# Contents

1 Preference Theory ........................................... 5
  1.1 Introduction ........................................... 5
  1.1.1 Behavior ........................................... 6
  1.1.2 Indifference Curves .................................. 8
  1.1.3 Economic Modeling .................................. 11

2 Applications of Choice Theory: The Theory of Demand 13
  2.1 Introduction ........................................... 13
  2.2 Consumer Theory ........................................ 16
    2.2.1 The Budget Set .................................... 17
    2.2.2 Using The Utility Theorem .......................... 18
  2.3 A Simple Example ........................................ 20
  2.4 How to Test Demand Theory .............................. 23
  2.5 Comparative Statics and The Envelope Theorem ........... 25
    2.5.1 Implicit Differentiation ............................ 26
    2.5.2 Graphical Methods .................................. 27
    2.5.3 The Envelope Theorem ............................... 29

3 Dealing With Discontinuities 33
  3.1 Introduction ........................................... 33
  3.2 Non-linear Pricing ...................................... 36
  3.3 Sign-up Fees ............................................ 40

4 Best Reply Behavior ........................................ 45
  4.1 Introduction ........................................... 45
  4.2 Consumption Theory with Externalities .................. 46
    4.2.1 The graphical Approach .............................. 47
    4.2.2 An algebraic Treatment .............................. 49

5 Uncertainty ................................................. 51
  5.1 Lotteries ............................................... 51
    5.1.1 Monty Hall ......................................... 52
    5.1.2 St. Petersburg ...................................... 53
  5.2 Choosing among lotteries ................................ 54
6 Expected Utility and Risk Aversion 63
   6.1 Introduction .............................................. 63
      6.1.1 Lotteries with a Continuum of Outcomes .......... 63
      6.1.2 Risk Aversion ........................................ 66
      6.1.3 Measuring Aversion to Risk ......................... 68
      6.1.4 The Portfolio Problem ............................... 69

7 Insurance 75
   7.1 Introduction .............................................. 75

8 First Welfare Theorem In Production Economies 79
   8.1 Profit Maximization ............................... 79
   8.2 Competitive (Walrasian) Equilibrium .......... 82
   8.3 First Welfare Theorem .................................. 82
   8.4 Distribution ............................................ 84

9 Public Goods 89
   9.1 Introduction .............................................. 89
      9.1.1 The voluntary contribution game ................. 90
   9.2 Resolving the Public Good Problem ................. 93

10 What Does Undergrad Micro Theory say about Music Downloading 97
   10.1 What is Wrong with Leaving Things Alone .......... 100
   10.2 The Lindahl Solution ............................... 103
   10.3 Some potential problems ............................. 105
      10.3.1 A Second Problem .............................. 109
   10.4 The Monopoly Problem ............................... 110
   10.5 Are there Any Alternatives? ......................... 111

11 Problems 113
   11.1 Preferences ............................................. 113
   11.2 Demand Theory .......................................... 117
   11.3 Discontinuous Budget Sets ......................... 120
   11.4 Best Reply Behavior .................................... 122
   11.5 Expected Utility ...................................... 123
   11.6 Risk Aversion ........................................... 126
   11.7 Insurance Theory ...................................... 129
   11.8 First Welfare Theorem ................................ 131
   11.9 Public Goods ........................................... 133
   11.10 Music Downloading .................................... 138
   11.11 Constrained Optimization ........................... 139
Chapter 1

Preference Theory

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1.1 Introduction

The foundation of all choice theory in economics is something called a preference relation. The idea is that if I present you with a pair of alternatives, then you could tell me which one you prefer, or possibly that you are indifferent between them. The word 'prefer' has different meanings in different contexts. For example, if I ask you whether you would prefer to see a movie or go to a hockey game your preference is expressing something about which you would enjoy. If I ask you whether you would like to have the Olympics in your local city, your preference may express something about what you think is best for everyone, or possibly something about what you think you are supposed to say. Sometimes you really can't say that one alternative is better than another. For example you might be equally happy with a ham sandwich or a tuna sandwich. If I allow for that possibility, it is hard to imagine a situation where you wouldn't be able to say something.

If I am trying to think about your choice behavior and how I might understand it, I could begin by trying to imagine all the alternatives that you could possibly choose. I would collect them together in a big set $X$. Then I could go about choosing different pairs of alternatives in $X$ and asking you to express your opinion about which of the two you prefer. Eventually (as long as you didn’t get tired of answering questions) I could learn which alternative you preferred among any pair of alternatives in $X$. This collection of information is your preference relation over $X$.

The set $X$ could be very general. For example, you might have guessed that we are going to be talking about preference relations over collections of
possible consumption bundles. There is no need to stop there. Much of modern microeconomic theory arises from thinking about preferences over things like political parties, environmental policies, business strategies, location decisions, and so on.

There are many kinds of preference relations you will encounter if you continue studying economics, but the most widely applied reasoning in economics assumes that preference relations have two properties - first, they must be complete in the sense that for any pair of alternatives in $X$, either you prefer one or the other, or are indifferent. There are some interesting preference relations that are incomplete, but let’s leave that for the moment and concentrate on another problem. Your preference relation could be ‘odd’. For example, suppose you like the Liberals more than the Conservatives because they are more socially progressive. You might like the NDP more than the Liberals for the same reason. However, you may prefer the Conservatives to the NDP because they are more fiscally responsible. Ignoring any other parties, then you have just expressed a complete and reasonable preference relation over the political parties. It does present something of a problem when you are trying to vote. You can’t vote Conservative because you prefer the Liberals to the Conservatives, you can’t vote Liberal because you prefer the NDP to the Liberals. Unfortunately you can’t vote NDP either because you prefer the Conservatives to the NDP.$^1$

We have a word for this kind of preference relation in economics, it is called an intransitive preference relation. To put this another way, a transitive preference relation is one such that for any 3 alternatives $x, y,$ and $z$ in $X$, if $x$ is preferred to $y$ and $y$ is preferred to $z$, then it must be the case that $x$ is preferred to $z$. A complete transitive preference relation is called a rational preference relation.

In fact, I have just described to you what rationality means in economics. A person is said to be rational in a particular economic environment if they have a complete and transitive preference relation over the alternatives that they face in that environment. In particular, it doesn’t mean that people are greedy or self interested. It doesn’t mean that they are super sophisticated calculators. It just means that they can express opinions about pairs of alternatives.

### 1.1.1 Behavior

So how do economists go about predicting what people will do? All they say is that whatever alternative $x$ is actually chosen from $X$, then there cannot be another alternative in $X$ that is preferred to $x$. It is true that in experiments, people sometimes exhibit intransitive preferences (though they quickly change their behavior when this is pointed out to them). There are also situations in which it seems impossible for people to make a choice. For the most part though, assuming that people are rational (have a complete transitive preference relation) is pretty innocuous.

It might also occur to you that if you accept that people are rational decision

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$^1$I suppose this could explain why so many people don’t vote.
makers, then you can’t really get yourself in too much trouble. I never said what these preference relations had to look like. To assert that an individual chooses the alternative that he or she most prefers is almost tautological. The real content of economic theory involves restrictions it imposes on $X$ and on the preference relation over $X$. Its failures and successes having nothing to do with the assumption of rationality.

Introductory economics courses focus on consumption and consumption bundles. A consumption bundle is a pair $(x, y)$ where the first component of this vector is some quantity that you consume of one good (just call it good $x$ for short), and the second component is the quantity you consume of the second good. Consumption doesn’t generate happiness or utility or utils or anything like that. If we follow your first year course, and imagine that good $x$ has a price $p$ and good $y$ has a price $q$, and that you have $W$ to spend, then the consumer faces a set of alternatives $X$ which consists of all pairs $(x, y)$ whose cost is less than or equal to $W$, i.e.,

$$X \equiv \{(x, y) \in \mathbb{R}^2_+ : px + qy \leq W\}$$

Here $\mathbb{R}^2_+$ is the set of all vectors with two non-negative components. Read the colon to mean "such that".

Well, since we have a set of alternatives, it is pretty safe to assume that for any pair of alternatives (a pair of vectors $(x, y)$ and $(x', y')$ here), the consumer can express a preference between them. Suppose for the moment that we could get the consumer to tell us what his or her preference relation is. But now we face a small problem. Suppose the consumer tells us that he prefers $(x, y)$ to $(x', y')$. Suppose that we now look at another budget set $X'$ where prices are $p'$ and $q'$, and maybe income is $W'$. Let’s pick this new set so that it contains both $(x, y)$ and $(x', y')$. Do we really need to ask the consumer if he prefers $(x, y)$ to $(x', y')$ in this new set? Of course his preference could well change. People have no use for telephones unless other people have telephones. The price changes might signal changes in quality of the goods that he is buying (suppose $x$ and $y$ are stocks or bonds or something like that).

Now we begin to impose some restrictions of preferences and economic theory begins to have some content (of course, we also study what happens when preference relations change with prices and income). We are going to assume that if $(x, y)$ and $(x', y')$ are in both $X$ and $X'$ and if $(x, y)$ is viewed by the consumer to be at least as good as $(x', y')$ in the preference relation relative to $X$, then it must also be at least as good as $(x', y')$ in the preference relation relative to $X'$.

The important point is that the assumption that our consumer was rational imposed no restriction whatsoever on his behavior. The added assumption about how his or her preferences are related across different budget sets does restrict what we should expect to see him do. For example, suppose that we could run a long series of experiments in which our consumer is repeatedly asked to choose

\footnote{This assumption is called the \textit{weak axiom of revealed preference}.}
something from $X$ and that he consistently chooses $(x, y)$. If our assumption is true, then it would be highly unlikely that if we had him choose repeatedly from $X'$ that he would consistently pick $(x', y')$. The predictive content of the theory comes from the assumption that his preference relation is independent of the prices and income that he faces, not from the assumption that he is rational.

You will see this repeatedly in economics - we will impose restrictions on $X$ and the preference relation over it, then make predictions (and test them). If you want to argue about economics the idea is to understand these restrictions and criticize them. It is a waste of time to argue about whether or not consumers are rational.

