

1 Theory of Auctions

1.1 Independent Private Value Auctions

- for the moment consider an environment in which there is a single seller who wants to sell one indivisible unit of output to one of n buyers whose valuations are private, a buyer whose valuation is x who trades at price p gets surplus $x - p$ the seller gets surplus p in this case
- auctions are implemented using a variety of indirect mechanisms
 1. in a first price auction each buyer submits a bid and the high bidder pays his bid, securites auctions, treasury bills, procurement, timber auctions
 2. in a second price auction each buyer submits a bid, the high bidder wins and pays the bid of the second high bidder, ebay, clearing house
 3. english auction - the auctioneer asks for a starting bid, then asks for a volunteer to raise the bid, he continues to do this

until no one offers to raise the bid, the last bidder offered to raise the bid wins and pays that bid

4. english button auction - each bidder who wants to bid begins by pressing and holding down a button. The price is then raised continuously and each bidder releases the button when the price gets too high. When there is only one bidder left holding down a button, that bidder wins and pays the price where the last bidder dropped out
 5. dutch auction - the price falls continuously until some bidder yells stop. The yelling bidder wins and pays the price at which the price stopped. Dutch flower auctions
 6. all pay auction - all bidders submit bids, the high bidder wins but all bidders pay what they bid.
 7. Cremer McLean auction - a second price auction in which all bidders who want to participate have to agree ex ante to pay a fee (that might depend on the bids of the other bidders)
- you can probably imagine other variants on this theme. All of them can be dealt with using the following theorem

1.2 Revenue Equivalence Theorem

- *Revenue Equivalence Theorem*: Suppose that buyer valuations are identically and *independently* distributed according to some known distribution F whose support is an interval in \mathbf{R} . Suppose further that the indirect mechanism that guides trade has an equilibrium in which the buyer with the highest valuation trades if and only if his valuation is at least r , and that a buyer with valuation r gets an expected payoff equal to zero. Then the seller's expected revenue from this indirect mechanism is

$$n \int_r^1 F^{n-1}(\theta) \left[\theta - \frac{1 - F(\theta)}{F'(\theta)} \right] F'(\theta) d\theta$$

Furthermore, each buyer's expected payment is given by

$$P(\theta) = F^{n-1}(\theta) \theta - \int_{\theta^*}^{\theta} F^{n-1}(x) dx$$

- *Proof*: This follows the mechanism design argument in the previous lecture, but I give it here for completeness

- each buyers' expected payoff is given by

$$Q_i(\theta)\theta - P_i(\theta)$$

where Q_i is the probability with which the buyer trades and P_i is the expected payment the buyer makes to the seller.

- by assumption the indirect mechanism has an equilibrium in which the buyer with the highest valuation trades, so this trading probability is the same for every one and equal to $F^{n-1}(\theta)$ for buyers whose valuations are at least r , and it is equal to zero otherwise
- since it cannot pay for a buyer to behave as if his type were different from his true type in any equilibrium, it must be that

$$F^{n-1}(\theta)' \theta = P_i'(\theta)$$

for every buyer whose valuation is at least r

- Integrating by parts gives

$$P_i(\theta) = \int_0^\theta F^{n-1}(s)' s ds = F^{n-1}(\theta)\theta - \int_r^\theta F^{n-1}(s) ds \quad (1)$$

- this gives the result for buyers' expected payoff. The seller's expected revenue is the sum of the expected revenue for each buyer, or

$$n \int_0^1 P_i(\theta) dF(\theta) = n \int_r^1 \left\{ F^{n-1}(\theta) \theta - \int_r^\theta F^{n-1}(s) ds \right\} dF(\theta)$$

- the final result follows by integrating the second term above by parts.

1.3 Using Revenue Equivalence - First Price Auction

- in the first lecture, we showed an example of a first price auction that possessed an equilibrium in increasing bidding rules. Now let us just suppose that such an equilibrium exists more generally. Then the expected payment is equal to the bid multiplied by the probability of winning, i.e.,

$$P_i(\theta) = Q_i(\theta) b_i(\theta) = F^{n-1}(\theta) b_i(\theta)$$

so

$$b_i(\theta) = \theta - \frac{\int_r^\theta F^{n-1}(s) ds}{F^{n-1}(\theta)} \quad (2)$$

for each $\theta \geq r$

- if all bidders use this bid function, the bidder with the high valuation will win because this function is increasing, it satisfies incentive compatibility, so no bidder using it would prefer to act like a bidder with another valuation. Check for yourself that it doesn't pay to bid prices that no other bidder would ever bid and that a buyer of valuation r gets zero expected payoff
- in other words, the revenue equivalence theorem can be used to calculate the equilibrium bidding strategy in a first price auction.

