consider many situations such as common agency, competing mechanisms and exclusive dealing. At date 3, each agent reports a message to a principal if the agent has participated in the principal's mechanism. Finally, each principal assigns an allocation. The assignment is determined by the mechanism that the principal has offered.

Formally, we consider the following model. Let J be the set of the principals, and I be the set of the agents. The number of the principals is $N \geq 2$ and the number of the agents is $n \geq 1$. Generic elements of the two sets are denoted by j and i, respectively. Agent $i \in I$ has private information $\theta_i \in \Theta_i$ ex ante, which we call the *type* of the agent. We assume that $\Theta = \prod_{i \in I} \Theta_i$ is finite. A type profile $\theta = (\theta_i)_{i \in I} \in \Theta$ is realized with probability $F(\theta)$. Assume $F_i(\theta_i) = \sum_{\theta_{-i} \in \prod_{h \neq i} \Theta_h} F(\theta_i, \theta_{-i}) > 0$ for all i and $\theta_i \in \Theta_i$.

An allocation of principal j is a vector $y_j \in \mathbb{R}^l$. For example, in an auction model with two buyers (agents), the set of feasible allocations for principal j is $\{(x_{j1}, x_{j2}, t_{j1}, t_{j2}) \in \mathbb{R}^4_+ | x_{j1} + x_{j2} \leq 1\}$, where agent i obtains the good with probability x_{ji} and pays t_{ji}. We are interested in which allocations are assigned for each type profile. A mapping $z : \Theta \to \mathbb{R}^{Nl}$ is called an *outcome*.

Let $U_j : \Theta \times \mathbb{R}^{Nl} \to \mathbb{R}$ represent principal j's utility function and $U_i : \Theta \times \mathbb{R}^{Nl} \to \mathbb{R}$ represent agent i's utility function.

The timing of the game is as follows. At date 1, the principals simultaneously offer mechanisms to the agents. Let M_{ji} denote a nonempty set for each i, which we call a *message space* for agent i. A mechanism of principal j is a mapping $\gamma_j : (P(J))^n \times \prod_{i \in I} M_{ji} \to \mathbb{R}^l$, where $P(J)$ is the power set of J. A set of principals $a_i \in P(J)$ is called agent i's *participation action*. Intuitively, we mean by $j \in a_i$ that agent i participates in principal j's mechanism, or that agent i is a *participant* of principal j's mechanism. We call

$m_{ji} \in M_{ji}$ a *message* of agent i to principal j. Therefore, a mechanism γ_j represents a rule which assigns an allocation for each profile of participation actions and each profile of messages. That is, for each $a = (a_i)_{i \in I} \in (P(J))^n$ and $(m_{ji})_{i \in I} \in \prod_{i \in I} M_{ji}$, the mechanism γ_j assigns $\gamma_j(a, (m_{ji})_{i \in I}) \in \mathbb{R}^l$. We sometimes specifically denote by $M_{ji}(\gamma_j)$ the message space for agent i under γ_j.

At date 2, each agent i simultaneously chooses a participation action $a_i \in P(J)$ after every agent has observed the profile of mechanisms $\gamma = (\gamma_j)_{j \in J}$. At date 3, each agent i simultaneously reports a message to each of the principals after every principal and every agent has observed the action profile $a = (a_i)_{i \in I}$. Finally, each principal j assigns an allocation $\gamma_j(a, (m_{ji})_{i \in I})$, and the game ends.

In some cases, the set of feasible allocations for each principal may be restricted by the agents' participation actions. For example, we may assume that principal j cannot make any monetary transfer to agent i if $j \notin a_i$. In general, let $Y_j(a) \subset \mathbb{R}^l$ represent the set of feasible allocations for principal j when the agents take the participation action profile $a = (a_i)_{i \in I}$. We assume that each $Y_j(a)$ is a continuum.

The set of feasible participation actions for each agent is also sometimes exogenously restricted. Let a nonempty set $A_i \subset P(J)$ represent the set of feasible participation actions for agent i. We can consider many situations such as delegated common agency ($A_i = P(J)$), intrinsic common agency ($A_i = \{\emptyset, J\}$), competing mechanisms ($A_i = \{\emptyset\} \cup J$) and exclusive dealing ($A_i = \{j(i), \emptyset\}$, where $j(i) \in J$ is exogenously given). Define $A = \prod_{i \in I} A_i$. The following assumption states that each principal has a "punishment allocation" for the participants.

Assumption 1. For each j and $a \in A$, there exists $\overline{y}_j(a) \in Y_j(a)$ such that for all $i \in \{h | j \in a_h\}$, θ_i and $y_{-j} \in \prod_{k \neq j} Y_k(a)$,

$$U_i(\theta_i, \overline{y}_j(a), y_{-j}) = \min_{y_j \in Y_j(a)} U_i(\theta_i, y_j, y_{-j}).$$

For example, $\overline{y}_j(a)$ consists of negative transfers to the participants. In some cases, it may be difficult to assign a negative transfer to all of the participants. Specifically, a firm (principal) may face limited liability constraints of the workers (agents). In this case, however, increasing working hours without payment may correspond to the punishment allocation.

We assume that each principal will offer a "feasible" mechanism in the following sense.

Definition 1. A mechanism γ_j of principal j is said to be *feasible* if:

1. for all $a \in A$ and $(m_{ji})_{i\in I} \in \prod_{i\in I} M_{ji}$,

$$\gamma_j(a, (m_{ji})_{i\in I}) \in Y_j(a),$$

2. for all $i \in I$, $a \in A$, all $m_{ji}, m'_{ji} \in M_{ji}$ and $(m_{jh})_{h\neq i} \in \prod_{h\neq i} M_{jh}$,

$$j \notin a_i \Rightarrow \gamma_j(a, m_{ji}, (m_{jh})_{h\neq i}) = \gamma_j(a, m'_{ji}, (m_{jh})_{h\neq i}).$$

Condition 1 requires that principal j assigns a feasible allocation. Condition 2 requires that an allocation assigned by principal j does not depend on the message reported by an agent who does not participates in γ_j.

To define the strategy space of principal j, let $\mathcal{M}_j$ be a nonempty set.[4] Principal j's strategy space $\Gamma_j^{\mathcal{M}_j}$ is a set of feasible mechanisms such that

$$\Gamma_j^{\mathcal{M}_j} = \{\gamma_j | \forall i,\ M_{ji}(\gamma_j) \subset \mathcal{M}_j\}.$$

Agent i's *action plan* after $\gamma = (\gamma_j)_{j\in J}$ has been offered is $(N+1)$-tuple $\sigma_i^\gamma = (\sigma_{ki}^\gamma)_{k=0,k\in J}$, where $\sigma_{0i}^\gamma : \Theta_i \to A_i$ is the agent's *participation plan* and $\sigma_{ji}^\gamma : \Theta_i \times A \to M_{ji}(\gamma_j)$ is the agent's *message plan* to each principal j. The set of action plans for agent i is denoted by

$$\Sigma_i(\gamma) = A_i^{\Theta_i} \times (\prod_{j\in J} M_{ji}(\gamma_j))^{\Theta_i \times A}.$$

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[4] Note that "the collection of all mechanisms with arbitrary message spaces" is not a set.

Define $\sigma^\gamma = (\sigma_i^\gamma)_{i \in I}$. If the principals offer γ and the agents conform to σ^γ, the following allocation profile will be realized: for each $\theta = (\theta_i)_{i \in I}$,

$$\gamma(\sigma^\gamma|\theta) = (\gamma_j(\sigma_0^\gamma(\theta), (\sigma_{ji}^\gamma(\theta_i, \sigma_0^\gamma(\theta)))_{i \in I}))_{j \in J},$$

where $\sigma_0^\gamma(\theta) = (\sigma_{0i}^\gamma(\theta_i))_{i \in I}$. Hence $\gamma(\sigma^\gamma) : \theta \mapsto \gamma(\sigma^\gamma|\theta) \in \mathbb{R}^{Nl}$ represents the outcome induced by pair (γ, σ^γ). We denote the jth component of $\gamma(\sigma^\gamma)$ by $\gamma_j(\sigma^\gamma) : \theta \mapsto \gamma_j(\sigma^\gamma|\theta) \in \mathbb{R}^l$.

The game in which the principals' strategy spaces are $\Gamma^{\mathcal{M}} = \prod_{j \in J} \Gamma_j^{\mathcal{M}_j}$ is referred to as a *mechanism design game* $\Gamma^{\mathcal{M}}$. We focus on the set of outcomes that are achieved at the pure strategy Bayesian equilibria of game $\Gamma^{\mathcal{M}}$.

The profile $\sigma^\gamma = (\sigma_i^\gamma)_{i \in I}$ is called a *rational response to* γ if for all i and $\tilde{\sigma}_i^\gamma \in \Sigma_i(\gamma)$,

$$\sum_{\theta \in \Theta} F(\theta) U_i(\theta, \gamma(\sigma^\gamma|\theta)) \geq \sum_{\theta \in \Theta} F(\theta) U_i(\theta, \gamma(\tilde{\sigma}_i^\gamma, (\sigma_h^\gamma)_{h \neq i}|\theta)). \tag{1}$$

The agents who conform to a rational response to γ play a Bayesian-Nash equilibrium given γ. Define the *accessible outcomes* for agent i under γ as follows:

$$D_i(\gamma, \sigma^\gamma) = \{\gamma(\tilde{\sigma}_i^\gamma, (\sigma_h^\gamma)_{h \neq i}) | \tilde{\sigma}_i^\gamma \in \Sigma_i(\gamma)\}.$$

Inequality (1) is then equivalent to the following: for all i and $z \in D_i(\gamma, \sigma^\gamma)$,

$$\sum_{\theta \in \Theta} F(\theta) U_i(\theta, \gamma(\sigma^\gamma|\theta)) \geq \sum_{\theta \in \Theta} F(\theta) U_i(\theta, z(\theta)).$$

A pure strategy profile of the agents $\sigma = (\sigma^\gamma)_{\gamma \in \Gamma^{\mathcal{M}}}$ is called a rational response to $\Gamma^{\mathcal{M}}$ if each σ^γ is rational to each $\gamma \in \Gamma^{\mathcal{M}}$.

Let $\gamma = (\gamma_j)_{j \in J}$ be a profile of feasible mechanisms. The profile of the mechanisms offered by the principals other than j is denoted by $\gamma_{-j} = (\gamma_k)_{k \neq j}$. A strategy profile (γ, σ) is called an *equilibrium* of game $\Gamma^{\mathcal{M}}$ if σ is rational to $\Gamma^{\mathcal{M}}$ and if for all j and $\gamma_j' \in \Gamma_j^{\mathcal{M}_j}$,

$$\sum_{\theta \in \Theta} F(\theta) U_j(\theta, \gamma_j(\sigma^\gamma|\theta), \gamma_{-j}(\sigma^\gamma|\theta)) \geq \sum_{\theta \in \Theta} F(\theta) U_j(\theta, \gamma_j'(\sigma^{\gamma_j', \gamma_{-j}}|\theta), \gamma_{-j}(\sigma^{\gamma_j', \gamma_{-j}}|\theta)).$$

The set of all equilibria of game $\Gamma^{\mathcal{M}}$ is represented by $E(\Gamma^{\mathcal{M}})$ and the equilibrium outcome set is represented by $Z(\Gamma^{\mathcal{M}})$. Throughout the paper, we assume that $E(\Gamma^{\mathcal{M}})$ is nonempty.