### 1.1.2 Indifference Curves

So let’s continue with first year economics. Since preference relations (let’s just say preferences from now on) are assumed to be independent of prices and income, we could sensibly take the consumer’s preference relation and collect together all the consumption bundles $(x', y')$ which are indifferent to some bundle $(x, y)$. As you remember from your first year course, this collection of consumption bundles is called an indifference curve. Please note that the indifference curve comes directly from the preference relation and has nothing to do with utils or satisfaction of anything like that. Since we can construct an indifference curve for any consumption bundle, there is really a family of indifference curves.

Pick two indifference curves in this family, say $C_1$ and $C_2$ and choose a bundle $(x, y)$ from $C_1$ (which is itself a set) and $(x', y')$ from $C_2$. If $(x, y)$ is preferred to $(x', y')$ then we say that the indifference curve $C_1$ is higher than $C_2$. Then of course, any bundle in $C_1$ will be preferred to any bundle in $C_2$.

There isn’t much that can be said about indifference curves at this point except that when a consumer is rational, two distinct indifference curves can’t have any point in common. To see this suppose that $C_1$ is higher than $C_2$. Let $(x'', y'')$ be the point that the curves have in common, with $(x, y)$ in $C_1$ and $(x', y')$ in $C_2$. Then $(x', y')$ is at least as good as $(x'', y'')$ since both are in $C_2$. $(x'', y'')$ is at least as good as $(x, y)$ since both are in $C_1$. Now transitivity requires that $(x', y')$ be at least as good as $(x, y)$ which is false if the consumer is rational.

At this point, we could try to describe graphic properties of the indifference curves. If we started to do that, we would end up spending considerable time trying to absorb graphic formalism and end up saying what we could have said with words. So it is time for me to introduce the theorem that makes economics

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4 He might do this once if he were indifferent, but would probably not do it consistently if he were indifferent.

4 To be formal, we could say that $(x, y)$ is indifferent to $(x', y')$ if $(x, y)$ is at least as good as $(x', y')$ and at the same time $(x', y')$ is at least as good as $(x, y)$.

5 A small digression - this simple argument is an example of a line of reasoning that you will see often in economics. If you want to show that some property $A$ implies that another property $B$ must be true, try to show that if $B$ isn’t true, then $A$ can’t be true either. This is called a proof by contradiction. Here we wanted to show that if a preference relation is transitive (A) then a pair of indifference curves couldn’t cross (B). We showed that if the curves did cross, the preference relation couldn’t be transitive.
Write the preference relation as $\succeq$, meaning that $(x, y) \succeq (x', y')$ whenever $(x, y)$ is preferred to $(x', y')$. A utility function is a relation that converts each bundle $(x, y)$ into a corresponding utility value or number. The utility function $u$ represents the preference relation $\succeq$ as long as $u(x, y) \geq u(x', y')$ if and only if $(x, y) \succeq (x', y')$. If we happened to be able to find a utility function to represent a preference relation then we would have a big leg up. To predict what a consumer will do so far, we need to scan all pairs of consumption bundles until we find a bundle such that no other bundle is preferred to it. This makes for a lot of tedious pairwise comparisons. There isn’t any obvious reason why this sort of reasoning is going to help us understand behavior. If preferences are represented by a utility function, we could take the function and find the bundle that produced the highest utility number in the set of alternatives. That would be relatively easy because we could use all the standard mathematical tricks we know about maximizing functions (like setting derivatives to zero and so on).

Yet the utility function yields something far more important. As I mentioned above, the content of economic theory doesn’t come from the rationality assumption. It comes from imposing restrictions on the preference relation and the feasible set. It is difficult to formulate ideas about preference relations since they are relative complex objects. On the other hand, it is much easier to impose and understand restrictions on utility functions.

Assuming that people have utility functions which they maximize is just about the last thing we want to do. If we did that, then all the people who accuse economists of being irrelevant because they assume that consumers are ‘rational’ would have a good point. We would be guilty of predicting behavior by assuming that people do something that they obviously don’t.

So why use a utility function? We need to add one important restriction on preference relations, and one simplifying restriction. The simplifying restriction is that our consumer likes more of both goods - i.e., if $(x, y)$ and $(x', y')$ are such that $x \geq x'$ and $y \geq y'$ and at least one of these inequalities is strict, then $(x, y) \succeq (x', y')$ but not the other way around. Having more of any good makes the consumer strictly better off. For short, let’s say that such a preference relation is monotonic.

Now for the important restriction. The set of bundles that are at least as good as $(x, y)$ is given by $B = \{(x', y') \in \mathbb{R}_+^2 : (x', y') \succeq (x, y)\}$. The set of consumption bundles that are no better than $(x, y)$ is given by $W = \{(x', y') \in \mathbb{R}_+^2 : (x, y) \succeq (x', y')\}$. The important assumption is that both $B$ and $W$ are closed sets. If the sets $B$ and $W$ are both closed for any $(x, y) \in \mathbb{R}_+^2$, then the preference relation is said to be continuous.

Now the following important theorem is true:

**Theorem 1.1.1** Let $\succeq$ be a continuous and monotonic rational preference re-
lation. Then there exists a utility function $u$ which represents the preference relation $\succeq$.

**Proof** We are going to prove this constructively by actually making up the function. We are going to construct a function that converts every point in $\mathbb{R}^2$ into a point in $\mathbb{R}$.

First some preliminaries. Let $Z$ represent the $45^0$ line (i.e., the set of all points in $\mathbb{R}_+^2$ which have the same horizontal and vertical coordinate). Let $(x, y)$ be any consumption bundle. Let $\varepsilon > 0$ be a small positive number. The bundle $(\max [x, y] + \varepsilon, \max [x, y] + \varepsilon)$ is in $Z$ and is strictly preferred to $(x, y)$ by the fact that preferences are monotonic. Similarly $(x, y)$ is preferred to $(\min [x, y] - \varepsilon, \min [x, y] - \varepsilon)$ by monotonicity. Therefore the sets $B \cap Z$ and $W \cap Z$ are non-empty. As preferences are continuous, these sets are both closed. This lets us deduce that the sets $P^+ = B \cap Z$ and $P^- = W \cap Z$ are both closed as the intersection of closed sets. In Figure 1.1.1 the set $P^+$ is marked in red. It is the intersection of the $45^0$ line and the set $B$ consisting of all bundles that are preferred to $(x, y)$. The set $P^-$ is marked in blue in the figure.

Now the sets $P^+$ and $P^-$ are made up of bundles (in $\mathbb{R}_+^2$) that have the same horizontal and vertical component. So, we can associate each bundle in $Z$ with this common component, which is just a positive real number. Since each bundle $z \in Z$ either has $z \succeq (x, y)$ or $(x, y) \succeq z$ by the completeness of preferences, (recall that completeness is part of rationality) each point in $Z$ is either in $P^+$ or $P^-$. Each point in $P^+$ or $P^-$ is also in $Z$ by construction, so $Z = P^+ \cup P^-$. By happy coincidence $P^+$ and $P^-$ share exactly one point in common. Part of the argument for this is an arcane point in set theory. Since $P^+ \cup P^-$ is all of $Z$, if they don’t share a common point, then $P^-$ must be the complement of $P^+$ in $Z$. Since the complement of a closed set is open, $P^-$ would have to be open which it cannot be. So there must be at least one common point. Could there be two? Again, suppose there were, say $z$ and $z'$. They are both in $Z$ so they are both on the $45^0$ line. If they are distinct then, say, $z >> z'$ (meaning each component of $z$ is strictly larger than the corresponding component of $z'$). Then by monotonicity $z \succeq z'$ but not the other way around. Then by transitivity $(x, y) \succeq z'$ but not the other way around. But this can’t be since $z' \in P^+$.

All this work leads to the conclusion that for every bundle $(x, y)$ we can find a point on the $45^0$ line which is indifferent to it. Let’s call the common coordinate of this point the utility $u(x, y)$ associated with the bundle $(x, y)$ (this emphasizes the point that utility is measured as some number of goods, not as utils or satisfaction).

Finally, all we need to do is check that this utility function $u(x, y)$ actually represents preferences. This is pretty straightforward. For example if $u(x, y) \geq u(x', y')$ then the $z$ associated with $(x, y)$ has a bigger common component than the $z'$ associated with $(x', y')$. Then $(x, y) \succeq z$ (since $z \in P^-$ for $(x, y)) \succeq z'$

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8 If you believe it is a coincidence, I have a bridge to sell you.
Figure 1.1.1: The sets $P^+$ and $P^-$

(by monotonicity) $\succeq (x', y')$ (since $z' \in P^+$ for $(x', y')$). The other direction is just as easy. □

So let’s collect our thoughts for a moment. When a consumer chooses a bundle from some budget set, she picks something such that if we offer her some other bundle from the same budget set, she will not want it. If her preferences are transitive and complete (and continuous) it will appear to be the case that she is choosing a bundle to maximize a utility function subject to the budget constraint. In the consumer’s own mind, there is no such thing as utility: rational utility maximization is an implication of simpler properties of consumer behavior. Nor is it assumed that there is any numerical way to measure happiness or satisfaction. These simply aren’t parts of modern microeconomic theory.

1.1.3 Economic Modeling

Why was this theorem so important? Well it shows first that economic methodology itself doesn’t rely on grand assumptions about human behavior. Of course, when we impose restrictions on the preference relation or the set of feasible alternatives, we are making assumptions. These assumptions are part of what we call economic models. When we formulate an economic model, we try to extract all the implications of the restrictions. These restrictions are predictions the model makes. We can collect data about the choices consumers actually do make, to check whether these predictions are right. When they are wrong, we know we need to reformulate the model (or change some of the restrictions).

The second thing is shows is that we can extract these restrictions using some fairly basic mathematical tools, like the theory of optimization (and of course, the dreaded calculus). The mathematization of economics occurred in the late 50’s and has had a remarkable impact on the way economists interact. To use mathematics, it is necessary that the concepts, sets, and functions involved be very precisely defined. There is no room for interpretation (though certainly there is room to fine tune and modify concepts). An economic concept must mean the same thing to everyone.