1.4 Second Price Auction

- in a second price auction, there is an equilibrium in which each buyer bids his true valuation. This bidding strategy is increasing,

so the buyer with the highest valuation will trade in a second price auction - a buyer who bids the reserve price will only win if no other buyers bid, but then he gets zero surplus

- thus from the revenue equivalence theorem a first and second price auction in which the reserve price is the same give the seller the same expected revenue.
- furthermore, the expected payment made by a bidder of type θ in the second price auction is equal to the probability of winning multiplied by the expectation of the second highest valuation conditional on θ being the highest valuation.
- the from (1), it follows that the equilibrium bid in the first price auction for a bidder of type θ is equal to the expected value of the second highest valuation conditional on θ being the highest valuation (you can also see this by undoing the integration by parts in (1) since

$$\theta - \frac{\int_r^\theta F^{n-1}(s) ds}{F^{n-1}(\theta)} = \frac{\int_r^\theta F^{n-2}(s) f(s) ds}{F^{n-1}(\theta)}$$

1.5 Dutch Auction

- a dutch auction resembles a first price auction, the buyer decides when to stop the clock -
- let $b(\theta)$ be a symmetric equilibrium strategy for the first price auction like the one given by (2)
- for any array of valuations for the buyers, the outcome of the dutch auction when all but one of the buyers use this strategy while the other buyer stops the clock at price b' is the same as the outcome of the first price auction when all but one of the buyers bids according to $b(\theta)$ while the other buyer bids b'
- it follows that the rule in which every buyer stops the clock (if it is stilling running) when the price hits $b(\theta)$ is an equilibrium for the dutch auction. In this equilibrium, the buyer with the highest valuation trades (since $b(\theta)$ is increasing) and a buyer whose valuation is equal to the seller's reserve price (the point where the seller stops the clock and refuses to trade) gets zero expected payoff.

- by revenue equivalence, the seller's expected revenue from the dutch auction is the same as it is from the first price auction

1.6 All-Pay Auction

- let us use the technique we employed for first price auctions to compute the equilibrium in the all pay auction, suppose there is an equilibrium in increasing bidding strategies so that the equilibrium outcome is always that the buyer with the high valuation ends up trading.
- since everyone pays whether or not they win the object, the expected payment is equal to the bid, i.e.,

$$b(\theta) = F^{n-1}(\theta)\theta - \int_r^\theta F^{n-1}(s) ds$$

which is increasing and the surplus of a bidder whose valuation is equal to the reserve price r gets zero surplus, hence $b(\theta)$ is an equilibrium satisfying the conditions of the revenue equivalence

theorem - so the all pay auction gives the seller exactly the same expected revenues as the first price, second price, dutch, english auction

- the seller's revenues rise because he gets a payment from each of the bidders, they fall because bidders take account of this and submit lower bids in equilibrium, the net result is a wash.

1.7 Applications outside auction theory

- (Klemperer) suppose two parties are involved in a lawsuit. The party that wins the lawsuit gets utility that is θ_i higher than it does when it loses (the payoff to losing is normalized to zero). These payoffs are private and drawn from a common monotonic distribution F
- each party spends b_i defending itself and the party that spends the most wins the lawsuit
- under standard rules (in the US) each party pays its own legal fees,

so the winner gets $\theta_i - b_i$

- could expenditures on legal fees be reduced by forcing the losing party to pay the winner some fraction of the loser's expenses? what about forcing the loser to pay some fraction of the winner's legal expenses, should there be a minimum legal expenditure required to win the case?
- the key insight is that the existing legal system is equivalent from a strategic viewpoint to an all pay auction - the party who spends (bids) the most wins the case (trades) but all parties pay what they spent (bid).
- total expected legal expenditures are equivalent to the seller's revenue in the auction problem
- the minimum expenditure requirement (for example forcing litigants to be represented by lawyers) is equivalent to the reserve price in the auction

- suppose first that there is an equilibrium in which parties expenditures are increasing functions of their gains θ , under the existing rules a party whose gain is exactly equal to the minimum expenditure requirement cannot gain by litigating, nor can they lose since they receive the default payoff 0 by spending nothing, so the revenue equivalence theorem implies that expected legal expenditures are

$$2 \int_r^1 F(\theta) \left[\theta - \frac{1 - F(\theta)}{F'(\theta)} \right] F'(\theta) d\theta \quad (3)$$

while parties expected expenditures are equal to the equilibrium legal expenses

$$F(\theta) \theta - \int_r^\theta F(s) ds$$

- legal expenditures go to lawyers, so assuming that objective of the legal system is to maximize lawyer income, the optimal expenditure

requirement is to set r such that

$$r = \frac{1 - F(r)}{F'(r)}$$

as in the optimal selling mechanism,

- on the other hand, if the objective is to maximize expected gains to litigation less expected expenditures (and assuming the virtual valuation is increasing), the reserve price 0 satisfies at least the necessary condition for optimization (just check the derivative of the payoff evaluated at $r = 0$).
- what about having the loser pay a portion of his own expenses to the winner as an additional penalty
- for simplicity assume the loser pays the winner whatever the loser actually spent litigating the case
- assume that the equilibrium bidding strategy is increasing, then the party with the highest valuation will win the case. Let s the

minimum expenditure required to litigate the case. A litigant who spends exactly s will win and get his value θ (without any transfer from the other player) if the other player decides not to contest