3 The Perfect Revelation Principle under Complete Information

We first consider the case in which the type space of every agent is a singleton in order to give intuition behind more general results under incomplete information.

When the set of available messages for principal j is so large that it satisfies $\mathcal{M}_j \supset \prod_{k \neq j} \Gamma_k^{\mathcal{M}_k}$ for all i, principal j may ask each agent to report the other principals' mechanisms directly. Suppose that this inclusion did hold for all j. Then, we could generalize the revelation principle for multi-principal games by simply constructing the "everything-revelation mechanism," in which each agent reports the other principals' mechanisms directly. In multi-principal games, however, we cannot construct such "everything-revelation" games unless we have ad hoc restrictions on the games, as McAfee (1993) and Katz (2004) pointed out. The next proposition shows this impossibility. Let $|S|$ denote the cardinality of a set S.

Proposition 1. For any game $\Gamma^{\mathcal{M}}$ with $N \geq 2$, there exists $j' \in J$ such that $|\mathcal{M}_{j'}| < |\prod_{k \neq j'} \Gamma_k^{\mathcal{M}_k}|$.

Proof. Suppose contrarily that we have $|\mathcal{M}_j| \geq |\prod_{k \neq j} \Gamma_k^{\mathcal{M}_k}|$ for all $j \in J$. Then for all j and $k \neq j$,

$$|\Gamma_j| \geq |2^{\mathcal{M}_j}| > |\mathcal{M}_j| \geq |\Gamma_k|,$$

We, however, similarly have $|\Gamma_k| > |\Gamma_j|$. This is a contradiction. □

Therefore, we need to construct other mechanisms than "everything-revelation mechanisms." We propose a new mechanism, which we call the perfect revelation mechanism, in which each agent is asked to report the outcome that will be realized.

The Perfect Revelation Game We now consider the game in which each principal asks each agent to report an outcome. A feasible mechanism ρ_j of principal j is called a *perfect revelation mechanism* if $M_{ji}(\rho_j) = \mathbb{R}^{Nl}$ for all i. A game in which each principal's strategy space is a set of perfect revelation mechanisms is called a *perfect revelation game*. Let R_j be a set of perfect revelation mechanisms of principal j. Let $R = \prod_{j \in J} R_j$ represent the perfect revelation game in which principal j's strategy space is R_j.

In a perfect revelation game, agent i reports an outcome $m_{ji} \in \mathbb{R}^{Nl}$ to principal j. Let $\rho = (\rho_j)_{j \in J}$ be a profile of perfect revelation mechanisms. The set of feasible actions for agent i under ρ is $\Sigma_i(\rho)$:

$$\Sigma_i(\rho) = A_i \times (\prod_{j \in J} \mathbb{R}^{Nl})^A.$$

A generic element of $\Sigma_i(\rho)$ is represented by $\overline{\sigma}_i^\rho = (\overline{\sigma}_{ki}^\rho)_{k=0,k\in J}$, where $\overline{\sigma}_{0i}^\rho \in A_i$ and $\overline{\sigma}_{ji}^\rho(a) \in \mathbb{R}^{Nl}$ for all j and a. The outcome induced by $(\rho, \overline{\sigma}^\rho)$ is denoted by $\rho(\overline{\sigma}^\rho)$. The jth component of $\rho(\overline{\sigma}^\rho)$ is denoted by $\rho_j(\overline{\sigma}^\rho)$. The agents' response $\overline{\sigma}^\rho$ is called a rational response to ρ if for all i and $y \in D_i(\rho, \overline{\sigma}^\rho)$,

$$U_i(\rho(\overline{\sigma}^\rho)) \geq U_i(y).$$

A strategy profile of the agents $\overline{\sigma} = (\overline{\sigma}^\rho)_{\rho \in R}$ is called a rational response to R if each $\overline{\sigma}^\rho$ is rational to each $\rho \in R$.

Define $\rho_{-j} = (\rho_k)_{k \neq j}$. A strategy profile $(\rho, \overline{\sigma})$ is called an equilibrium of game R if $\overline{\sigma}$ is rational to R and if for all j and $\rho'_j \in R_j$,

$$U_j(\rho_j(\overline{\sigma}^\rho), \rho_{-j}(\overline{\sigma}^\rho)) \geq U_j(\rho'_j(\overline{\sigma}^{\rho'_j, \rho_{-j}}), \rho_{-j}(\overline{\sigma}^{\rho'_j, \rho_{-j}})).$$

We do not study all equilibrium outcomes achieved at perfect revelation games; rather we focus on "truthful" equilibria. Before presenting the formal definition, we informally discuss the idea of the truthfulness. For example, let us assume $J = \{1, 2\}$ and assume that $(\rho, \overline{\sigma})$ is an equilibrium. On the equilibrium path, agent i will take the participation action $\overline{\sigma}^{\rho}_{0i}$ and report the message $\overline{\sigma}^{\rho}_{ji}(\overline{\sigma}^{\rho}_{0})$ to principal j. Then outcome $z = \rho(\overline{\sigma}^{\rho})$ will be realized. Thus, we may say that "the agents are truthful on the equilibrium path" if $\overline{\sigma}^{\rho}_{ji}(\overline{\sigma}^{\rho}_{0}) = z$ for each $j = 1, 2$. On the other hand, suppose that principal 2 has deviated to $\tilde{\rho}_2 \neq \rho_2$, while principal 1 conforms to the equilibrium strategy ρ_1. Define $\tilde{\rho} = (\rho_1, \tilde{\rho}_2)$. Then the outcome $\tilde{z} = \tilde{\rho}(\overline{\sigma}^{\tilde{\rho}})$ will be realized. We may say that "the agents are truthful off the equilibrium path" if $\overline{\sigma}^{\tilde{\rho}}_{1i}(\overline{\sigma}^{\tilde{\rho}}_{0}) = \tilde{z}$. Note that we do not require any condition on messages for the deviating principal. Intuitively, the agents' report informs principal 1 whether and how principal 2 has deviated from the equilibrium strategy.

Formally, an equilibrium $(\rho, \overline{\sigma})$ is said to be *truthful* if for all i and $\tilde{\rho} = (\tilde{\rho}_j)_{j \in J} \in R$,

$$\tilde{\rho}_j = \rho_j \Rightarrow \overline{\sigma}^{\tilde{\rho}}_{ji}(\overline{\sigma}^{\tilde{\rho}}_{0}) = \tilde{\rho}(\overline{\sigma}^{\tilde{\rho}}).$$

Let $E_T(R)$ and $Z_T(R)$ be respectively the set of truthful equilibria and the set of truthful equilibrium outcomes of a perfect revelation game R.

The following theorem which we call the *perfect revelation principle* implies that we find each equilibrium outcome of the game $\Gamma^{\mathcal{M}}$ at a truthful equilibrium of a perfect revelation game $R^{\mathcal{M}}$.

Theorem 1. For each $\Gamma^{\mathcal{M}}$, there exists a perfect revelation game $R^{\mathcal{M}}$ such that $Z(\Gamma^{\mathcal{M}}) \subset Z_T(R^{\mathcal{M}})$. That is, for each $(\gamma, \sigma) \in E(\Gamma^{\mathcal{M}})$, there exists a truthful equilibrium $(\rho, \overline{\sigma}) \in E_T(R^{\mathcal{M}})$ such that $\gamma(\sigma^{\gamma}) = \rho(\overline{\sigma}^{\rho})$.

Some additional definitions and a lemma are required to prove Theorem 1. First, we define the set of *accessible allocations* for each agent. Let $a \in A$ be the agents' participation action profile and $(m_{ji})_{i \in I} \in \prod_{i \in I} M_{ji}(\gamma_j)$ be a message profile reported in mechanism γ_j of principal j. The set of all *accessible allocations* for agent i under γ_j is defined as

$$D_{ji}(\gamma_j, a, (m_{jh})_{h \in I}) = \{\gamma_j(a, \tilde{m}_{ji}, (m_{jh})_{h \neq i}) | \tilde{m}_{ji} \in M_{ji}(\gamma_j)\}.$$

The set of accessible allocations for agent i is the set of principal j's allocations that the agent can attain by deviation from m_{ji} to other messages. Note that for all nonparticipants $h \notin \{i|j \in a_i\}$, we have $D_{jh}(\gamma_j, a, (m_{ji})_{i\in I}) = \{\gamma_j(a, (m_{ji})_{i\in I})\}$.

Let γ'_j be another mechanism of principal j and $(m'_{ji})_{i\in I} \in \prod_{i\in I} M_{ji}(\gamma'_j)$ be a message profile reported to principal j. We say that $(\gamma_j, a, (m_{ji})_{i\in I})$ *is simpler than* $(\gamma'_j, a, (m'_{ji})_{i\in I})$ if $\gamma_j(a, (m_{ji})_{i\in I}) = \gamma'_j(a, (m'_{ji})_{i\in I})$ and if each agent has "fewer" accessible allocations under γ_j than under γ'_j in the following sense: for all $i \in I$,

$$D_{ji}(\gamma_j, a, (m_{jh})_{h\in I}) \subset D_{ji}(\gamma'_j, a, (m'_{jh})_{h\in I}) \cup \{\overline{y}_j(a)\}.$$

Let $\gamma = (\gamma_j)_{j\in J}$ and $\gamma' = (\gamma'_j)_{j\in J}$ be two profiles of mechanisms and σ be a pure strategy profile of the agents. If $\sigma^{\gamma}_0 = \sigma^{\gamma'}_0$ and $(\gamma_j, a, (\sigma^{\gamma}_{ji}(a))_{i\in I})$ is simpler than $(\gamma'_j, a, (\sigma^{\gamma'}_{ji}(a))_{i\in I})$ for all j and a, then we say that $(\gamma, \sigma^{\gamma})$ is simpler than $(\gamma', \sigma^{\gamma'})$.

Lemma 1. Let σ be a pure strategy profile of the agents where $(\gamma, \sigma^{\gamma})$ is simpler than $(\gamma', \sigma^{\gamma'})$. Suppose that $\sigma^{\gamma'}$ is rational to γ'. Then σ^{γ} is rational to γ.