This has had an impact that you might not expect. Anyone who under-
stands basic mathematics should be able to understand the most advanced ideas in economic theory. Oddly enough mathematics makes economics very inclusive. This has had great benefits for economists, since other fields have been moving in much the same direction. Computer science, biology, ecology, environmental science, all use methods similar to those used by economists. The level of interaction among practitioners in these different fields is increasing to the enrichment of all.

Most of this course tries to develop the mathematical and conceptual tools you need to formulate and analyze economic models on your own. As we go about this, you will see some models that have worked out pretty well in the sense that they give very good insight into some pretty applied problems. You will also get a chance to see some models that don’t work so well. These ‘failures’ give a good deal of insight into how theoretical and empirical work interact. Though these applications are important in the overall scheme of things, they are not the main focus of the course. It is the art of building the models themselves that is the concern here. Once you begin to appreciate this approach, your subsequent studies in more applied areas will make more sense.

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9 You might like to compare the definition of utility I have given above with definitions you will hear for important concepts like capitalism or post modernism.
Chapter 2

Applications of Choice Theory: The Theory of Demand

2.1 Introduction

Preference Theory tells us that individuals who can express opinions about the various alternatives available to them will act as if they are maximizing a utility function. It is important to remember that this isn’t meant to be a description of what people actually do when they make decisions. Obviously people don’t consciously maximize anything when they make choices. Consumers who never got the hang of finding $x$ in high school, nonetheless seem perfectly capable of deciding how to spend their money. The basic presumption in economics is that there is no way to know what people are actually thinking when they make decisions.¹

The fact that individuals’ choices will look just like solutions to maximization problems allows us to use methods and concepts from mathematics to help describe behavior. This mathematical representation of behavior ultimately leads to the greatest contribution of economics, the concept of equilibrium behavior. We will begin the description of equilibrium behavior later on in this course.

¹Not everyone agrees with this. For example, polling at elections is done under the assumption that respondents will truthfully reveal who they want to vote for - they are often pretty close. Psychologists often run experiments in which they simply ask people what they would do if ... . Neuroeconomists believe that new brain scanning technology will make it possible to observe preferences directly.
To make use of the method that the utility theorem provides, we have to add something to what we have so far. Suppose we are trying to figure out how people will react to a price change. At the initial price, we can use the theorem that we proved in the chapter on preferences to show that there is a utility function and that the choices our consumer makes maximize this utility function subject to whatever constraints she faces. The construction of this utility function depends on the alternatives over which the consumer has to decide. There is nothing in the theorem that says that the consumer's preferences won't change when the choice set does. Our consumer might believe that a higher price means that the good she is buying has a higher quality than she initially thought. After the price change, she might 'want' the good more than before. Perhaps more importantly, the price change might affect what other people do. Some goods are more desirable when other people like and use them, for example.

To make use of the maximization approach, we need to make assumptions about utility and how it changes when we change the environment. These assumptions are called an economic model. We use our economic model to make a prediction. We will start with one of the oldest and perhaps simplest economic models in the next section, and I will explain these extra assumptions and how to use the maximization approach to understand it.

You might wonder about this. Does this mean that economic predictions are just elaborate assumptions about the way people behave? Why should I believe these assumptions? If you are thinking this way, you are on the right track. Economists spend an enormous amount of time and effort collecting and analyzing data - often with the purpose of testing some economic model. You'll be learning how to use models in this course, so we won't say much more about testing, but we might find that our prediction is inconsistent with what appears to be going on in the data we have collected. This may require that we go back and revise the assumptions of our model to try to get things to work out. So, the assumptions evolve with our knowledge of how people behave.

Perhaps this leads you to a second, closely related question. If models are just elaborate guesses about preferences designed to generate predictions, why not just start off with the predictions? For example, suppose we are interested in the impact of an increase in price. It seems perfectly reasonable to guess that if the price of a good rises, then people will buy less. Then why bother to write

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2My favourite example of this is Adobe Acrobat Software used for making pdf files (like the file you are currently reading). Many free programs will produce pdf files. However, Adobe had the idea to offer an expensive software package to do the same thing, so that people would incorrectly believe that it was higher quality software. This strategy worked brilliantly, at least among my colleagues who have jointly shelled out thousands of dollars from their research grants to Adobe for free software - thousands of dollars they could have paid to graduate students.

3Telephones are an obvious example. The fashion industry seems to work on this principle, as well. Companies advertise their brand heavily (for example, product placement in a popular movie like I, Robot, or The Italian Job), then raise the product price to make it exclusive. Suddenly, everyone wants the product and is willing to pay a lot for it.

4This is called the Law of Demand. In October 1981, American Senator William Proxmire gave his Golden Fleece Award to the National Science Foundation for funding an empirical
down a maximization model, find Lagrange multipliers, take derivatives, and do all that other tedious stuff? After all, we can always test our guess, and refine it if we are wrong.

There are basically two answers. Part of the answer is that mathematics is universal: everyone, no matter what their field of study, knows math. Formal mathematical models can, in principal, be understood by everyone, not just specialists in economics. Apart from the obvious connection with math and statistics, the modeling approach in economics is similar to that used in some branches of computer science, theoretical biology, and zoology. In an odd way, formal modeling makes economic theory more inclusive.

The real benefit of formal modeling (to all these fields) is that it helps make up for the deficiencies in our own intuition. Our intuition is rarely wrong, but it is almost always incomplete. It is also lazy. It wants to push every new and challenging fact into an existing ‘intuitive’ box, which makes us very conservative intellectually. Careful mathematical analysis of well-defined models makes up for this. It helps us to see parts of the story that we might otherwise have missed. Often, those insights-gained through painstaking mathematical analysis-lead to the most fundamental changes in thinking. So, don’t despair if you spend hours thinking through the logic of one of the problems in the problem set without actually getting the answer. You are often laying the groundwork for important leaps in your understanding that will often transcend the particular problem you are working on.

At a more practical level, mathematical analysis of a model will often reveal implications that your intuition would never have imagined. These implications can often be critical. For example, it isn’t hard to show that the law of demand mentioned above need not be true. Nothing in the nature of preferences or the characteristics of markets requires it to be true. If our model doesn’t tell us anything about demand curves, what use is it? Rational behavior does impose restrictions on demand that are amenable to econometric test. I will show you enough of the argument below for you to see that the real implications of rational behavior in a market like environment can not be understood using intuition, you need formal analysis.

Bear in mind as we go along, that the content of economics is not the particular models we study, but the method of using models like this to generate predictions, then modifying these until the predictions match the information we have in our data.

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test of the Law of Demand. Pigeons in a laboratory would receive food by pecking a lever. Once the scientists had trained the pigeons to peck on the lever to get food (the first ten years of the project), they changed the rules so that the pigeons had to peck twice on the lever to get food, instead of only once. The idea was that if the law of demand holds, then pigeons should eat less when they have to peck twice than they would if they only had to peck once.
2.2 Consumer Theory

A consumer is an individual who wants to buy some stuff. The "stuff" will be a list of quantities of the goods that she wants. We express this list as a vector, that is, an ordered list of real numbers \(x_1, x_2, \ldots, x_n\) where \(x_1\) is the total units of good 1 she wants, and so on. We refer to a generic bundle of goods as \(x \in \mathbb{R}^n\), where this latter notation means that \(x\) is an ordered list consisting of exactly \(n\) real numbers.

For the moment, let \(\mathcal{B}\) be the set of bundles that our consumer can afford to buy. If we propose different alternatives in \(\mathcal{B}\) to our consumer, she will be able to tell us which one she prefers. If these preferences are transitive, along with an appropriate continuity assumption (see the previous chapter), then there will be a utility function \(u\) which converts bundles in \(\mathbb{R}^n\) into real numbers, and our consumer will look just like she is maximizing \(u\) when she chooses a bundle from \(x\).

Now, let \(x\) and \(y\) be a pair of alternatives in \(\mathcal{B}\). For the sake of argument, suppose that \(x \succeq y\) (which means that the consumer prefers \(x\) to \(y\)). Classical consumer theory makes two very strong assumptions. First, the preferences of our consumer are independent of the preferences and choices of all other consumers. Second, the preferences are independent of the budget set that the consumer faces. The first assumption just means that we can think about one consumer in isolation. No one really believes this is a good assumption, and we will begin to relax it later on. It does make it much easier to explain the approach.

The second assumption can be stated more formally given the notation we have developed. If the consumer prefers \(x\) to \(y\) when these are offered as elements of \(\mathcal{B}\), then the consumer will still prefer \(x\) to \(y\) if these are offered as choices from any other budget set \(\mathcal{B}'\).

What does this mean in words? Well as a good Canadian, you no doubt drink foreign beer like Molson (Coors, USA), or Labatt (Interbrew, Belgium). Suppose you would prefer a Molson to a Labatt if you are given a choice. If you suddenly won a lottery that gave you $1 million for life, would you still prefer Molson to Labatt? Probably. You might not want a Molson or Labatt because you could then afford to buy champagne or something—but, if you are given a choice between those two only, you would probably still choose Molson.

Whatever you think of these two assumptions, let us accept them for the moment and try to show how to draw out their implications.

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5The presidents of Molson, Labatt and Big Rock Brewery (Calgary) once went for a beer after attending a conference together. The waiter asked the president of Molson what he wanted to drink. He said proudly, “I’ll have a Canadian.” “Fine,” said the waiter. Then, he asked the president of Labatt, who said he would like a Labatt Blue. “Fine,” said the waiter, “good choice.” Then, he asked the president of Big Rock. “I’ll have a Coke,” she said. “Pardon?” asked the waiter. “They aren’t drinking beer so I don’t think I will either,” she replied.
2.2.1 The Budget Set

The budget set refers to the set of consumption bundles that the consumer can afford. We can provide a mathematical characterization of this set fairly easily. Let’s assume that the consumer knows the prices of each of the goods, and that these prices can be represented as a vector \( p \in \mathbb{R}^n \), where \( p \) is an ordered list \( \{p_1, \ldots, p_n\} \). Let’s assume further that the consumer has a fixed amount of money \( W \) to spend on stuff. The set of consumption bundles that the consumer can afford to buy is the set

\[
\{ x : x_i \geq 0 \forall i ; \sum_{i=1}^{n} p_i x_i \leq W \} \tag{2.2.1}
\]

The brackets around the expression are used to describe the set. The notation inside the bracket means the set of \( x \) such that (:) each component of \( x \) is at least as big as zero, and such that (;) if you sum up the product of the price and quantity across all components you end up with something less than, or equal to, the amount of money you have to spend in the first place. Hopefully, you find the mathematical expression a lot more compact. However, the real benefit of using the math is yet to come.