- on the other hand, if the other player decides to contest, the litigant who makes the minimum expenditure s will lose for sure and be forced to pay $2s$, so the expected payment is

$$F(r) s + (1 - F(r)) 2s = s + (1 - F(r)) s$$

which should equal $F(r) r$ in order that the marginal participant get exactly 0 surplus

- then by the revenue equivalence theorem, a legal system with minimum expenditure $s = \frac{F(r)r}{2-F(r)}$ in which the loser pays his own expenditure to the winner yields the same expected expenditures as a system where each litigant pays his own costs and where minimum expenditures are r

2 Multi-Unit Private Value Auctions

- maintaining the assumption that each bidder wants only a single unit and that each bidders' valuation is independently drawn from the distribution F
- suppose that the seller has $n > K > 1$ units to offer for sale.
- analogously to the case with a single unit, there are a number of different ways that the good could be allocated
 1. goods could be allocated to the K highest bidders at the highest rejected price
 2. again the K highest bidders at the lowest accepted price
 3. K highest bidders are allocated, each pays the price that he or she bids
 4. K objects could be auctioned one at a time to the K highest bidders

- *Revenue Equivalence Theorem for Multiple Units:* Suppose that the auction rules and equilibrium are such that for every vector $\theta \in \Theta^n$ of valuations, the buyers with the K highest valuations trade if and only if their valuations are at least r , while buyers whose valuations are exactly equal to r get zero expected payoff. Then the expected payment by a buyer of type θ is given by

$$\int_r^\theta s \binom{n-1}{K} F^{n-1-K}(s) (1-F(s))^{K-1} f(s) ds \quad (4)$$

- **Proof:** By assumption, only the highest K valuation buyers will trade in equilibrium, so if the K^{th} highest valuation among the other buyers exceeds θ the buyer will fail to trade. Conversely, if this K^{th} highest valuation is less than θ , then the buyer will be one of the winning bidders provided his own valuations exceeds the seller's reserve price r . The density for the K^{th} highest valuation is

$$\binom{n-1}{K} F^{n-1-K}(s) (1-F(s))^{K-1} f(s)$$

so the probability with which the buyer trades is

$$Q(\theta) = \int_0^\theta \binom{n-1}{K} F^{n-1-K}(s) (1-F(s))^{K-1} f(s) ds$$

when the buyers' valuation exceeds r and $Q(\theta) = 0$ otherwise.

- incentive compatibility gives

$$P_i(\theta) = \int_0^\theta Q(s)' ds$$

as above, so substituting for Q' gives the result.

2.1 Highest Rejected Bid Auction (uniform price auction)

- suppose the K highest bidders trade and pay the price bid by the $K+1$ highest bidder (that is the highest bid that fails to win) and that the seller sets reserve price r .

- then the price paid by a buyer who trades is independent of the price that he bids, and bidding true valuation is a weakly dominant strategy, and a Bayesian equilibrium. Since this bidding function is monotonically increasing, the buyers with the highest n valuations will trade as required by the revenue equivalence theorem
- since a buyer with valuation r will only trade when the K^{th} highest bid among the other buyers is below r a buyer with this valuation will pay r when he wins and nothing if he doesn't so his expected payoff is zero as required by the revenue equivalence theorem to expected payments are given by (4) and seller revenue is just n times the expectation of this payment over θ

2.2 Pay your own bid (Discriminatory Price Auction)

- suppose K highest bidders trade, and each pays his own bid, again with reserve price r
- assume for the moment that this indirect mechanism and the equi-

equilibrium associated with it satisfy the assumptions of the revenue equivalence theorem

- then the expected payment is equal to the bid multiplied by the probability of winning, or

$$b(\theta) \int_0^\theta \binom{n-1}{K} F^{n-1-K}(s) (1-F(s))^{K-1} f(s) ds =$$

$$\int_r^\theta s \binom{n-1}{K} F^{n-1-K}(s) (1-F(s))^{K-1} f(s) ds$$

by (4), and this can be solved for the equilibrium bidding rule