Proof. Let $a_{-i} = (\sigma^{\gamma}_{0h})_{h\neq i} = (\sigma^{\gamma'}_{0h})_{h\neq i}$ denote the profile of participation actions of the agents other than i. Define

$$\begin{aligned}\overline{D}_i(\gamma', \sigma^{\gamma'}) \;=\; & \{y \in \prod_{j\in a_i}[D_{ji}(\gamma'_j, (a_i, a_{-i}), (\sigma^{\gamma'}_{jh}(a_i, a_{-i}))_{h\in I}) \cup \overline{y}_j(a_i, a_{-i})] \\ & \times \prod_{j\notin a_i}\{\gamma'_j(a_i, a_{-i}, (\sigma^{\gamma'}_{jh}(a_i, a_{-i}))_{h\in I})\}|a_i \in A_i\}.\end{aligned}$$

The set of accessible outcomes for agent i under γ is

$$\begin{aligned}D_i(\gamma, \sigma^{\gamma}) &= \{\gamma(\tilde{\sigma}^{\gamma}_i, (\sigma^{\gamma}_h)_{h\neq i})|\tilde{\sigma}^{\gamma}_i \in \Sigma_i(\gamma)\} \\ &= \{y \in \prod_{j\in a_i} D_{ji}(\gamma_j, (a_i, a_{-i}), (\sigma_{jh}(a_i, a_{-i}))_{h\in I}) \\ &\qquad \times \prod_{j\notin a_i}\{\gamma_j(a_i, a_{-i}, (\sigma_{jh}(a_i, a_{-i}))_{h\in I})\}|a_i \in A_i\} \\ &\subset \overline{D}_i(\gamma', \sigma^{\gamma'}).\end{aligned}$$

Since $\sigma^{\gamma'}$ is rational to γ' and $\overline{y}_j(a)$ is the least preferred allocation for each participant in principal j's mechanism, we have

$$\begin{aligned} U_i(\gamma(\sigma^{\gamma})) &= U_i(\gamma'(\sigma^{\gamma'})) \\ &= \max_{y \in D_i(\gamma', \sigma^{\gamma'})} U_i(y) \\ &= \max_{y \in \overline{D}_i(\gamma', \sigma^{\gamma'})} U_i(y) \\ &\geq \max_{y \in D_i(\gamma, \sigma^{\gamma})} U_i(y). \end{aligned}$$

Therefore, σ^{γ} is rational to γ. □

For example, assume that $I = \{1, 2\}$ and $a_1 = a_2 \ni j$. Let us consider two mechanisms $\overline{\gamma}_j$ and $\tilde{\gamma}_j$ such that $M_{ji}(\overline{\gamma}_j) = \{s^1, s^2\}$ and $M_{ji}(\tilde{\gamma}_j) = \{s^1, s^2, s^3\}$ for $i = 1, 2$. The allocations in each mechanism for each message profile are denoted in Tables 1 and 2.

Specifically, $\overline{\gamma}_j(a, s^1, s^2) = x \in Y_j(a)$, $\tilde{\gamma}_j(a, s^2, s^3) = w \in Y_j(a)$ and so on. In this example, $(\overline{\gamma}_j, a, (s^1, s^1))$ is simpler than $(\tilde{\gamma}_j, a, (s^1, s^1))$ because

$$\overline{\gamma}_j(a, s^1, s^1) = \tilde{\gamma}_j(a, s^1, s^1) = w,$$

and

$$D_{ji}(\overline{\gamma}_j, a, (s^1, s^1)) = \{w, x\} \subset \{w, x, y\} = D_{ji}(\tilde{\gamma}_j, a, (s^1, s^1)).$$

for $i = 1, 2$. (Table 1 and 2 will be here)

	s^1	s^2
s^1	w	x
s^2	x	y

Table 1: $\overline{\gamma}_j$

	s^1	s^2	s^3
s^1	w	x	y
s^2	x	y	w
s^3	y	w	x

Table 2: $\tilde{\gamma}_j$

Proof. (proof of Theorem 1)

Let $E_j \subset \Gamma_j^{\mathcal{M}_j}$ be the set of equilibrium strategies of principal j in the game $\Gamma^{\mathcal{M}}$:

$$E_j = \{\gamma_j \in \Gamma_j^{\mathcal{M}_j} | \exists(\gamma_{-j}, \sigma),\ (\gamma_j, \gamma_{-j}, \sigma) \in E(\Gamma^{\mathcal{M}})\}.$$

For each $\gamma = (\gamma_j)_{j \in J}$, define $\gamma(\sigma^\gamma | a)$ so that

$$\gamma(\sigma^\gamma | a) = (\gamma_j(a, (\sigma_{ji}^\gamma(a))_{i \in I}))_{j \in J}.$$

We say that an outcome z is *conceivable* under $\gamma_j \in E_j$ and $a \in A$ if there exists $\gamma_{-j} \in \prod_{k \neq j} \Gamma_k^{\mathcal{M}_k}$ and a rational response σ to $\Gamma^{\mathcal{M}}$ such that

$$z = \gamma(\sigma^\gamma | a),$$

where $\gamma = (\gamma_j, \gamma_{-j})$.

Let $Z_j(\gamma_j, a)$ be the set of all conceivable outcomes under γ_j and a. Define a perfect revelation mechanism $\rho_j^{\gamma_j}$ of principal j so that for each a,

$$\rho_j^{\gamma_j}(a, (z^i)_{i \in I}) = \begin{cases} \overline{z}_j & \text{if } \forall i \in \{h | j \in a_h\},\ z^i = \overline{z} \in Z_j(\gamma_j, a), \\ \overline{y}_j(a) & \text{otherwise,} \end{cases}$$

where $z^i \in \mathbb{R}^{Nl}$ for each i. The set of perfect revelation mechanisms constructed from each element in E_j is denoted by $R_j^1 = \{\rho_j^{\gamma_j} | \gamma_j \in E_j\}$.

If $\mathcal{M}_j$ has cardinality at least as large as the continuum, then let $M \subset \mathcal{M}_j$ be an arbitrary set of messages which has the cardinality of the continuum. If $\mathcal{M}_j$ has less cardinality than the continuum, then let $M = \mathcal{M}_j$. We define a set of principal j's mechanisms $\Gamma_j^M \subset \Gamma_j^{\mathcal{M}_j}$ so that

$$\Gamma_j^M = \{\gamma_j \in \Gamma_j^{\mathcal{M}_j} \setminus E_j | \forall i \in I,\ M_{ji}(\gamma_j) = M\}.$$

If $\mathcal{M}_j$ has cardinality at least as large as the continuum, then let $f_i : \mathbb{R}^{Nl} \to M$ be a bijection. If $\mathcal{M}_j$ has less cardinality than the continuum, then let $f_i : \mathbb{R}^{Nl} \to M$ be a surjection. We define $\rho_j^{\gamma_j}$ for each $\gamma_j \in \Gamma_j^M$ so that for all $a \in A$,

$$\rho_j^{\gamma_j}(a, (z^i)_{i \in I}) = \gamma_j(a, (f_i(z^i))_{i \in I}).$$

The set of perfect revelation mechanisms constructed from each element in Γ_j^M is denoted by $R_j^2 = \{\rho_j^{\gamma_j} | \gamma_j \in \Gamma_j^M\}$.

Define $R_j = R_j^1 \cup R_j^2$. Since each element in R_j is a perfect revelation mechanism of principal j, we have constructed a perfect revelation game represented by $R^{\mathcal{M}} = \prod_{j \in J} R_j$.

Let $(\gamma^*, \sigma^*) \in E(\Gamma^{\mathcal{M}})$ be an arbitrary equilibrium of the game $\Gamma^{\mathcal{M}}$. For each $\rho_j \in R_j$, we define $\gamma_j^{\rho_j}$ so that (1) $\rho_j = \rho_j^* \Rightarrow \gamma_j^{\rho_j} = \gamma_j^*$ and (2) $\rho_j \neq \rho_j^* \Rightarrow \gamma_j^{\rho_j} \in \{\gamma_j | \rho_j = \rho_j^{\gamma_j}\}$. Define $\gamma^\rho = (\gamma_j^{\rho_j})_{j \in J}$. We construct $\overline{\sigma}^{*\rho}$ as follows: (1) define $\overline{\sigma}_{0i}^{*\rho} = \sigma_{0i}^{*\gamma^\rho}$ for all i, and (2) define[5]

$$\overline{\sigma}_{ji}^{*\rho}(a) \begin{cases} = \gamma^\rho(\sigma^{*\gamma^\rho} | a) & \text{if} \quad \gamma_j^{\rho_j} \in E_j \\ \in f_i^{-1}(\sigma_{ji}^{*\gamma^\rho}(a)) & \text{if} \quad \gamma_j^{\rho_j} \in \Gamma_j^M, \end{cases}$$

for all j, i and $a \in A$.

Let $(\rho, \overline{\sigma}^*)$ be a strategy profile of the game $R^{\mathcal{M}}$. If principal j has offered ρ_j such that $\gamma_j^{\rho_j} \in \Gamma_j^M$, then principal j who has observed $a \in A$ will assign

$$\gamma_j^{\rho_j}(a, (\sigma_{ji}^{*\gamma^\rho}(a))_{i \in I}),$$

since each agent i reports an element in $f_i^{-1}(\sigma_{ji}^{*\gamma^\rho}(a))$. If principal j has offered ρ_j such that $\gamma_j^{\rho_j} \in \Gamma_j^M$, then principal j who has observed $a \in A$ will assign an allocation

$$\gamma_j^{\rho_j}(a, (\sigma_{ji}^{*\gamma^\rho}(a))_{i \in I}),$$

which is jth component of $\gamma^\rho(\sigma^{*\gamma^\rho} | a)$, since each agent i reports $\gamma^\rho(\sigma^{*\gamma^\rho} | a)$ and this is conceivable under $(\gamma_j^{\rho_j}, a)$.

As agent i takes the participation action $\overline{\sigma}_{0i}^{*\rho} = \sigma_{0i}^{*\gamma^\rho}$, we obtain $\rho(\overline{\sigma}^{*\rho}) = \gamma^\rho(\sigma^{*\gamma^\rho})$. Because $\gamma^{\rho^*} = \gamma^*$, we especially have

$$\rho^*(\overline{\sigma}^{*\rho^*}) = \gamma^*(\sigma^{*\gamma^*}).$$

Note that $\overline{\sigma}^*$ is truthful to ρ^*: each agent i takes the participation action $\sigma_{0i}^{*\gamma^\rho}$ and reports $\gamma^\rho(\sigma^{*\gamma^\rho})$ to each principal who conforms to ρ^* when the agent has observed ρ, and then the outcome $\gamma^\rho(\sigma^{*\gamma^\rho})$ will actually be realized.

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[5] $f_i^{-1}(m_{ji})$ represents the inverse image of f_i at m_{ji}.

Moreover, $(\rho, \overline{\sigma}^{*\rho})$ is simpler than $(\gamma^{\rho}, \sigma^{*\gamma^{\rho}})$. Let $a \in A$ be an arbitrary profile of participation actions. Under ρ_j such that $\gamma_j^{\rho_j} \in \Gamma_j^M$,

$$\begin{aligned}
&D_{ji}(\rho_j, a, (\overline{\sigma}^{*\rho}_{jh}(a))_{h \in I}) \\
&= \{\gamma_j^{\rho_j}(a, m_{ji}, (\sigma^{*\gamma^{\rho}}_{jh}(a))_{h \neq i}) | m_{ji} \in f_i(\mathbb{R}^{Nl}) = M_{ji}(\gamma_j)\} \\
&= D_{ji}(\gamma_j^{\rho_j}, a, (\sigma^{*\gamma^{\rho}}_{jh}(a))_{h \in I}).
\end{aligned}$$

Under ρ_k such that $\gamma_k^{\rho_k} \in E_k$, we have

$$\begin{aligned}
&D_{ki}(\rho_k, a, (\overline{\sigma}^{*\rho}_{kh}(a))_{h \in I}) \\
&\subset \begin{cases} Z_k(\gamma_k^{\rho_k}, a) \cup \{\overline{y}_k(a)\} & \text{if} \quad \{i\} = \{h | j \in a_h\} \\ \{\gamma_k^{\rho_k}(a, (\sigma^{*\gamma^{\rho}}_{kh}(a))_{h \in I})\} \cup \{\overline{y}_k(a)\} & \quad \text{otherwise} \end{cases} \\
&\subset D_{ki}(\gamma_k^{\rho_k}, a, (\sigma^{*\gamma^{\rho}}_{kh}(a))_{h \in I}) \cup \{\overline{y}_k(a)\}.
\end{aligned}$$

Hence $\overline{\sigma}^*$ is rational to $R^{\mathcal{M}}$ by Lemma 1.