It helps to mix formal arguments together with pictures like the ones you saw in your first-year course. To do this, imagine that there are only two goods. Call them good \( x \) and good \( y \). The price of good \( x \) will be \( p_x \) and the price of \( y \) will be \( p_y \). The amount of money you spend buying good \( x \) is \( p_x x \). The amount you spend on \( y \) is \( p_y y \). Total spending is \( p_x x + p_y y \), which can be no larger than the money you have, \( W \). That is exactly what the math says in equation (2.2.1).

To help you think about this, let’s draw the following picture.

In the picture above, our consumer has $W to spend on two different goods.
called $x$ and $y$. If she spends her entire income on good $x$, she can actually purchase $\frac{W}{p_x}$ units in all. This point is labeled on the horizontal axis, and represents one feasible consumption bundle; i.e., $\frac{W}{p_x}$ units of good $x$, and no units of good $y$. By the same token, she could spend all her money on good $y$, and purchase $\frac{W}{p_y}$ units of good $y$, and no $x$. This point is labeled on the vertical axis as another feasible consumption bundle.

Any combination of these two would also work. For example, spending half her income on each good would yield the consumption bundle $(\frac{1}{2}W, \frac{1}{2}W)$. This bundle lies halfway along the line segment that joins the points $(\frac{W}{p_x}, 0)$ on the horizontal axis, and $(0, \frac{W}{p_y})$ on the vertical axis.

She doesn’t really have to spend all her money either. Since she doesn’t have any good $x$ or $y$ to sell, the set of feasible consumption bundles consists of all the points in the triangle formed by the axis and the line segment joining the point $(\frac{W}{p_x}, 0)$ to the point $(0, \frac{W}{p_y})$.

The budget line is the upper right face of the triangle. The slope of this line (rise over run) is $-\frac{p_x}{p_y}$. The relative price of good $x$ is the ratio of the price of good $x$ to the price of good $y$ ($-1$ times the slope of the budget line).

2.2.2 Using The Utility Theorem

Implicitly, when we say that the bundle $(x, y)$ is at least as good as $(x', y')$, we interpret this to mean that, given the choice between the bundles $(x, y)$ and $(x', y')$, our consumer would choose $(x, y)$. If that is true, then once we describe the budget set, we must expect the consumer to choose a point in the budget set that is at least as good as every other point in the budget set. Our ‘utility function’ theorem says that-as preferences are complete, transitive and, continuous-there will be a function $u$ such that a bundle $(x, y)$ will be at least as good as every other bundle in the budget set if and only if $u(x, y)$ is at least as large as $u(x', y')$ for every other bundle $(x', y')$ in the budget set. If we knew the function $u$, then we could find the bundle by solving the problem

$$\max u(x, y)$$

subject to the constraints

$$p_x x + p_y y \leq W$$

$$x \geq 0$$

$$y \geq 0$$

Now, before we try to use the mathematical formulation, let’s go back for a moment to the characterization you learned in first-year economics.

As we have assumed that our consumer’s preferences are independent from the budget set he faces, we can construct a useful conceptual device. Take any bundle $(x, y)$. Form the set

$$\{(x', y') : (x, y) \succeq (x', y') \text{ and } (x', y') \succeq (x, y) \}$$
In words, this is the set of all bundles \((x', y')\) such that the consumer is indifferent between \((x', y')\) and \((x, y)\). This set is referred to as an indifference curve. If the bundle \((x, y)\) is preferred to the bundle \((x', y')\), then every bundle in the indifference curve associated with \((x, y)\) will be preferred to every bundle in the indifference curve associated with \((x', y')\). This follows by the transitivity of preferences (remember that preferences are transitive if \(x \succeq y\) and \(y \succeq z\) implies that \(x \succeq z\)). So, the consumer's choice problem outlined above is equivalent to choosing the highest indifference curve that touches his or her budget set. This gives the tangency condition that you are familiar with, as in Figure 2.2.2.

The two bundles \((x^*, y^*)\) and \((x^* + dx, y^* - dy)\) both lie on the same indifference curve. The vertical distance \(dy\) is the amount of good \(y\) that this consumer is willing to give up in order to get \(dx\) additional units of good \(x\). When \(dx\) is very small, the ratio of \(dy\) to \(dx\) is referred to as the marginal rate of substitution of \(y\) for \(x\). Using your elementary calculus, notice that this marginal rate of substitution is the same thing as the slope of the consumer’s indifference curve.

Now, we can bring our utility theorem to bear. Assuming that the consumer’s preferences are complete, transitive, and continuous, they must be represented by some utility function: let’s call it \(u(x, y)\). Then, the indifference curve must be the set of solutions to the equation

\[
u(x', y') = u(x^*, y^*)
\]

We could then calculate the slope of the indifference curve (that is, the marginal rate of substitution) from the total differential

\[
u_x(x, y)\,dx + \nu_y(x, y)\,dy = 0
\]

or

\[
\frac{dy}{dx} = -\frac{\nu_x(x, y)}{\nu_y(x, y)}
\]
where $u_x(x, y)$ means the partial derivative of our utility function $u$ with respect to $x$ evaluated at the point $(x, y)$.

Since the highest indifference curve touching the budget set is the one that is just tangent to it, the marginal rate of substitution of $y$ for $x$ must be equal to the slope of the budget line, $-\frac{p_x}{p_y}$.

Now, let’s take the utility function that we know exists, go back to the purely mathematical formulation and maximize (2.2.2) subject to the constraints (2.2.3) through (2.2.5). By the Lagrangian theorem, there are three multipliers (one for each of the three constraints) $\lambda_1, \lambda_2,$ and $\lambda_3$ such that the Lagrangian function can be written as

$$u(x, y) + \lambda_1(p_x x + p_y y - W) - \lambda_2 x - \lambda_3 y$$

At the optimal solution to the problem, the following first order conditions must hold

1. $u_x(x, y) + \lambda_1 p_x - \lambda_2 = 0$ (2.2.6)
2. $u_y(x, y) + \lambda_1 p_y - \lambda_3 = 0$ (2.2.7)
3. $p_x x + p_y y - W \leq 0; \lambda_1 \leq 0$ (2.2.8)
4. $-x \leq 0; \lambda_2 \leq 0$ (2.2.9)
5. $-y \leq 0; \lambda_3 \leq 0$ (2.2.10)

where the last three conditions holding with complementary slackness.

Suppose that we knew for some reason that the solution must involve positive amounts of both $x$ and $y$ (you will see an example like this below). Then by complementary slackness, the multipliers associated with both of these variables would have to be zero. Then (2.2.6) and (2.2.7) would simplify to

$$u_x(x, y) = -\lambda_1 p_x$$

and

$$u_y(x, y) = -\lambda_1 p_y$$

Dividing the first condition by the second gives exactly the same result that we deduced from the picture

$$\frac{u_x(x, y)}{u_y(x, y)} = \frac{p_x}{p_y}$$

### 2.3 A Simple Example

If we know more about the utility function, then the mathematical approach can be quite helpful. For example, in the section on Lagrangian theory it was assumed that the utility function had the form

$$u(x, y) = x^\alpha y^{(1-\alpha)}$$

(2.3.1)
Then the first order conditions become

\[ \alpha x^{(\alpha-1)y^{(1-\alpha)}} + \lambda_1 p_x - \lambda_2 = 0 \]  \hspace{1cm} (2.3.2)
\[ (1 - \alpha)x^\alpha y^{-\alpha} + \lambda_1 p_y - \lambda_3 = 0 \]  \hspace{1cm} (2.3.3)
\[ p_x x + p_y y - W \leq 0; \lambda_1 \leq 0 \]  \hspace{1cm} (2.3.4)
\[ -x \leq 0; \lambda_2 \leq 0 \]  \hspace{1cm} (2.3.5)
\[ -y \leq 0; \lambda_3 \leq 0 \]  \hspace{1cm} (2.3.6)

where (2.3.4), (2.3.5), and (2.3.6) hold with complementary slackness. At first glance, this mess doesn’t look particularly useful. However, notice that if either \( x \) or \( y \) are zero, then utility is zero on the right hand side of (2.3.1). If the consumer has any income at all, then she can do strictly better than this by purchasing any bundle where both \( x \) and \( y \) are positive. As a consequence, we can be sure that, in any solution to the consumer’s maximization problem, both \( x \) and \( y \) are positive. Then, by the complementary slackness conditions (2.3.5) and (2.3.6), \( \lambda_2 \) and \( \lambda_3 \) must both be zero.

In addition, the solution will also require that the consumer use up her whole budget since the right hand side of (2.3.1) is strictly increasing in both its arguments. Complementary slackness in (2.3.4) unfortunately doesn’t tell us that \( \lambda_1 \) is positive, it is possible, but unlikely that both the constraint and its multiplier could be zero.

Let’s continue. The logic of the Lagrange theorem is that the first order conditions have to hold at a solution to the problem. Remember that the converse is not true: a solution to the first order conditions may not give a solution to the maximization problem. Now, as long as both prices are strictly positive and both \( x \) and \( y \) must also be so, a solution to the maximization problem (if it exists) must satisfy

\[ \alpha x^{(\alpha-1)y^{(1-\alpha)}} = -\lambda_1 p_x \]  \hspace{1cm} (2.3.7)

and

\[ (1 - \alpha)x^\alpha y^{-\alpha} = -\lambda_1 p_y \]  \hspace{1cm} (2.3.8)

Now, divide (2.3.7) by (2.3.8) (which means divide the left hand side of (2.3.7) by the left hand side of (2.3.8) and the same for the right hand sides). You will get

\[ \frac{\alpha y}{1 - \alpha x} = \frac{p_x}{p_y} \]  \hspace{1cm} (2.3.9)

or \( p_x x = p_y y^{1-\alpha} \). Again, this last equation has to be true at any solution to the maximization problem. Since it also has to be true that \( p_x x + p_y y = W \), then \( p_y y^{1-\alpha} + p_y y = W \). This means that is has to be true that

\[ y = W(1 - \alpha)/p_y \]  \hspace{1cm} (2.3.10)

Similarly, \( p_x = W\alpha/p_x \). These two equations are great because they tell us the solution to the maximization problem for all different values of \( p_x \), \( p_y \), and \( W \). These last two equations are ‘demand curves,’ just like the ones you saw in your
first-year economics course. You can easily see that the ‘law of demand’ holds for this utility function: an increase in price lowers demand.

This simple example takes us a long way along the road to understanding what it is that economists do differently from many other social scientists. We started with some very plausible assertions about behavior; in particular, given any pair of choices, consumers could always make one, and these choices would be transitive. This showed us that we could ‘represent’ these preferences with a utility function. Using this utility function, we can conclude that the consumer’s choice from any set of alternatives will be the solution to a maximization problem.

By itself, this seems to say very little - if you give a consumer a set of choices, she will make one. However, we now have the wherewithal to formulate models - additional assumptions that we can add to hone our predictions. We added two of them. The first is basic to all the old-fashioned consumer theory - the way the consumer ranks any two bundles does not depend on the particular budget set in which the alternatives are offered. The second assumption was that the utility function has a particular form as given by (2.3.1).

Putting these together we were able to apply some simple mathematics to predict what the consumer would do in all the different budget sets that we could imagine the consumer facing. This is the demand function (2.3.10) that we derived above. As promised above, the mathematics has delivered all the implications of our model. The demand function shows that there are a lot of implications, so it shouldn’t be too hard for us to check whether the model is right.6

The utility function theorem allows us to unify our approach (though not our model) to virtually all behavioral problems. We don’t even need to confine ourselves to human behavior. For instance, animals make both behavioral and genetic choices. Transitivity is arguably plausible and we can assume that they are always able to make some choice (completeness). So, we could also represent their choices as solutions to utility maximization problems. Genetics involves choices made by biological systems in response to changes in environmental conditions. Completeness and transitivity of these choices are both compelling. Completeness is immediate. The idea that organisms evolve seems to rule out the kind of cyclic choices implied by intransitivity (which would require that one evolves then eventually reverts back again). So we could try to model genetic behavior using the maximization approach.7

This unified approach is nice, but not necessarily better. After all, we need to add a model (assumptions about utility, for example) that could quite well be

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6This is both good and bad when a model has lots of implications. This is good because the model is easy to test. That may make it a bad model, as well, if its predictions are obviously wrong. The utility function in (2.3.1) is like this. It predicts that the consumer will consume positive amounts of every good - no sensible consumer would pay for Microsoft Windows, or buy an SUV.

7I can’t resist suggesting one of my favourite arguments by Arthur Robson (http://www.sfu.ca/robson/wwgo.pdf). The formal title is “Why we grow Large and then grow old: Biology, Economics and Mortality”; the informal title of his talk was “Why we Die”. Yes, it is the solution to a maximization problem.
wrong. Fortunately, the econometricians have taught us how to test our models and reject the ones that are wrong, so that we can refine them. If you are taking econometrics, you might want to learn how. If you take logs of equation (2.3.10) you will get

$$\log(y) = \log(1 - \alpha) + \log(W) - \log(p_y)$$

(2.3.11)

If you add an error term to this, you get a simple linear regression equation in which the coefficient associated with the log of price is supposed to be 1. That is very easy to test (and reject).

### 2.4 How to Test Demand Theory

If we make assumptions about the utility function, we can say a lot about how consumers behave. As with the formulation given by (2.3.1), these strong predictions often won’t be borne out in whatever data we have. For example, an econometric test of (2.3.11) will almost surely fail. Then we can reject our model. However, we will most likely be rejecting our assumption that the utility function has the form given in (2.3.1). What if we wanted to test the assertion that preferences are independent of the budget set the consumer faces? To do that, we need to find a prediction that will be true no matter what form the utility function has, then find a situation where the consumer doesn’t obey that prediction.

This creates a bit of a problem. Suppose our consumer simply doesn’t care what consumption bundle she gets. Then our model is consistent with any pattern of behavior at all, and we could never reject it. Neither would we find such a model useful, because it doesn’t really make any predictions. So a useful and testable economic model will inevitably involve some assumptions about the utility function.

Fortunately, if we simply add the assumption that consumers always prefer more of a good to less of it, we get a prediction that is true no matter what other properties the consumer’s preferences have. It goes the following way - suppose we observe at particular array of prices, a level of income, and the choice the consumer makes under those circumstances. Then, suppose that, at another time, we observe a new array of prices, and a new level of income such that the consumer could just afford to buy the consumption bundle that she purchased in the first case. Of course, along this new budget line we will get to observe another choice by the consumer. Along this new budget line there will be some consumption bundles that would have been inside (strictly) the budget set at the old prices and level of income. If the consumer picks one of these then she is not acting as predicted by our model, and we can reject our model.

Let me illustrate this in the simple case where there are only two goods. The basic idea is depicted in Figure 2.4.1.

The point \((x^*, y^*)\) is the solution to the consumer’s problem at the initial set of prices. Here we simplify a bit by assuming that at the initial situation, the price of good \(x\) is \(p\) while the price of good \(y\) is just 1. The budget set for
the consumer is the triangle formed by the axis and the line between the points \((0, W)\) and \((\frac{W}{p})\).

Now we present the consumer with a new higher price for good \(x\). The new price is \(p'\). At this new price, good \(x\) is more expensive than it was before, so our consumer could not afford to buy the bundle \((x^*, y^*)\) unless there is some change in her income. So, let’s suppose we can give her just enough income to buy the bundle \((x^*, y^*)\) that she bought before the change in prices. The compensated income is denoted \(W'\). The new income, along with the new price \(p'\), gives her the blue budget line. By construction, this budget line just passes through the point \((x^*, y^*)\).

This is all reasoning from your first-year economics course. Along the new budget line, the consumer should pick a point like \((x', y')\). If she picks a point like \((x_0, y_0)\) instead, then she would be choosing a point that she could have afforded to buy at the initial price \(p\) before her income changed.

What would be wrong with that? Well, remember, we are trying to figure out whether our model is true. The model consists of three kinds of assumptions. The first are our most basic axioms - completeness, transitivity and continuity of preferences. The second is our assumption that preferences are independent of the budget set that is presented to the consumer. The third is the assumption that the consumer prefers more of each good.

Since \((x_0, y_0)\) is inside the budget set when the price of \(x\) is \(p\) (we leave out the additional qualifier “and when income is \(W\)” to make the argument is little shorter), then whatever the consumer’s indifference curves actually look like, there must be other bundles in the initial budget set that are strictly preferred to \((x_0, y_0)\). We have no idea what all these bundles are, but suppose that one such bundle is \((x'', y'')\) (which isn’t marked in the picture). Since the consumer chose \((x^*, y^*)\) from that budget set, it must be that \((x^*, y^*)\) is at least as good from the consumer’s point of view as \((x'', y'')\). Yet \((x'', y'')\) is strictly better
than \((x^0, y^0)\). By transitivity, \((x^*, y^*)\) is strictly better for the consumer than \((x^0, y^0)\). Then, if preferences are the same in every budget set, the consumer could do strictly better in the new budget set at prices \(p'\) by choosing \((x^*, y^*)\). If our consumer chooses a bundle like \((x^0, y^0)\) then there must be something wrong with our story.

So, if our model of the consumer is correct, we should observe that an *income compensated* increase in the price of any commodity will result in a fall in demand for that commodity. I will leave it to your econometrics courses to tell you how the tests of consumer demand theory have worked out.

### 2.5 Comparative Statics and The Envelope Theorem

To appreciate most modern economic theory, you need to understand that the consumer’s choice depends on the constraint set she faces. If we characterize the choice as the solution to a maximization problem, then the consumer’s choice could be thought of as a *function* of the parameters of the constraint set she faces. In general, we refer to this as a *best reply* function. In consumer theory, the best reply function is called a demand function. More generally, the parameters that affect the choice sets may not be prices. In game theory, the parameters that affect the individual’s choice behavior are the actions that she thinks others will take.

You have seen a best reply function already. When preferences are given by (2.3.1) then the amount of good \(y\) the consumer will buy for *any* pair of prices \((p_x, p_y)\) and *any* level of income \(W\) is given by (2.3.10). The demand for good \(y\) is a function of its price and the consumer’s income.

It is actually pretty unusual to have the demand function in such a complete form. To get such a thing, you actually need to be able to find a complete solution to the first order conditions. That requires assumptions about utility that are unlikely to pass any kind of empirical test. However, it is often possible to use mathematical methods to say useful things.

Let’s go back to the case where preferences are represented by a function \(u(x, y)\) and assume there is a demand function, \(D(p_x, p_y, W)\), that tells us for each possible argument what quantity of good \(x\) the consumer will choose to buy. This function probably looks something like (2.3.10), but we can’t really say exactly what it is like. Let’s make the heroic assumption that this function looks like (2.3.10) in the sense that it is differentiable; that is, \(D(p_x, p_y, W)\) has exactly three partial derivatives, one for each of its arguments.

In particular, for preferences given by (2.3.1), the demand function for good \(y\) is

\[
D(p_x, p_y, W) = \frac{(1 - \alpha)W}{p_y}
\]
The three partial derivatives are given by
\[
\frac{\partial D(p_x, p_y, W)}{\partial p_x} \equiv D_{p_x}(p_x, p_y, W) = 0
\]
\[
\frac{\partial D(p_x, p_y, W)}{\partial p_y} \equiv D_{p_y}(p_x, p_y, W) = -(1 - \alpha)W \left( \frac{1}{p_y} \right)^2
\]
\[
\frac{\partial D(p_x, p_y, W)}{\partial W} \equiv D_W(p_x, p_y, W) = \frac{1 - \alpha}{p_y}
\]

More generally, we can just refer to the partial derivatives as \( D_{p_x}, D_{p_y}, \) and \( D_W \) as long as you remember that these derivatives depend on their arguments.