Finally, a deviation from ρ_j^* to some perfect revelation mechanism ρ_j attains the outcome $(\gamma_j^{\rho_j}(\sigma^{*\gamma_j^{\rho_j}, \gamma^*_{-j}}), \gamma^*_{-j}(\sigma^{*\gamma_j^{\rho_j}, \gamma^*_{-j}}))$, which is not preferable for principal j to conforming to ρ_j^* and achieving $\gamma^*(\sigma^{*\gamma^*})$. Therefore, $(\rho^*, \overline{\sigma}^*) \in E_T(R^{\mathcal{M}})$. $\square$

Let $\overline{R}_j$ denote the set of all perfect revelation mechanisms of principal j. The "largest" perfect revelation game is represented by $\overline{R} = \prod_{j \in J} \overline{R}_j$. If we consider a mechanism design game $\Gamma^{\mathcal{M}} = \prod_{j \in J} \Gamma_j^{\mathcal{M}_j}$ such that each $\mathcal{M}_j$ has cardinality at least as large as the continuum, then we have $R^{\mathcal{M}} = \overline{R}$.

Corollary 1. For each $\Gamma^{\mathcal{M}}$ such that $\mathcal{M}_j$ has cardinality at least as large as the continuum for all j, we have $Z(\Gamma^{\mathcal{M}}) \subset Z_T(\overline{R})$.

We can also show the converse. If we consider a mechanism design game $\Gamma^{\mathcal{M}}$ such that $\mathcal{M}_j$ has cardinality at least as large as the continuum for all j, then we have $Z_T(\overline{R}) = Z(\Gamma^{\mathcal{M}})$. This result corresponds to the weak robustness introduced by Peters (2001). It implies that we obtain "just enough" equilibrium outcome set of $\Gamma^{\mathcal{M}}$ by seeking truthful equilibrium outcomes of the largest perfect revelation game $\overline{R}$. It also implies $Z(\Gamma^{\mathcal{M}}) = Z(\Gamma^{\mathcal{M}'})$ if each $\mathcal{M}_j$ and each $\mathcal{M}'_j$ have cardinality at least as large as the continuum. We discuss it more rigorously in Section 5.

4 The Perfect Revelation Principle under Incomplete Information

The Perfect Revelation Game The main results obtained under complete information can be extended to the incomplete information model. In the perfect revelation game under incomplete information, each agent is asked to report a type and an outcome. A feasible mechanism ρ_j of principal j is called a *perfect revelation mechanism* if $M_{ji}(\rho_j) = \Theta_i \times (\mathbb{R}^{Nl})^{\Theta}$ for each i.

A game in which each principal's strategy space is a set of perfect revelation mechanisms is called a *perfect revelation game*. Let R_j be a set of principal j's perfect revelation mechanisms. Let $R = \prod_{j \in J} R_j$ represent the perfect revelation game in which principal j's strategy space is R_j.

The outcome induced by a profile of mechanisms ρ and the agents' response $\overline{\sigma}^{\rho} = (\overline{\sigma}_i^{\rho})$ is denoted by $\rho(\overline{\sigma}^{\rho})$. The jth component of $\rho(\overline{\sigma}^{\rho})$ is denoted by $\rho_j(\overline{\sigma}^{\rho})$. The agents' response $\overline{\sigma}^{\rho}$ is called a rational response to ρ if

$$\sum_{\theta \in \Theta} F(\theta) U_i(\theta, \rho(\overline{\sigma}^{\rho}|\theta)) \geq \sum_{\theta \in \Theta} F(\theta) U_i(\theta, z(\theta)),$$

for all i and $z \in D_i(\rho, \overline{\sigma}^{\rho})$. A strategy profile of the agents $\overline{\sigma} = (\overline{\sigma}^{\rho})_{\rho \in R}$ is called a rational response to R if each $\overline{\sigma}^{\rho}$ is rational to each $\rho \in R$.

Let $\rho = (\rho_j)_{j \in J}$ be a profile of perfect revelation mechanisms. Define $\rho_{-j} = (\rho_k)_{k \neq j}$. A strategy profile $(\rho, \overline{\sigma})$ is called an equilibrium of game R if $\overline{\sigma}$ is rational to R and

$$\sum_{\theta \in \Theta} F(\theta) U_j(\theta, \rho_j(\overline{\sigma}^{\rho}|\theta), \rho_{-j}(\overline{\sigma}^{\rho}|\theta)) \geq \sum_{\theta \in \Theta} F(\theta) U_j(\theta, \rho'_j(\overline{\sigma}^{\rho'_j, \rho_{-j}}|\theta), \rho_{-j}(\overline{\sigma}^{\rho'_j, \rho_{-j}}|\theta)),$$

for all j and $\rho'_j \in R_j$.

An equilibrium $(\rho, \overline{\sigma})$ is said to be *truthful* if

$$\tilde{\rho}_j = \rho_j \Rightarrow \overline{\sigma}_{ji}^{\tilde{\rho}}(\theta_i, \overline{\sigma}_0^{\tilde{\rho}}(\theta)) = (\theta_i, \tilde{\rho}(\overline{\sigma}^{\tilde{\rho}})),$$

for all i, $\theta = (\theta_i)_{i \in I}$ and $\tilde{\rho} = (\tilde{\rho}_j)_{j \in J} \in R$. Let $E_T(R)$ and $Z_T(R)$ be respectively the set of truthful equilibria and the set of truthful equilibrium outcomes of the perfect revelation game R.

The following theorem, which we call the *perfect revelation principle* under incomplete information, implies that we find each equilibrium outcome of the game $\Gamma^{\mathcal{M}}$ at a truthful equilibrium of a perfect revelation game $R^{\mathcal{M}}$.

Theorem 2. For each $\Gamma^{\mathcal{M}}$, there exists a perfect revelation game $R^{\mathcal{M}}$ such that $Z(\Gamma^{\mathcal{M}}) \subset Z_T(R^{\mathcal{M}})$. That is, for each $(\gamma, \sigma) \in E(\Gamma^{\mathcal{M}})$, there exists a truthful equilibrium $(\rho, \overline{\sigma}) \in E_T(R^{\mathcal{M}})$ such that $\gamma^*(\sigma^{*\gamma^*}) = \rho^*(\overline{\sigma}^{*\rho^*})$.

We also require some additional definitions and a lemma to prove Theorem 2. First, we define *type-contingent allocations* and the set of *accessible type-contingent allocations* for each agent. Let $\alpha_i : \Theta_i \to A_i$ represent agent i's participation action plan. Define $\alpha = (\alpha_i)_{i \in I}$. Let $\mu_{ji} : \Theta_i \times A \to M_{ji}(\gamma_j)$ represent the agent's message plan in a mechanism γ_j of principal j. Principal j then assigns

$$\gamma_j(\alpha, (\mu_{ji})_{i \in I} | \theta) = \gamma_j(\alpha(\theta), (\mu_{ji}(\theta_i, \alpha(\theta)))_{i \in I}),$$

for each $\theta = (\theta_i)_{i \in I}$.

Hence the mapping $\gamma_j(\alpha, (\mu_{ji})_{i \in I}) : \theta \mapsto \gamma_j(\alpha, (\mu_{ji})_{i \in I} | \theta)$ represents principal j's *type-contingent allocation.* The set of all *accessible type-contingent allocations* for agent i under γ_j is defined as

$$D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I}) \quad = \quad \{\gamma_j(\alpha, \tilde{\mu}_{ji}, (\mu_{jh})_{h \neq i}) | \tilde{\mu}_i \in (M_{ji}(\gamma_j))^{\Theta_i \times A}\}.$$

The set of accessible type-contingent allocations for agent i is the set of principal j's allocations that the agent can attain by deviation from message plan μ_{ji} to other message plans under γ_j.

Let $I_{j,i,a} : \mathbb{R}^l \to \mathbb{R}^l$ be a mapping such that $I_{j,i,a}(y_j) = y_j$ if $j \notin a_i$ or $y_j \neq \overline{y}_j(a)$. The set of all $I_{j,i,a}$ is denoted by $\mathcal{I}_{j,i,a}$. Define

$$\begin{aligned} \overline{D}_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I}) \quad = \quad & \{\overline{z}_j \in (\mathbb{R}^l)^{\Theta} | \exists z_j \in D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I}), \\ & \forall \theta, \ \exists I \in \mathcal{I}_{j,i,\alpha(\theta)}, \ I(\overline{z}_j(\theta)) = z_j(\theta)\}. \end{aligned}$$

Note that $D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I}) \subset \overline{D}_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I})$. Each element in $\overline{D}_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I})$ is a type-contingent allocation constructed from an element in $D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I})$ by replacing some components to the punishment allocations. For each $\overline{z}_j \in \overline{D}_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h\in I})$, thus, there exists an element in $D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h\in I})$ that is more preferable than $\overline{z}_j$ for agent i regardless of the types and the other principals' allocations.

Let γ'_j be another mechanism of principal j and $(\mu'_{ji})_{i\in I} \in \prod_{i\in I}(M_{ji}(\gamma'_j))^{\Theta_i \times A}$ be a message profile under γ'_j. We say that $(\gamma_j, \alpha, (\mu_{ji})_{i\in I})$ is simpler than $(\gamma'_j, \alpha, (\mu'_{ji})_{i\in I})$ if $\gamma_j(\alpha, (\mu_{ji})_{i\in I}) = \gamma'_j(\alpha, (\mu'_{ji})_{i\in I})$ and

$$D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h\in I}) \subset \overline{D}_{ji}(\gamma'_j, \alpha, (\mu'_{jh})_{h\in I}),$$

for all $i \in I$. Let $\gamma = (\gamma_j)_{j\in J}$ and $\gamma' = (\gamma'_j)_{j\in J}$ be two profiles of mechanisms and σ be a pure strategy profile of the agents. If $\sigma_0^{\gamma} = \sigma_0^{\gamma'}$ and $(\gamma_j, \alpha, (\sigma_{ji}^{\gamma})_{i\in I})$ is simpler than $(\gamma'_j, \alpha, (\sigma_{ji}^{\gamma'})_{i\in I})$ for all j and α, then we say that $(\gamma, \sigma^{\gamma})$ is simpler than $(\gamma', \sigma^{\gamma'})$.

Lemma 2. Let σ be a pure strategy profile of the agents in which $(\gamma, \sigma^{\gamma})$ is simpler than $(\gamma', \sigma^{\gamma'})$. Suppose that $\sigma^{\gamma'}$ is rational to γ'. Then σ^{γ} is rational to γ.