### 2.5.1 Implicit Differentiation

The method of implicit differentiation will sometimes give you a lot of information about a best reply function. To be honest, it doesn’t really work very well in demand theory, but I will explain it anyway. We will use this method in our discussion of portfolio theory below.

Let’s simplify things a bit and hold the price of good \( y \) constant at 1 and vary only the price \( p \) of good \( x \), and the level of income \( W \). Let’s suppose as well that for some price \( p \) and level of income \( W \), the solution to the consumer’s maximization problem involves strictly positive amounts of both goods \( x \) and \( y \). Then by the Lagrangian theorem, there must be a multiplier \( \lambda \) such that the first order conditions
\[
\begin{align*}
    u_x(x, y) + \lambda p &= 0 \\
    u_y(x, y) + \lambda &= 0 \\
    px + y &= W
\end{align*}
\]
hold.

As we vary \( p \) slightly, the values of \( x, y, \) and \( \lambda \) will change so that (2.5.1) to (2.5.3) continue to hold. Then, by the chain rule of calculus,
\[
\begin{align*}
    u_{xx}(x, y) \frac{dx}{dp} + u_{xy}(x, y) \frac{dy}{dp} + \lambda + p \frac{d\lambda}{dp} &= 0 \\
    u_{yx}(x, y) \frac{dx}{dp} + u_{yy}(x, y) \frac{dy}{dp} + \frac{d\lambda}{dp} &= 0 \\
    x + p \frac{dx}{dp} + \frac{dy}{dp} &= 0
\end{align*}
\]
In this notation, the terms like \( u_{xx}(x, y) \) are second derivatives. For example, when preferences are given by (2.3.1), \( u_{xx}(x, y) = \alpha(\alpha - 1)x^{\alpha - 2}y^{1-\alpha} \). The terms like \( \frac{dx}{dp} \) are the derivatives of the implicit functions that satisfy the first order conditions (2.5.1) to (2.5.3) as \( p \) changes a little.

We are interested in trying to figure out properties of \( \frac{dx}{dp} \). In principle, we could use these last three equations to learn about it. There are three equations
and three unknowns. They are non-linear, so there is no guarantee they will have a solution, but they probably will. The complication is that this solution is complicated and won’t actually say much. For what it is worth, pure brute force gives the following

\[
\frac{dx}{dp} = \frac{(u_{xy} - pu_{yy})x - \lambda}{u_{xx} - 2pu_{xy} - p^2u_{yy}} \quad (2.5.7)
\]

This is pretty bleak, because there is too much in the expression that we don’t know. The sign of the expression could be either positive or negative depending on the sizes of the cross derivatives. Then, there is the mysterious multiplier term \(\lambda\).

The one advantage of this approach is that it will often tell you what you need to assume in order to get the result that you want. Since the irritating terms are the cross derivatives, suppose that we make the utility function separable. For example, it might have the form \(u(x, y) = v(x) + w(y)\) where \(v\) and \(w\) are concave functions (which means that their derivatives get smaller as their arguments get larger). Then \(u_{xy} = u_{yx} = 0\) and (2.5.7) reduces to

\[
\frac{dx}{dp} = \frac{-pu_{yy}x - \lambda}{v_{xx} - p^2w_{yy}} \quad (2.5.8)
\]

This still not enough. If we assume that the function \(w\) and \(v\) are both concave, then their second derivatives can’t be negative. The multiplier is less than or equal to zero by the complementary slackness conditions, so the numerator is non-negative. The denominator can be either positive or negative depending on the magnitudes of the second derivatives.

This leads us to the second most famous special functional form in economics. If we assume that \(w(y) = y\), we get something called a quasi-linear utility function. Then \(w_{yy} = 0\) and we know that the demand function is at least downward sloping. Quasi-linear utility functions are widely used in the theory of mechanism design and auctions.

### 2.5.2 Graphical Methods

The arguments above are a bit obscure. Graphical methods will often provide some more insight. The methods in the previous section are also local methods, since they assume that all the changes that are occurring are small. Graphical analysis won’t really give you a full solution to the problem you are trying to solve, you will ultimately need to return to the math for a full solution. Yet graphical analysis will often point in the right direction.

If you simply want to understand why the demand function doesn’t slope downward, a graphical trick will show you. Go back to Figure 2.4.1 where the consumer was faced with an increase in the price of good \(x\), but was given enough income to allow her to afford her initial consumption bundle. We concluded that this combination of changes in her budget set would induce her to lower her demand for good \(x\). We can decompose these changes into their constituent
parts - an increase in price, followed by an increase in income. The two changes together appear in Figure 2.5.1.

The picture shows a problem similar to the one in Figure 2.4.1. The initial price for good $x$ is $p$. At the price and her initial income, the consumer selects the bundle $(x^*, y^*)$. As we saw before, if we raise the price of good $x$ to $p'$ but give the consumer enough extra income that she could just purchase the original bundle $(x^*, y^*)$, then she must respond by purchasing more good $y$. In other words, her compensated demand for good $x$ must fall. For example, she might choose the new bundle $(x', y')$ as in the Figure.

If we want to know how the impact of the price increase by itself will influence her demand, we need to take away the extra income we gave her so that she could afford her initial bundle. In the picture we do this by shifting the budget line downward (toward the origin) from the red line to the blue line. Since we are holding both prices constant as we take away this income, the slope of the budget line doesn’t change as we shift it in. (Make sure you understand why the blue line goes through the point $(0, W)$.

As the picture is drawn, our consumer chooses the bundle $(x'', y'')$. The remarkable thing about this bundle is that it actually involves more good $x$ than there is in the initial bundle $(x^*, y^*)$. An increase in the price of good $x$ has actually caused an increase in demand for good $x$. The diagram illustrates why. As our consumers income rises (shifting the budget line up from the blue line to the red line, her demand for good $x$ actually falls. Goods that have this property are called *inferior goods* as you may recall from your first-year course.
2.5.3 The Envelope Theorem

There is one special theorem associated with the Lagrangian that is sometimes quite useful. Suppose that we are trying to solve the problem

$$\max_x u(x)$$ (2.5.9)

subject to

$$G_1(x, y) \leq 0$$ (2.5.10)

$$\vdots$$

$$G_m(x, y) \leq 0$$ (2.5.11)

where $$x \in \mathbb{R}^n$$, $$m \geq 1$$, and $$y$$ is some parameter that affects our constraints, for example, the price of one of the goods, or the consumer’s income. If we could find a solution to this problem, the we could call the value of the solution $$V(y)$$. This value is a function of the parameter $$y$$. If $$y$$ were a price, for instance, then the maximum value of utility would be a decreasing function of price. Suppose we are interested in finding out how a change in $$y$$ will change this maximum value - i.e., we want to know something about $$\frac{dV(y)}{dy}$$. One way to do this to use implicit differentiation as we did above. The vector $$x^*$$ that solves the problem is an implicit function of $$y$$. Imagine that $$x^*[y]$$ is the function that gives us the solution to the problem. For example, in the consumer’s problem, if we think of $$y$$ as the price of good $$x$$, then $$x^*[y]$$ is the bundle that provides the maximum utility. Whatever the actual interpretation, it should be clear that $$V[y] = u[x^*[y]]$$. We could then compute the impact of a change in $$y$$ by finding all the partial derivatives of $$u$$ with respect to each of the $$x$$‘s evaluated at the initial optimal solution, multiplying each of these by the total derivative of the corresponding solution with respect to a change in $$y$$, then summing everything up. In math

$$\frac{dV(y)}{dy} = \sum_{i=1}^n \frac{\partial u(x^*[y])}{\partial x_i} \frac{dx_i^*[y]}{dy}$$

This would require not only that we take a lot of partial derivatives, but also that we compute function $$x^*[y]$$ and find its total derivatives - a daunting amount of work.

Fortunately, there is a very nice way around this. Recall that the Lagrangian function associated with this maximization problem is

$$L(x, \lambda, y) = u(x) + \sum_{j=1}^m \lambda_j G_j(x, y)$$

Then the envelope theorem says that

**Theorem 2.5.1**

$$\frac{dV(y)}{dy} = \frac{\partial L(x, \lambda, y)}{\partial y} \bigg|_{x=x^*; \lambda=\lambda^*}$$ (2.5.12)
This says that to compute the total derivative of the maximum value, then we only need to compute the partial derivative of the Lagrangian evaluated at the optimal solution. This is much easier. I am going to show you why this is true, and how nicely it works. Our consumer solves the problem
\[
\max u(x, y)
\]
subject to
\[
px + y - W \leq 0
\]
\[
-x \leq 0
\]
\[
-y \leq 0
\]
The Lagrangian is
\[
u(x, y) + \lambda_1(px + y - W) - \lambda_2x - \lambda_3y
\]
Suppose I want to find out the impact of an increase in wealth on the consumer’s optimal utility starting from an initial price \( p_0 \) and wealth level \( W_0 \). The Envelope theorem says that we first need to solve the consumer’s problem and find the utility maximizing demands, call them \( x_0^* \) and \( y_0^* \), as well as the multipliers that satisfy the first order conditions at the optimal solution, \( \lambda_1^0, \lambda_2^0, \) and \( \lambda_3^0 \). The Lagrangian is generally a complicated function of \( W \) because all the multipliers and the optimal \( x \) and \( y \) are changing with \( W \). Nonetheless the derivative of this optimal value is simply
\[
\frac{\partial L}{\partial W} = -\lambda_1
\]
The significance of the \( \partial L \) instead of \( dL \) is that we don’t have to worry about all the implicit functions.

Here is the proof of the envelope theorem:

**Proof** First observe that
\[
V(y) = u(x^*) \equiv L(x^*, \lambda^*, y) = u(x^*) + \sum_{j=1}^{m} \lambda_j^* G_j(x^*, y) \quad (2.5.13)
\]
It might seem that this would be false because of the sum that we add to \( u(x^*) \). However, by the complementary slackness conditions, the product of the multiplier and the constraint will always be zero at the solution to the first order conditions. So, the sum is exactly zero.

As long as we think of \( x^* \) and \( \lambda^* \) as implicit functions of \( y \), then this is an identity, so we find the derivative using the chain rule.
\[
\frac{dV(y)}{dy} = \sum_{i=1}^{n} \frac{\partial u(x^*)}{\partial x_i} \frac{dx_i^*}{dy} + \sum_{j=1}^{m} \left[ \frac{d\lambda_j^*}{dy} G_j(x^*, y) + \lambda_j^* \sum_{i=1}^{n} \frac{\partial G_j(x^*, y)}{\partial x_i} \frac{dx_i^*}{dy} + \lambda_j^* \frac{dG_j(x^*, y)}{dy} \right]
\]

30
First consider the terms \( \frac{d\lambda^*_j}{dy} G_j(x^*, y) \). By complementary slackness, either \( G_j(x^*, y) \) is zero, or \( \lambda^*_j \) is zero, or both are zero. In the first case, and the last case, we can forget about the term \( \frac{d\lambda^*_j}{dy} G_j(x^*, y) \) because it will be zero. What happens when \( G_j(x^*, y) < 0 \)? Then \( \lambda^*_j \) is zero. In that event, changing \( y \), say by \( dy \), will not change the solution very much and we can rely on continuity to ensure that \( G_j(x^*[y + dy], y + dy) \) is still negative. If that is the case, then again using complementary slackness, it must be that \( \lambda^*_j[y + dy] = 0 \), which means that \( \frac{d\lambda^*_j}{dy} = 0 \).

Using this, we can rewrite the derivative as follows

\[
\frac{dV(y)}{dy} = \sum_{i=1}^{n} \left( \frac{\partial u(x^*)}{\partial x_i} + \sum_{j=1}^{m} \lambda^*_j \frac{\partial G_j(x^*, y)}{\partial x_i} \right) \frac{dx^*_i}{dy} + \sum_{j=1}^{m} \lambda^*_j \frac{dG_j(x^*, y)}{dy}
\]

Now notice that the terms in the first sum over \( i \) are all derivatives of the Lagrangian with respect to some \( x_i \) evaluated at the optimal solution. Of course the optimal solution has the property that the derivatives of the Lagrangian with respect to the \( x_i \) are all equal to zero. Consequently the derivative reduces to

\[
\frac{dV(y)}{dy} = \sum_{j=1}^{m} \lambda^*_j \frac{\partial G_j(x^*, y)}{\partial y}
\]

which is just the partial derivative of the Lagrangian with respect to the parameter \( y \).

\( \square \)
Chapter 3

Dealing With Discontinuities

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3.1 Introduction

The Lagrangian method for solving maximization problems that we have studied so far is sometimes the only way to get insight into the solution to your problem. Most of the time, it is only part of the technique you need. The reason is that the constraint sets consumers face often come with built-in discontinuities of various sorts. You will see these every day. Your phone company offers 100 minutes of long distance calling for a fixed fee, but if you talk for more than 100 minutes, you pay big time. The local restaurant runs a burger-beer special $6 for a burger and beer - and charges $5 for each additional beer. Buy one issue of The Economist magazine, and the price is about $5; buy a year’s subscription, and the price falls to about $2 per issue.

To solve problems like these, you need to use a variety of different techniques in combination. When approaching a problem, you may be tempted to simply write down the Lagrangian, derive all the first order conditions, and then, try to solve. This chapter is a warning that this will fail in most of the problems you are likely to encounter. You will need the Lagrangian method, but only as part of a larger toolbox.

Again, you may be tempted to rush to the ‘write down the first order conditions’ approach—especially during an exam when time pressure keeps you from being your most creative—but this is the wrong way to start in most situations. I am a big fan of graphical methods for guessing the solution to problems. They are usually the most useful at helping you to see the special properties of the
problem you are trying to solve. In a previous chapter, we solved a consumer’s problem using Lagrangian methods, where preferences were given by

\[ u(x, y) = x^\alpha y^{1-\alpha} \quad (3.1.1) \]

Graphical methods would start with a picture like the one you saw so much in your first-year course.

![Graphical representation of indifference curves and budget set](image)

**Figure 3.1.1:** The best bundle for the consumer to choose is the one that lies on the highest indifference curve that just touches the budget set. This indifference curve could look like the green curve in Figure 3.1.1. You have to remember, though, that it could also resemble the red curve that touches the budget line at the point \((0, \frac{W}{p_y})\), or resemble the blue curve that touches the budget line at the point \((\frac{W}{p_x}, 0)\). These are referred to as *corner solutions*. In the case where preferences are given by \((3.1.1)\), we have already explained why corner solutions are not possible. If either coordinate of the chosen bundle is zero, then overall utility is zero, and the consumer will be able to do strictly better by picking *any* bundle in her budget set that has strictly positive coordinates.

So the picture, along with this latter insight, tells us the solution has to look like the point \((x^*, y^*)\) in the diagram where the indifference curve is tangent to the budget line. If we think of the indifference curve as a function that converts each value of \(x\) into a different value of \(y\), then the solution is going to be at the point where this function has the same slope as the budget line.

The slope of budget line is easy to compute: it is linear so you can just use the rise over run formula from high school to get the slope equal to \(-\frac{p_x}{p_y} \)\(^1\).

What about the slope of the indifference curve? One way to proceed is to find the function explicitly by trying to solve the equation

\[ u(x, y) = u(x^*, y^*) \]

\(^1\)The rise is \(\frac{W}{p_y}\) while the 'run' is \(\frac{W}{p_x}\).
for $y$. A better way is to find the slope of the function implicitly by solving the equation
\[ u_x(x, y)dx + u_y(x, y)dy = 0 \]
for $\frac{dy}{dx}$. The term $u_x(x, y)$ represents the amount that utility goes up when you increase $x$ a bit, while $dx$ is the amount that you are going to change $x$. We don’t know or care what $dx$ is exactly. It is going to be very small for one thing. In any case, the first term gives the impact on utility of small changes in $x$. The second term does the same for $y$. If we pick $dx$ and $dy$ so that we move along the indifference curve, then the total change in utility will have to be zero, which is what right-hand side of the equation says. Solving this gives us the slope of the indifference curve at any point equal to
\[
-\frac{u_x(x, y)}{u_y(x, y)}
\]
When preferences are given by (3.1.1), then this becomes
\[
-\frac{\alpha x^{\alpha-1}y^{1-\alpha}}{(1-\alpha)x^{\alpha}y^{(-\alpha)}} = -\frac{\alpha y}{(1-\alpha)x}
\]
Then we want
\[
-\frac{\alpha y}{(1-\alpha)x} = -\frac{p_x}{p_y}
\]
and the budget equation is $p_x x + p_y y = W$. Solving this gives the same equation we derived in the last chapter
\[
y^* = \frac{(1-\alpha)W}{p_y}
\]
The Lagrangian and graphical methods are pretty much interchangeable when preferences are given by (3.1.1). To see a problem where the graphical method works considerably better, suppose that preferences are given by $u(x, y) = ax + by$. If you remember your first-year course, preferences that have this kind of representation are such that the consumer views goods $x$ and $y$ to be perfect substitutes. Using our above approach, the slope of the indifference curve at the point $(x, y)$ is given by
\[
-\frac{u_x(x, y)}{u_y(x, y)} = -\frac{a}{b}
\]
As soon as you try to draw the indifference curves, you see that their slopes are independent of the point $(x, y)$ at which you try to evaluate them. In other words, the indifference curves are all straight lines. Our picture now looks like Figure 3.1.2.
Figure 3.1.2:
If the consumer’s indifference curves are all flatter than the budget line, then the solution to the maximization problem has to occur at the point \((0, \frac{W}{p_y})\). Then, the consumer’s indifference curves will look like the red line. They are flatter when their slope \((-\frac{a}{b})\) is closer to zero than \(-\frac{p_x}{p_y}\), or when \(\frac{a}{b} < \frac{p_x}{p_y}\).

Now, we know the entire demand curve for the consumer: demand for good \(y\) is equal to \(\frac{W}{p_y}\) when \(\frac{a}{b} < \frac{p_x}{p_y}\); 0 when \(\frac{a}{b} > \frac{p_x}{p_y}\); and is anything between 0 and \(\frac{W}{p_y}\) otherwise. You should derive this result using Lagrangian methods to see how the graphical technique is much more straightforward.

### 3.2 Non-linear Pricing

The main purpose of this chapter isn’t to take you back to things you learned in your first-year course. The purpose is to show you that the Lagrangian technique is not the way to solve maximization problems, it is just one technique that may prove useful. You will encounter problems where it doesn’t work well quite often in practise. A practise called non-linear pricing is one example. This exotic name means that the per unit price that you pay may depend on how much you want to buy. There are two ways that this can work. Let’s look first at the case where the marginal price is increasing. It works this way: the price of good \(y\) is held constant at 1. The price for each of the first \(n\) units of good \(x\) is \(p\), but if you want to buy more than \(n\) units, then each additional unit beyond \(n\) will cost a higher price \(p + dp\).

Cable television companies often use this kind of pricing scheme. There is a basic package consisting of around 50 channels. Specialty channels can be added to the package, but a bundle of specialty channels might involve only another 10 channels. The cost of the extra 10 channels will often be about as high as the first 50. Apparently, tickets to World Cup soccer matches also work this way. You can participate in a lottery, and win the right to buy a pair of tickets to a World Cup match. If you want four tickets, however, you have to buy the
additional tickets from resellers at much higher prices.

Our consumer faces a budget line given in Figure 3.2.1.

![Budget Line Diagram](image)

**Figure 3.2.1:**

The important point in the diagram is the bundle \((n, W - pn)\). The line segment connecting this point with \((0, W)\) has slope \(-p\). The line segment connecting this point to \(\left( \frac{W - np}{p + dp}, \frac{W}{p} \right)\) is steeper and has slope \(-p - dp\). The arrows that point outward from the edges of the budget set represent a convenient way to illustrate the slope of the line segment. Notice that these arrows are perpendicular to the line segments that they touch. You can imagine that the arrow labeled \((p, 1)\) on the upper part of the budget set is itself a little vector in \(\mathbb{R}^2\). The \(x\)-coordinate of this vector is \(p\) and the \(y\)-coordinate is 1, as illustrated. This is a convenient way to say that the slope of the line segment that the arrow is touching is \(p\).