Proof. Let $\alpha_{-i} = (\sigma_{0h}^{\gamma})_{h\neq i} = (\sigma_{0h}^{\gamma'})_{h\neq i}$. Define

$$\overline{D}_i(\gamma', \sigma^{\gamma'}) \;=\; \{z \in \prod_{j\in J} \overline{D}_{ji}(\gamma'_j(\alpha_i, \alpha_{-i}, (\sigma_{jh}^{\gamma'})_{h\in I})|\alpha_i \in (A_i)^{\Theta_i}\}.$$

Observe that the most preferable outcomes for agent i in $\overline{D}_i(\gamma', \sigma^{\gamma'})$ coincides with the most preferable outcomes in $D_i(\gamma', \sigma^{\gamma'})$.

The set of accessible outcomes for agent i under γ is

$$\begin{aligned} D_i(\gamma, \sigma^{\gamma}) &= \{\gamma(\tilde{\sigma}_i^{\gamma}, (\sigma_h^{\gamma})_{h\neq i}))|\tilde{\sigma}_i^{\gamma} \in \Sigma_i(\gamma)\} \\ &= \{z \in \prod_{j\in J} D_{ji}(\gamma_j(\alpha_i, \alpha_{-i}, (\sigma_{jh}^{\gamma})_{h\in I})|\alpha_i \in (A_i)^{\Theta_i}\} \\ &\subset \{z \in \prod_{j\in J} \overline{D}_{ji}(\gamma'_j(\alpha_i, \alpha_{-i}, (\sigma_{jh}^{\gamma'})_{h\in I}))|\alpha_i \in (A_i)^{\Theta_i}\} \\ &= \overline{D}_i(\gamma', \sigma^{\gamma'}). \end{aligned}$$

Since $\sigma^{\gamma'}$ is rational to γ',

$$\begin{aligned}\sum_{\theta\in\Theta} F(\theta)U_i(\theta,\gamma(\sigma^{\gamma}|\theta)) &= \sum_{\theta\in\Theta} F(\theta)U_i(\theta,\gamma'(\sigma^{\gamma'}|\theta)) \\ &= \max_{z\in D_i(\gamma',\sigma^{\gamma'})} \sum_{\theta\in\Theta} F(\theta)U_i(\theta,z(\theta)) \\ &= \max_{z\in \overline{D}_i(\gamma',\sigma^{\gamma'})} \sum_{\theta\in\Theta} F(\theta)U_i(\theta,z(\theta)) \\ &\geq \max_{z\in D_i(\gamma,\sigma^{\gamma})} \sum_{\theta\in\Theta} F(\theta)U_i(\theta,z(\theta)),\end{aligned}$$

for each i. Therefore, σ^{γ} is rational to γ. □

Proof. (proof of Theorem 2)

Let $E_j \subset \Gamma_j^{\mathcal{M}_j}$ be the set of equilibrium strategies of principal j in the game $\Gamma^{\mathcal{M}}$:

$$E_j = \{\gamma_j \in \Gamma_j^{\mathcal{M}_j} | \exists(\gamma_{-j},\sigma),\ (\gamma_j,\gamma_{-j},\sigma) \in E(\Gamma^{\mathcal{M}})\}.$$

For each $\gamma = (\gamma_j)_{j\in J}$, define $\gamma(\sigma^{\gamma}|a) : \theta \mapsto \gamma(\sigma^{\gamma}(\theta)|a) \in \mathbb{R}^{Nl}$ so that

$$\gamma(\sigma^{\gamma}(\theta)|a) = (\gamma_j(a,(\sigma_{ji}^{\gamma}(\theta_i,a))_{i\in I}))_{j\in J},$$

for all $\theta = (\theta_i)_{i\in I}$. We say that an outcome z is *conceivable* under $\gamma_j \in E_j$ and $a \in A$ if there exists $\gamma_{-j} \in \prod_{k\neq j} \Gamma_k^{\mathcal{M}_k}$ and a rational response σ to $\Gamma^{\mathcal{M}}$ such that

$$z = \gamma(\sigma^{\gamma}|a),$$

where $\gamma = (\gamma_j,\gamma_{-j})$.

Let $Z_j(\gamma_j,a)$ be the set of all conceivable outcomes under γ_j and a. Define a perfect revelation mechanism $\rho_j^{\gamma_j}$ of principal j so that for each a,

$$\rho_j^{\gamma_j}(a,(\theta^i,z^i)_{i\in I}) = \begin{cases} \overline{z}_j((\theta^i)_{i\in I}) & \text{if} \quad \forall i \in \{h|j\in a_h\},\ z^i = \overline{z} \in Z_j(\gamma_j,a), \\ \overline{y}_j(a) & \text{otherwise,}\end{cases}$$

where $(\theta^i,z^i) \in \Theta_i \times (\mathbb{R}^{Nl})^{\Theta}$ for each i. The set of perfect revelation mechanisms constructed from each element in E_j is denoted by $R_j^1 = \{\rho_j^{\gamma_j} | \gamma_j \in E_j\}$.

If $\mathcal{M}_j$ has cardinality at least as large as the continuum, then let $M \subset \mathcal{M}_j$ be an arbitrary set of messages that has the cardinality of the continuum. If $\mathcal{M}_j$ has less cardinality than the continuum, then let $M = \mathcal{M}_j$. We define a set of principal j's mechanisms $\Gamma_j^M \subset \Gamma_j^{\mathcal{M}_j}$ so that

$$\Gamma_j^M = \{\gamma_j \in \Gamma_j^{\mathcal{M}_j} \setminus E_j | \forall i \in I,\ M_{ji}(\gamma_j) = M\}.$$

If $\mathcal{M}_j$ has cardinality at least as large as the continuum, then let $f_i : \Theta_i \times (\mathbb{R}^{Nl})^{\Theta} \to M$ be a bijection. If $\mathcal{M}_j$ has less cardinality than the continuum, then let $f_i : \Theta_i \times (\mathbb{R}^{Nl})^{\Theta} \to M$ be a surjection. We define $\rho_j^{\gamma_j}$ for each $\gamma_j \in \Gamma_j^M$ so that

$$\rho_j^{\gamma_j}(a, (\theta^i, z^i)_{i \in I}) = \gamma_j(a, (f_i(\theta^i, z^i))_{i \in I}),$$

for each a. The set of perfect revelation mechanisms constructed from each element in Γ_j^M is denoted by $R_j^2 = \{\rho_j^{\gamma_j} | \gamma_j \in \Gamma_j^M\}$.

Define $R_j = R_j^1 \cup R_j^2$. Since each element in R_j is a perfect revelation mechanism of principal j, we have constructed a perfect revelation game represented by $R^{\mathcal{M}} = \prod_{j \in J} R_j$.

Let $(\gamma^*, \sigma^*) \in E(\Gamma^{\mathcal{M}})$ be an arbitrary equilibrium of the game $\Gamma^{\mathcal{M}}$. For each $\rho_j \in R_j$, we define $\gamma_j^{\rho_j}$ so that (1) $\rho_j = \rho_j^* \Rightarrow \gamma_j^{\rho_j} = \gamma_j^*$ and (2) $\rho_j \neq \rho_j^* \Rightarrow \gamma_j^{\rho_j} \in \{\gamma_j | \rho_j = \rho_j^{\gamma_j}\}$. Let $\gamma^\rho = (\gamma_j^{\rho_j})_{j \in J}$. We construct $\overline{\sigma}^{*\rho}$ as follows: (1) define $\overline{\sigma}_{0i}^{*\rho} = \sigma_{0i}^{*\gamma^\rho}$ for all i, (2) define[6]

$$\overline{\sigma}_{ji}^{*\rho}(\theta_i, a) \begin{cases} = (\theta_i, \gamma^\rho(\sigma^{*\gamma^\rho} | a)) & \text{if} \quad \gamma_j^{\rho_j} \in E_j, \\ \in f_i^{-1}(\sigma_{ji}^{*\gamma^\rho}(\theta_i, a)) & \text{if} \quad \gamma_j^{\rho_j} \in \Gamma_j^M, \end{cases}$$

for all j, i, θ_i and $a \in A$.

Let $(\rho, \overline{\sigma}^*)$ be a strategy profile of the game $R^{\mathcal{M}}$. If principal j has offered ρ_j such that $\gamma_j^{\rho_j} \in \Gamma_j^M$, then principal j who has observed $a \in A$ will assign

$$\gamma_j^{\rho_j}(a, (\sigma_{ji}^{*\gamma^\rho}(\theta_i, a))_{i \in I}),$$

---

[6] $f_i^{-1}(m_{ji})$ represents the inverse image of f_i at m_{ji}.

for each $\theta = (\theta_i)_{i \in I}$, since each agent i of type θ_i reports an element in $f_i^{-1}(\sigma_{ji}^{*\gamma^\rho}(\theta_i, a))$.

If principal j has offered ρ_j such that $\gamma_j^{\rho_j} \in E_j$, then principal j who has observed $a \in A$ will assign

$$\gamma_j^{\rho_j}(a, (\sigma_{ji}^{*\gamma^\rho}(\theta, a))_{i \in I}),$$

for all $\theta = (\theta_i)_{i \in I}$, since each agent i of type θ_i reports $(\theta_i, \gamma^\rho(\sigma^{*\gamma^\rho}|a))$ and $\gamma^\rho(\sigma^{*\gamma^\rho}|a)$ is conceivable under $(\gamma_j^{\rho_j}, a)$.

As agent i of θ_i takes the participation action $\overline{\sigma}_{0i}^{*\rho}(\theta_i) = \sigma_{0i}^{*\gamma^\rho}(\theta_i)$, we obtain $\rho(\overline{\sigma}^{*\rho}) = \gamma^\rho(\sigma^{*\gamma^\rho})$. Because $\gamma^{\rho^*} = \gamma^*$, we especially have

$$\rho^*(\overline{\sigma}^{*\rho^*}) = \gamma^*(\sigma^{*\gamma^*}).$$

Note that $\overline{\sigma}^*$ is truthful to ρ^*: each agent i of type θ_i takes participation action $\sigma_{0i}^{*\gamma^\rho}(\theta_i)$ and reports $(\theta_i, \gamma^\rho(\sigma^{*\gamma^\rho}))$ to each principal who conforms to ρ^* when the agent has observed ρ, and then outcome $\gamma^\rho(\sigma^{*\gamma^\rho})$ will actually be realized.

Moreover, $(\rho, \overline{\sigma}^{*\rho})$ is simpler than $(\gamma^\rho, \sigma^{*\gamma^\rho})$. Let $\alpha_i : \Theta_i \to A_i$ be an arbitrary participation action plan of agent i and define $\alpha = (\alpha_i)_{i \in I}$. Under ρ_j such that $\gamma_j^{\rho_j} \in \Gamma_j^M$,

$$\begin{aligned}
&D_{ji}(\rho_j, \alpha, (\overline{\sigma}_{jh}^{*\rho})_{h \in I}) \\
&= \{\gamma_j^{\rho_j}(\alpha, \mu_{ji}, (\sigma_{jh}^{*\gamma^\rho})_{h \neq i}) | \forall \theta_i,\ \mu_{ji}(\theta_i) \in f_i(\Theta_i \times (\mathbb{R}^{Nl})^\Theta) = M_{ji}(\gamma_j)\} \\
&= D_{ji}(\gamma_j^{\rho_j}, \alpha, (\sigma_{jh}^{*\gamma^\rho})_{h \in I}),
\end{aligned}$$

for each $i \in I$. Under ρ_k such that $\gamma_k^{\rho_k} \in E_k$, we have

$$\begin{aligned}
&D_{ki}(\rho_k, \alpha, (\overline{\sigma}_{kh}^{*\rho})_{h \in I}) \\
&\subset \{\overline{z}_k \in (\mathbb{R}^l)^\Theta | \exists z_k \in D_{ki}(\gamma_k^{\rho_k}, \alpha, (\sigma_{kh}^{*\gamma^\rho})_{h \in I}),\ \forall \theta,\ \exists I \in \mathcal{I}_{k,i,\alpha(\theta)},\ I(\overline{z}_k) = z_k\} \\
&= \overline{D}_{ki}(\gamma_k^{\rho_k}, \alpha, (\sigma_{kh}^{*\gamma^\rho})_{h \in I}).
\end{aligned}$$

Hence $\overline{\sigma}^*$ is rational to $R^{\mathcal{M}}$ by Lemma 2.

Finally, a deviation from ρ_j^* to some perfect revelation mechanism ρ_j attains the outcome $(\gamma_j^{\rho_j}(\sigma^{*\gamma_j^{\rho_j},\gamma_{-j}^*}), \gamma_{-j}^*(\sigma^{*\gamma_j^{\rho_j},\gamma_{-j}^*}))$, which is not preferable for principal j to conforming to ρ_j^* and achieving $\gamma^*(\sigma^{*\gamma^*})$. Therefore, $(\rho^*, \overline{\sigma}^*) \in E_T(R^{\mathcal{M}})$. $\square$

Let $\overline{R}_j$ denote the set of all perfect revelation mechanisms of principal j. The "largest" perfect revelation game is represented by $\overline{R} = \prod_{j \in J} \overline{R}_j$. If we consider a mechanism design game $\Gamma^{\mathcal{M}} = \prod_{j \in J} \Gamma_j^{\mathcal{M}_j}$ such that each $\mathcal{M}_j$ has cardinality at least as large as the continuum, then we have $R^{\mathcal{M}} = \overline{R}$.

Corollary 2. For each $\Gamma^{\mathcal{M}}$ such that $\mathcal{M}_j$ has cardinality at least as large as the continuum for all j, we have $Z(\Gamma^{\mathcal{M}}) \subset Z_T(\overline{R})$.

The converse is not necessarily guaranteed in the case of incomplete information. There may exist an outcome which is achieved at some truthful equilibria of the perfect revelation game $\overline{R}$ but not achieved at any equilibrium of a mechanism design game $\Gamma^{\mathcal{M}}$ with large $\mathcal{M}$.[7]

Remark. There are two characteristics that are common in the perfect revelation mechanisms at equilibria of the perfect revelation games defined in the proof of the theorems. First, if there are multiple participants in a perfect revelation mechanism offered by a principal, then the principal applies the logic of "shoot the liars." Because every agent has symmetrically observed all of the offered mechanisms, a truthful strategy profile makes the agents report the same outcome. Hence, if different outcomes are reported, at least one agent must be a liar. By "shooting" doubtful agents, therefore, the principal effectively prevents the agents from reporting a false outcome and will know the true outcome with no information rent. That is, only the type-incentive constraints have to be satisfied.

If only one agent participates in a perfect revelation mechanism, it is impossible to apply the logic of "shoot the liars." In this case, instead,

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[7] Those equilibria of the perfect revelation game $\overline{R}$ are said to be unrobust in the sense of the weak robustness introduced by Peters (2001).

the perfect revelation mechanism is equivalent to a menu mechanism, and hence, the dimension of incentive constraints is at most the dimension of the allocation space.

More precisely, let $a \in A$ be a profile of participation actions such that only agent i participates in principal j's mechanism. Let $Y_{ji} \subset \mathbb{R}^l$ be the range of $\rho_j(a)$. We call each element in Y_{ji} a *menu* from principal j to agent i. Since $\rho_j(a, (m_{ji})_{i \in I})$ depends only on agent i's message m_{ji}, choosing one message in $M_{ji}(\rho_j)$ and choosing one menu in Y_{ji} are equivalent for agent i.

5 Robustness of the Perfect Revelation Mechanism

Theorems 1 and 2 have shown that we find each equilibrium outcome of each mechanism design game at a truthful equilibrium outcome of a corresponding perfect revelation game. Here, we consider whether each equilibrium outcome of the largest perfect revelation game $\overline{R}$ is also achieved at some equilibrium of a mechanism design game $\Gamma^{\mathcal{M}}$ such that each $\mathcal{M}_j$ has cardinality at least as large as the continuum.

Definition 2. (1) An equilibrium outcome of game $\overline{R}$ is said to be *robust* if the outcome is achieved at an equilibrium of any mechanism design game $\Gamma^{\mathcal{M}}$ in which each $\mathcal{M}_j$ has cardinality at least as large as the continuum. (2) An equilibrium outcome of game $\overline{R}$ is said to be *rationalizable* if the outcome is achieved at an equilibrium of some mechanism design game $\Gamma^{\mathcal{M}}$ in which each $\mathcal{M}_j$ has cardinality at least as large as the continuum.

The definition of the robustness corresponds to the weak robustness introduced by Peters (2001).[8] We may say that a robust equilibrium outcome

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[8] Of course, the agents' response to each of the mechanisms in $\Gamma^{\mathcal{M}}$ is critical for each equilibrium outcome of game $\overline{R}$ to survive at an equilibrium of game $\Gamma^{\mathcal{M}}$. Our definition of the robustness requires that there exists a rational response to $\Gamma^{\mathcal{M}}$ that attains the outcome at some equilibrium. In this sense our approach corresponds to the weak robustness.

is more "plausible" than unrobust equilibrium outcomes in the sense that this outcome will survive even if each principal is allowed to offer more complicated mechanisms. On the other hand, if an equilibrium outcome is not even rationalizable, then the outcome should be excluded because the outcome is never achieved at any mechanism design game in which the set of feasible messages for each principal has cardinality at least as large as the continuum.

Fortunately, every equilibrium outcome of the largest perfect revelation game $\overline{R}$ is proved to be rationalizable.

Proposition 2. Let $\Gamma^{\mathcal{M}} = \prod_{j \in J} \Gamma_j^{\mathcal{M}_j}$ represent a mechanism design game such that each $\mathcal{M}_j$ has the cardinality of the continuum for all j. Then every outcome in $Z_T(\overline{R})$ is rationalizable, i.e., $Z(\Gamma^{\mathcal{M}}) = Z_T(\overline{R})$.

Proof. It suffices to show $Z(\Gamma^{\mathcal{M}}) \supset Z_T(\overline{R})$.

Let $\varepsilon_{ji} : \Theta_i \times (\mathbb{R}^{Nl})^{\Theta} \to \mathcal{M}_j$ be a bijection. For each $\gamma_j \in \Gamma_j^{\mathcal{M}_j}$, let $e_{ji}^{\gamma_j} : \mathcal{M}_j \to M_{ji}(\gamma_j)$ denote the identity mapping if $M_{ji}(\gamma_j) = \mathcal{M}_j$ and be a surjection otherwise. Define $\rho_j^{\gamma_j}$ so that

$$\rho_j^{\gamma_j}(a, (\theta^i, z^i)_{i \in I}) = \gamma_j(a, (e_{ji}^{\gamma_j} \circ \varepsilon_{ji}(\theta^i, z^i))_{i \in I}),$$

for all a and $(\theta^i, z^i) \in \Theta_i \times (\mathbb{R}^{Nl})^{\Theta}$ of each agent i. Observe that $\{\rho_j^{\gamma_j} | \gamma_j \in \Gamma_j^{\mathcal{M}_j}\} = \overline{R}_j$. For each $\gamma = (\gamma_j)_{j \in J}$, we define $\rho^{\gamma} = (\rho_j^{\gamma_j})_{j \in J}$.

Let $(\rho^*, \overline{\sigma}^*)$ be an arbitrary equilibrium of the game $\overline{R}$. For each j, let $\gamma_j^* \in \Gamma_j^{\mathcal{M}_j}$ be a mechanism such that $M_{ji}(\gamma_j^*) = \mathcal{M}_j$ for all i and $\rho_j^{\gamma_j^*} = \rho_j^*$.

Define $\sigma^{*\gamma}$ for each $\gamma = (\gamma_j)_{j \in J}$ as follows: (1) define $\sigma_{0i}^{*\gamma} = \overline{\sigma}_{0i}^{*\rho^{\gamma}}$ for all i, and (2) define

$$\sigma_{ji}^{*\gamma}(\theta_i, a) = e_{ji}^{\gamma_j} \circ \varepsilon_{ji} \circ \overline{\sigma}_{ji}^{*\rho^{\gamma}}(\theta_i, a),$$

for all j, i, θ_i and $a \in A$.

Epstein and Peters (1999) offered a stronger requirement. See Epstein and Peters (1999) and Peters (2001) for more precise analysis.

We show that $(\gamma, \sigma^{*\gamma})$ is simpler than $(\rho^{\gamma}, \overline{\sigma}^{*\rho^{\gamma}})$. Observe that $\gamma(\sigma^{*\gamma}) = \rho^{\gamma}(\overline{\sigma}^{*\rho^{\gamma}})$ because $\sigma^{*\gamma}_{0i} = \overline{\sigma}^{*\rho^{\gamma}}_{0i}$ and

$$\begin{aligned} \gamma_j(a, (\sigma^{*\gamma}_{ji}(\theta_i, a))_{i\in I}) &= \gamma_j(a, (e^{\gamma_j}_{ji} \circ \varepsilon_{ji} \circ \overline{\sigma}^{*\rho^{\gamma}}_{ji}(\theta_i, a))_{i\in I}) \\ &= \rho^{\gamma_j}_j(a, (\overline{\sigma}^{*\rho^{\gamma}}_{ji}(\theta_i, a))_{i\in I}), \end{aligned}$$

for all j, $\theta = (\theta_i)_{i\in I}$ and all $a \in A$. Especially we have $\gamma^*(\sigma^{*\gamma^*}) = \rho^*(\overline{\sigma}^{*\rho^*})$ since $\rho^{\gamma^*} = \rho^*$. Observe also that

$$\begin{aligned} &D_{ji}(\rho^{\gamma_j}_j, \alpha, (\overline{\sigma}^{*\rho^{\gamma}}_{jh})_{h\in I}) \\ &= \{\rho^{\gamma_j}_j(\alpha, \overline{\mu}_{ji}, (\overline{\sigma}^{*\rho^{\gamma}}_{jh})_{h\neq i} | \forall(\theta_i, a),\ \overline{\mu}_{ji}(\theta_i, a) \in \Theta_i \times (\mathbb{R}^{Nl})^{\Theta}\} \\ &= \{\gamma_j(\alpha, \tilde{\mu}_{ji}, (\sigma^{*\gamma}_{jh})_{h\neq i} | \forall(\theta_i, a),\ \tilde{\mu}_{ji}(\theta_i, a) \in M_{ji}(\gamma_j)\} \\ &= D_{ji}(\gamma_j, \alpha, (\sigma^{*\gamma}_{jh})_{h\in I}), \end{aligned}$$

for all $i \in I$ and α. Hence, σ^* is rational to $\Gamma^{\mathcal{M}}$ by Lemma 2.

Finally, a deviation from γ^*_j to some mechanism $\gamma_j \in \Gamma^{\mathcal{M}_j}_j$ attains the outcome $(\rho^{\gamma_j}_j(\overline{\sigma}^{*\rho^{\gamma_j}_j, \rho^*_{-j}}), \rho^*_{-j}(\overline{\sigma}^{*\rho^{\gamma_j}_j, \rho^*_{-j}}))$, which is not preferable for principal j to conforming to γ^*_j and achieving $\rho^*(\overline{\sigma}^{*\rho^*})$. Therefore, $(\gamma^*, \sigma^*) \in E(\Gamma^{\mathcal{M}})$. $\square$

We now show that every truthful equilibrium outcome of the game $\overline{R}$ is robust if the agents have perfectly correlated types. We additionally assume the following condition.

Assumption 2. There exists a finite set T such that $\Theta_i = T$ for all $i \in I$. The distribution function F over Θ satisfies

$$\sum_{t\in T} F(t, \ldots, t) = 1.$$

For example, the complete information model and the single agent model satisfy Assumption 2. Proposition 3 in the single agent model corresponds to the results obtained by Peters (2001) and Martimort and Stole (2002).

Let $(\rho^*, \overline{\sigma}^*)$ be a truthful equilibrium of the perfect revelation game $\overline{R}$. In an arbitrary game $\Gamma^{\mathcal{M}}$ in which each $\mathcal{M}_j$ has cardinality at least as large as the continuum, we show that the outcome $\rho^*(\overline{\sigma}^{*\rho^*}) \in Z_T(\overline{R})$ is also achieved at some equilibrium of $\Gamma^{\mathcal{M}}$.

Proposition 3. Under Assumption 4, every outcome in $Z_T(\overline{R})$ is robust. That is, for each $(\rho^*, \overline{\sigma}^*) \in E_T(\overline{R})$ and for every mechanism design game $\Gamma^{\mathcal{M}} = \prod_{j \in J} \Gamma_j^{\mathcal{M}_j}$ in which $\mathcal{M}_j$ has cardinality at least as large as the continuum for all j, there exists $(\gamma^*, \sigma^*) \in E(\Gamma^{\mathcal{M}})$ such that $\rho^*(\overline{\sigma}^{*\rho^*}) = \gamma^*(\sigma^{*\gamma^*})$.

To prove Proposition 3, we require some additional notation and a lemma. Let $d_{ji}(\gamma_j, a, (m_{jh})_{h \in I}) = \{\gamma_j(a, \tilde{m}_{ji}, (m_{jh})_{h \neq i}) | \tilde{m}_{ji} \in M_{ji}(\gamma_j)\}$ be the set of accessible allocations for agent h under γ_j when a participation actions $a \in A$ has been observed and the agents other than i will report $(m_{jh})_{h \neq i}$ to principal j. Under complete information (i.e., T is a singleton), we have $d_{ji}(\cdot) = D_{ji}(\cdot)$. In general under Assumption 2,

$$D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I}) = \prod_{t \in T} d_{ji}(\gamma_j, \alpha(t, \ldots, t), (\mu_{jh}(t, \alpha(t, \ldots, t)))_{h \in I}),$$

for all α and $(\mu_{jh})_{h \in I}$.

Let $R(\gamma_j(a)) = \{\gamma_j(a, (m_{ji})_{i \in I}) | \forall i, \ m_{ji} \in M_{ji}(\gamma_j)\}$ be the range of the mapping $\gamma_j(a)$. Since d_{jh} represents the set of accessible allocations by unilateral deviation of agent i under $\gamma_j(a)$, we have

$$d_{ji}(\gamma_j, a, (m_{jh})_{h \in I}) \subset R(\gamma_j(a))$$

for all i and $(m_{jh})_{h \in I}$.

If $d_{ji}(\gamma_j, a, (m_{jh})_{h \in I}) = R(\gamma_j(a)) = \tilde{Y}_j(a)$ for all $i \in I$, $a \in A$ and all $(m_{jh})_{h \in I}$, then γ_j is said to be $\tilde{Y}_j(\cdot)$-complex (e.g., see $\tilde{\gamma}_j$ in Table 2). Let γ'_j be another mechanism such that $R(\gamma'_j(a)) = \tilde{Y}_j(a)$ for each a (e.g., see $\overline{\gamma}_j$ in Table 1) and let $\mu_{ji} : T \times A \to M_{ji}(\gamma_j)$ and $\mu'_{ji} : T \times A \to M_{ji}(\gamma'_j)$ be agent i's message plans in γ_j and in γ'_j respectively. Suppose $\gamma_j(a, (\mu_{ji}(t, a))_{i \in I}) = \gamma'_j(a, (\mu'_{ji}(t, a))_{i \in I})$ for all t and a. Since γ_j is $\tilde{Y}_j(\cdot)$-complex, we have

$$D_{ji}(\gamma'_j, \alpha, (\mu'_{jh})_{h \in I}) \subset D_{ji}(\gamma_j, \alpha, (\mu_{jh})_{h \in I}) = \prod_{t \in T} \tilde{Y}_j(\alpha(t, \ldots, t)),$$

and hence, $(\gamma'_j, (\mu'_{ji})_{i \in I})$ is simpler than $(\gamma_j, (\mu_{ji})_{i \in I})$.

Lemma 3. For each $\tilde{Y}_j(\cdot)$, there exists a $\tilde{Y}_j(\cdot)$-complex mechanism in $\overline{R}_j$.

Proof. Observe that $\tilde{Y}_j(a) \subset Y_j(a)$ for each a. It hence suffices to show that there exists a $Y_j(\cdot)$-complex mechanism in $\overline{R}_j$.

Let $g_i^a : \Theta_i \times (\mathbb{R}^{Nl})^{\Theta} \to \mathbb{R}$ and $G^a : \mathbb{R} \to Y_j(a)$ be bijections. Let $\rho_j \in \overline{R}_j$ be a perfect revelation mechanism such that

$$\rho_j(a, (\theta^i, z^i)_{i\in I}) = G^a\left(\sum_{i\in I} g_i^a(\theta^i, z^i)\right),$$

for each a and $(\theta^i, z^i) \in \Theta_i \times (\mathbb{R}^{Nl})^{\Theta}$. The perfect revelation mechanism ρ_j is $Y_j(\cdot)$-complex because

$$\begin{aligned} d_{ji}(\rho_j, a, (\theta^h, z^h)_{h\in I}) &= \{G^a(\sum_{h\neq i} g_h^a(\theta^h, z^h) + g) | g \in \mathbb{R}\} \\ &= \{G^a(g) | g \in \mathbb{R}\} \\ &= Y_j(a), \end{aligned}$$

for all $i \in I$, a and $(\theta^h, z^h) \in \Theta_h \times (\mathbb{R}^{Nl})^{\Theta}$ of each $h \in I$. $\square$

Proof. (proof of Proposition 3)

Let $M_{ji} \in \mathcal{M}_j$ be an arbitrary set which has the set of continuum and $e_i : M_{ji} \to \Theta_i \times (\mathbb{R}^{Nl})^{\Theta}$ be a bijection. Define $\gamma_j^* \in \Gamma_j^{\mathcal{M}_j}$ so that

$$\gamma_j^*(a, (m_{ji})_{i\in I}) = \rho_j^*(a, (e_i(m_{ji}))_{i\in I}),$$

for each a and $(m_{ji})_{i\in I} \in \prod_{i\in I} M_{ji}$.

Let γ_j be an arbitrary mechanism in $\Gamma_j^{\mathcal{M}_j} \setminus \{\gamma_j^*\}$ and let $\tilde{Y}_j(a) = R(\gamma_j(a))$ denote the range of $\gamma_j(a)$ for each a. By Lemma 3, we can find a $\tilde{Y}_j(\cdot)$-complex mechanism in $\overline{R}_j$, which we denote by $\rho_j^{\gamma_j}$.

Define $\rho_j^{\gamma_j^*} = \rho_j^*$ and $\rho^{\gamma} = (\rho_j^{\gamma_j})_{j\in J}$. The agents' strategy profile σ^* is defined as follows: (1) Define $\sigma_{0i}^{*\gamma} = \overline{\sigma}_{0i}^{*\rho^{\gamma}}$ for each i. (2) If $\gamma_j = \gamma_j^*$, define for each i, θ_i and a,

$$\sigma_{ji}^{*\gamma}(\theta_i, a) \in e_i^{-1}(\overline{\sigma}_{ji}^{*\rho^{\gamma}}(\theta_i, a)).$$

(3) If $\gamma_j \neq \gamma_j^*$, then define $(\sigma_{ji}^{*\gamma}(a))_{i\in I}$ so that for each i and a,

$$\gamma_j(a, (\sigma_{ji}^{*\gamma}(a))_{i\in I}) = \rho_j^{\gamma_j}(a, (\overline{\sigma}_{ji}^{*\rho^{\gamma}}(a))_{i\in I}).$$

$(\gamma, \sigma^{*\gamma})$ is clearly simpler than $(\rho^{\gamma}, \overline{\sigma}^{*\rho^{\gamma}})$ for each $\gamma \in \Gamma^{\mathcal{M}}$, and thus, we have $\gamma^*(\sigma^{*\gamma^*}) = \rho^*(\overline{\sigma}^{*\rho^*})$ and we obtain that σ^* is rational to $\Gamma^{\mathcal{M}}$ by Lemma 2.

Finally, a deviation from γ_j^* to some mechanism $\gamma_j \in \Gamma_j^{\mathcal{M}_j}$ attains the outcome $(\rho_j^{\gamma_j}(\overline{\sigma}^{*\rho_j^{\gamma_j}, \rho^*_{-j}}), \rho^*_{-j}(\overline{\sigma}^{*\rho_j^{\gamma_j}, \rho^*_{-j}}))$, which is not preferable for principal j to conforming to γ_j^* and achieving $\rho^*(\overline{\sigma}^{*\rho^*})$. Therefore, $(\gamma^*, \sigma^*) \in E(\Gamma^{\mathcal{M}})$. $\square$

If we do not impose Assumption 2, agent i of type θ_i does not in general know the other agents' types. The set of accessible allocations d_{ji} of agent i under γ_j is then not represented by the set of allocations, but by the set of lotteries over allocations. We can no longer define the mechanisms that are "complex" in the proper sense, and therefore, the robustness of each outcome in $Z_T(\overline{R})$ is not guaranteed.

Even in this case, however, we believe that the perfect revelation principle is useful. We will choose some equilibrium outcomes in $Z_T(\overline{R})$ that seem to be plausible in view of criteria other rather than robustness and rationalizability.

6 Example

In this section, we consider a mechanism design game with two principals, whom we call the retailers, and two agents, whom we call the manufacturers. Each manufacturer sells goods to the retailers, and each retailer sells the goods to consumers in a Cournot market. The manufacturers (agents) independently have private information about the demand curve.

We show that the retailers can achieve the cartel outcome in the perfect revelation game, while they cannot achieve the cartel outcome if their strategy spaces are restricted to the set of type-revelation mechanisms or the set of menu mechanisms.

Intuitively, to attain the cartel outcome in equilibria, the following two conditions are needed. First, each principal must detect the other principal's deviation from the cartel and then punish the deviator. A principal who conforms a type-revelation mechanism cannot detect the deviation as in

Martimort and Stole (2002)'s example. Second, the cartel production levels in our example are different with respect to the type profile (parameters of the demand curve) of the agents. Because an agent (say agent 1) does not know the other agent's type when agent 1 chooses a menu, agent 1 cannot choose the cartel production level depending on the other agent's type. Hence, if a principal conforms a menu mechanism, then the realized production level does not perfectly depend on both agents' type profile.

Note that the perfect revelation mechanisms satisfy both of the conditions. First, each principal would detect the other principal's deviation by asking the agents to report the outcome which would be realized. Second, the production level will differ depending on the agents' type profile, because each principal will ask each agent's type.

Formally, we consider the following Cournot market. There are two retailers ($j = 1, 2$) and two manufacturers ($i = 1, 2$). The retailers cannot produce goods, and thus, they purchase goods from the manufacturers. More precisely, retailer j assigns an allocation $(q_{j1}, q_{j2}, t_{j1}, t_{j2}) \in [0, \overline{Q}]^2 \times \mathbb{R}_+^2$ to the manufacturers, where q_{ji} denotes the quantity of good i supplied for retailer j from manufacturer i and t_{ji} denotes the transfer from retailer j to manufacturer i.

If the manufacturer produces q_{ji} units for each j, then it costs $q_{1i} + q_{2i}$ for manufacturer i. For simplicity, we assume $A_i = \{J, \emptyset\}$ for each i, that is, agent i may participate in both mechanisms or neither.

We assume that both goods are essential for the retailers in the sense that each consumer always buys a pair of the goods made by the manufacturers (e.g., manufacturer 1 makes personal computers and manufacturer 2 makes software). Thus it is never optimal for each retailer to supply $q_{j1} \neq q_{j2}$ units of each good. We hence redefine retailer j's allocation space as

$$Y_j = [0, \overline{Q}] \times \mathbb{R}_+^2 \ni (q_j, t_{j1}, t_{j2}),$$

where q_j denotes the quantity of both goods supplied by both manufacturers for retailer j (i.e., $q_j = q_{j1} = q_{j2}$). Define $Y = Y_1 \times Y_2$.

Let $P(q,\theta) = a(\theta) - q$ be the price when q units are supplied, where $a(\theta) \in (0, \overline{Q})$ reflects information about the size of the market for each $\theta \in \Theta = \{\underline{\theta}, \overline{\theta}\}^2$. Let $\underline{a} = \min_\theta a(\theta)$ and $\overline{a} = \max_\theta a(\theta)$. Each type profile is realized with equal probability, that is, $F(\theta) = \frac{1}{4}$ for every θ. We assume $a(\theta) \neq a(\theta')$ for $\theta \neq \theta'$, $\underline{a} > 2$ and $\overline{a} - \underline{a}$ is sufficiently small.[9] The second condition assures the existence of interior solutions, and the third condition implies that the demand fluctuation is not so large.

If retailer j supplies q_j units of both goods to the market and transfers t_{j1}, t_{j2} to each manufacturer, then the profit for principal j is denoted by

$$\pi_j = (a(\theta) - q_1 - q_2)q_j - t_{j1} - t_{j2}.$$

Benchmark 1 First, suppose that two retailers have successfully formed a cartel. Thanks to the type-revelation principle, the cartel finds an optimal mechanism in the set of type-revelation mechanisms. Let $(q(\theta), t_1(\theta), t_2(\theta)) \in [0, \overline{Q}] \times \mathbb{R}_+^2$ for each θ represent a type-revelation mechanism. Given $q(\theta)$, it is clearly optimal for the cartel to set $t_1(\theta) = t_2(\theta) = q(\theta)$ for each θ. Because $U_i = t_i(\theta) - q(\theta) = 0$ for all θ, this payment scheme implements $q(\theta)$. The optimal levels of production and transfers are given by

$$(q^m(\theta), t_1^m(\theta), t_2^m(\theta)) = (\frac{a(\theta)-2}{2}, \frac{a(\theta)-2}{2}, \frac{a(\theta)-2}{2}),$$

for each θ. Note that $q^m(\theta) \neq q^m(\theta')$ and $t_i^m(\theta) \neq t_i^m(\theta')$ for each $\theta \neq \theta'$.

Benchmark 2 Suppose that there is no successful cartel between the retailers. We call an outcome the *cartel outcome* if each retailer supplies $\frac{q^m(\theta)}{2}$ units of the goods and pays $\frac{t_i^m(\theta)}{2}$ to each manufacturer i for each θ.

First, we analyze the game in which the set of feasible mechanisms for each retailer is restricted to the set of type-revelation mechanisms. Suppose that the cartel outcome is achieved at some equilibrium of this game. The expected profit for each retailer in the cartel is given by $\Pi_1^* = \sum_\theta \frac{(a(\theta)-2)^2}{32}$.

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[9] More precisely, we assume $\overline{a} - \underline{a} \geq \frac{3-2\sqrt{2}}{4}(\overline{a} - 2)$, or equivalently, $\underline{a} \geq \frac{1+2\sqrt{2}}{4}\overline{a} + \frac{3-2\sqrt{2}}{2}$.

Now, suppose that retailer 1 deviates to $(q'(\theta), t_1'(\theta), t_2'(\theta))$ such that $q'(\theta) = \frac{1}{8}(4\underline{a} - \overline{a} - 6)$ and $t_i'(\theta) = q'(\theta) + \varepsilon$ for each i and each θ, where $\varepsilon > 0$. Each manufacturer would strictly like to participate in both mechanisms, but indifferent among all messages because the manufacturer would always obtain ε in the case of participation. Retailer 1's expected profit is then given by

$$\begin{aligned}\Pi_1' &= \frac{1}{64}(4\underline{a} - \overline{a} - 6)^2 \\ &\geq \frac{(\overline{a} - 2)^2}{8} \\ &> \Pi_1^*.\end{aligned}$$

Therefore, retailer 1 will be better off by deviation. The logic of this cartel failure is the same as that under complete information.

Second, let us consider the game in which the set of feasible mechanisms for each retailer is restricted to the set of menu mechanisms. Suppose without loss of generality that the decision of production level of retailer 1 is delegated to manufacturer 1. In this case, retailer 1's production level does not depend on manufacturer 2's message. It implies that $q_1(\theta_1, \theta_2) = q_1(\theta_1, \theta_2')$ for each θ_1 on the equilibrium path, because manufacturer 1 does not know manufacturer 2's type. Therefore, the cartel outcome cannot be achieved.

The Perfect Revelation Game We investigate the possibility of supporting the cartel outcome in the perfect revelation game. Let $z^*(\theta) = (\frac{1}{2}q^m(\theta), \frac{1}{2}t^m(\theta), \frac{1}{2}t^m(\theta))$ and $\overline{z}(\theta) = (a(\theta), a(\theta), a(\theta))$. We show that the following "grim" mechanism ρ_j^G can be retailer j's equilibrium strategy of the perfect revelation game that achieves the cartel production level. For each agent i's message $(\theta_i, z^i) = (\theta_i, z_1^i, z_2^i) \in \Theta_i \times Y_1^\Theta \times Y_2^\Theta$, we define $\rho_j^G(\cdot) \in [0, \overline{Q}] \times \mathbb{R}_+^2$ so that

$$\rho_j^G(\theta_1, z_j^1, z_{-j}^1, \theta_2, z_j^2, z_{-j}^2) = \begin{cases} z^*(\theta_1, \theta_2) & \text{if} \quad z_{-j}^1 = z_{-j}^2 = z^*, \\ \overline{z}(\theta_1, \theta_2) & \text{otherwise.} \end{cases}$$

Since each manufacturer always obtains utility zero, it is rational for the manufacturers to participate always in both retailers' mechanisms and conform the truthful report. Hence, manufacturer i of type θ_i will report (θ_i, z^*, z^*) on the equilibrium path, and will report $(\theta_i, \overline{z}, \overline{z})$ at least to retailer 2, if retailer 1 has deviated from ρ_1^G and retailer 2 has offered ρ_2^G.

Suppose retailer 2 will offer ρ_2^G. Retailer 1's expected profit is the cartel profit Π_1^* if the retailer conforms ρ_1^G, and obtain at most zero profit if the retailer deviates, because retailer 2 will produce $a(\theta)$ units of the goods for each θ. Therefore, retailer 1 has no incentive for deviation, and we have shown that the cartel outcome is achieved at an equilibrium of the perfect revelation game.

7 Conclusion

In this paper, we have investigated general mechanism design games with multiple principals and agents. We introduced the perfect revelation mechanisms, in which each agent is asked to report a type and an outcome. We showed the perfect revelation principle: each equilibrium outcome of a mechanism design game with any strategy space of the principals is achieved at a truthful equilibrium of a corresponding perfect revelation game. Moreover in the model with perfectly correlated types of the agents (Assumption 2), every equilibrium outcome in the largest perfect revelation game is also achieved at any mechanism design game in which the strategy space of each principal is sufficiently large.

In addition, we examine two properties of the perfect revelation mechanisms in equilibria. When there are multiple participants in the mechanism, via "shooting the liars," only type-incentive compatibilities need to be satisfied. Hence, the dimension of the incentive constraints is at most that of the agent's type space. When there is only one participant, we cannot apply "shooting the liars," but there is a menu mechanism that is equivalent to each perfect revelation mechanism. The dimension of the incentive constraints is therefore at most that of the allocation space.

There are several topics that remain to be done. First, we believe that it would be interesting to investigate the characteristics of equilibrium outcomes and equilibrium mechanisms. Specifically, which incentive constraints bind is an interesting question. It is not a trivial question because, as Champsaur and Rochet (1989), Stole (1995) and Biglazer and Mezzetti (2000) showed in their analysis, the principals may face "endogenous" countervailing incentives under competing mechanisms settings. Second, analyzing relationships between the perfect revelation mechanisms and other mechanisms (especially the type-revelation mechanisms and the menu mechanisms) is also of some value. In some multi-principal situations, it is known that any equilibrium outcome is achieved at a truthful equilibrium under type-revelation mechanisms.[10] Finally, mutual dependence among mechanisms may have a considerable influence on outcomes in many economic situations. There are many examples in which mutual dependence among mechanisms may have great influence on economic outcomes, such as sales competitions, tax competitions to attract enterprises, coordination problems among several regulating bodies and headhunting. The perfect revelation principle will be useful in analyzing these situations.

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[10] For instance, although our model differs from theirs, Calzolari and Pavan (2000) investigate games with two principals who sequentially offer mechanisms, and several papers including Myerson (1982) and Gal-Or (1991) analyzed exclusive dealing settings where each agent can observe only one principal's mechanism, who is the exogenously fixed partner.

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