Where does the point \((n, W - pn)\) come from? Well, if our consumer decided to purchase exactly \(n\) units of good \(x\), she would spend \(pn\) and have \(W - pn\) dollars left over to spend on good \(y\). If she decides to go further and spend all her money on \(x\), then she will have \(W - np\) dollars left over after she buys her first \(n\) units of good \(x\). This will allow her to afford \(\frac{W - np}{p + dp}\) more units of good \(x\).

Let’s stick with preferences of the form (3.1.1) so that we don’t have to worry about corner solutions. Figure 3.2.2 indicates the three possible solutions to the consumer’s problem.
The highest indifference curve could be tangent on the upper section, the lower section, or the indifference curve may not be tangent to either section if it just touches the kink as with the green curve in the figure. In this case, the optimal solution does not have to satisfy the first order conditions because the constraint set is not \textit{differentiable}.

Now, let’s use the special properties of the utility function given by (3.1.1) to give a complete solution to this problem. We will basically whittle the problem down piece by piece until we find a solution. It is a bit more complex than simply writing first order conditions, but still pretty algorithmic. First, simply ask what the consumer would do if she had income $W$ and could buy all she wanted at price $p$. There are two possibilities here: either $\frac{aW}{p} < n$ or not. You can see these two in Figure 3.2.3.

If the tangency occurs on the upper segment of the budget line (the red indifference curve), then you are finished. The consumer will simply purchase
\(\frac{\alpha W}{p}\) units of good \(x\) just as she would have without the non-linear price. On the other hand, if the tangency occurs on the dashed segment of the budget line (the green indifference curve), then the consumer would have liked to purchase more than \(n\) units of good \(x\) at the original price. This isn’t going to be feasible for her, since each unit beyond the \(n^{th}\) actually costs here \(p + dp\).

In this case, we can try a trick. Let’s take the lower segment of the budget line and extend it so that it looks like the budget line the consumer would face if she could buy all the good \(x\) she wants at the constant price \(p + dp\). In addition, let’s adjust her income so that she is able to afford exactly \(n + \frac{W - np}{p + dp}\) units of good \(x\) if she decides to spend all her money on good \(x\). This is pretty easy. To find this, just solve

\[
\frac{W'}{p + dp} = n + \frac{W - np}{p + dp}
\]

for \(W' = W + ndp\). Now, solve the consumer’s problem under these new circumstances (using (3.1.1)) and you get the choice for \(x\) to be

\[
\frac{\alpha(W + ndp)}{p + dp}
\]

If this solution is larger than \(n\), you are finished. As you can see from Figure 3.2.4, our consumer can’t do any better by cutting consumption of good \(x\) below \(n\).

On the other hand, if \(\frac{\alpha(W + ndp)}{p + dp} < n\) (and you have already checked that \(\frac{\alpha W}{p} > n\)), then you are left with one remaining possibility: the solution is right at the kink in the budget line with \(x^* = n\).

This gives us a pretty complete solution to the problem. The important thing to observe is that we used the Lagrangian technique (implicitly because we used it to solve for the demands when preferences are given by (3.1.1)), but only as part of a more complicated algorithm. The more complicated algorithm became necessary because the budget set that we are dealing with has a kind of discontinuity.
Here is a picture that very nicely summarizes the information we have learned from this process.

Figure 3.2.5:

We have determined that if $\frac{\alpha W}{p} \leq n$ then our consumer will just buy fewer than $n$ units and won’t have to pay the higher price. This is like a ticket buyer who simply buys one or two tickets when he wins the world cup lottery. This inequality is the same as $\alpha W \leq pn$. So, the diagram draws the graph of the function $\alpha W$ and $pn$. The latter, of course, doesn’t depend on $\alpha$ so the graph is just a horizontal line. The point where these two functions intersect is $\frac{np}{W}$ as is shown in the diagram.

We also figured out that if $\frac{\alpha(W + ndp)}{p + dp} > n$, then our consumer would buy strictly more than $n$ units, paying the lower price $p$ for the first $n$ units, then shelling out the higher price for the others. This is like the soccer fan who buys additional higher-priced tickets from resellers (scalpers).

This inequality is the same thing as $\alpha(W + ndp) > np + (1 - \alpha)ndp$. The figure adds the graph of the function $np + (1 - \alpha)ndp$ which is the downward sloping curve. As you can see, this curve starts out above $pn$ and eventually equals it. This curve intersects $\alpha W$ at the point $\frac{pn + ndp}{W + ndp}$ as marked in the Figure. So, in the interval between $\frac{np}{W}$ and $\frac{pn + ndp}{W + ndp}$, our consumer buys exactly $n$ units of the good.

Perhaps from this analysis, you can tell why firms will use non-linear prices. The people who choose to buy $n$ or fewer units don’t care what price you charge for extra units of output, because they don’t buy any extra. That means that you can raise the price you charge your fanatic customers without losing any business from those who are a little less fanatic.

### 3.3 Sign-up Fees

Another common pricing technique is the sign-up fee. Long distance-phone charges work this way: you pay a fixed fee for 100 free minutes. Each minute
thereafter will cost you an extra 5 cents. You may want only 50 minutes of long
distance service, but you will still be forced to pay the same fixed fee.

So, let’s consider a sign-up fee problem. Let’s begin by fixing notation. As
before, we will assume that every unit of good $y$ has the same per unit price
of $1. The sign-up fee for delivery of good $x$ will be $K$, which will provide $n$
units of good $x$. Each additional unit of good $x$ will cost $p$. The budget set the
consumer faces in this case is given in Figure 3.3.1.

\[0 \leq x \leq n + \frac{W-K}{p}\]

\[y \leq W - K\]

\[y \leq W - K\]

Figure 3.3.1:

Now, if the consumer only wants to buy good $y$, then she can afford $W$
units. If her indifference curves look like the red curve that cuts through the
point $(0, W)$, then she will do exactly that. If she wants to buy any good $x$ at
all, she must pay the fixed fee $K$. This will cause her budget set to jump down
discontinuously to the point $(0, W - K)$. She would never choose this point if
she likes good $x$ because she can have up to $n$ free units of the good once she
has paid the fixed fee. So, her budget set must also shift discontinuously, up to
the right, to the point $(n, W - K)$. If she wants more than $n$ units of good $x$,
then she will choose a tangency point on the downward-sloping portion of the
budget line (whose slope is $-p$).

Let’s suppose first that our consumer has preferences given by (3.1.1). Then,
we can use the method we used for the last problem: combining the Lagrangian
with a systematic scan of the possibilities. To do this, let’s first pretend that
the consumer simply has a fixed income equal to $W - K + np$, and that she can
buy all the good $x$ she wants, even small amounts of it, at price $p$. Applying
the Lagrangian method with preferences given by (3.1.1), we get demand equal
to $\alpha(W - K + np)/p$. If this is larger than $n$, then we are finished and have
found the solution to the problem. If it is less than $n$, it means the solution is
not feasible given the firm’s pricing scheme. When preferences are given by
(3.1.1), we know that the consumer will never choose 0 units of good $x$. So if
$\alpha(W - K + np)/p \leq n$, our consumer will simply choose $n$ units of good $x$.

It may be apparent to you that this pricing scheme helps the firm because
it induces some consumers who would have purchased fewer than $n$ units at a
constant price $p$ to increase the amount they buy to $n$. Of course, firms always tout their pricing schemes as being designed to allow consumers to choose the plan that is best suited to their needs. Perhaps, our analysis indicates that this claim typically means best suited to the firm’s needs.

To make the argument in a little stronger way, suppose that our consumer has a slightly different utility function given by $u(x, y) = y + \log(x)$. This is a special case of the famous quasi-linear utility function that is now just about the most widely used utility function in economics.\(^2\) Let’s use the Lagrangian method to figure out our consumer’s demand function in this case.

She is trying to solve the problem

$$\max_x y + \log(x)$$

subject to the usual constraints

$$px + y - W \leq 0$$

$$-x \leq 0$$

$$-y \leq 0$$

The first order conditions are

$$1 + \lambda_1 p - \lambda_2 = 0$$

$$1 + \lambda_1 - \lambda_3 = 0$$

along with the three complementary slackness conditions. Since $\log(0)$ is undefined (or equal to $-\infty$), we know our consumer will always choose a strictly positive amount of good $x$. So, by the complementary slackness condition, $\lambda_2$ must be equal to zero. Let’s suppose for the moment, that the optimal $y$ is also positive. Then, from the first order conditions, $x = \frac{1}{p}$. This is an odd property of quasi-linear functions, as long as $W > 1$, the consumer’s demand will be independent of her income. Cigarettes are a commodity (I can’t really call them a good) that have this property for low income people.\(^3\)

So, let’s look at the entry fee problem using this result. The next figure just reproduces the last one with this extra information about quasi-linearity.

\(^2\)You may not see this utility function much for a while. It is most widely used in the theory of auctions and mechanism design.

\(^3\)Though, as income rises cigarette demand eventually falls. Cigarette demand is also very insensitive to price changes, at least when income is low.

42
The picture is drawn so that the consumer initially has income $W > 1$ and so chooses to buy $\frac{1}{p}$ units of good $x$. Our Lagrangian analysis says that if the fixed fee varies a little bit, the consumer continues to purchase exactly $\frac{1}{p}$ units of good $x$.

Then, as we raise the fixed fee $K$, this will have no effect on our consumer’s demand for good $x$. Of course, the revenue that the firm gets selling good $x$ to this consumer consists of $px$ and the fixed fee, so the firm’s profit is rising as it raises the fee. If the firm raises the fixed fee to $W - 1$, the consumer’s budget line (when she pays the fixed fee) will shift down until it is equal to the blue line in the figure. This line which connects the point $(0, 1)$ to the point $(\frac{1}{p}, 0)$. Then, our consumer will still buy $\frac{1}{p}$ units of output, but will now give all the rest of her income to the firm as well. When demand is insensitive to changes in consumer income, an entry fee is a good way for a firm to raise its revenues.

You should make sure you that work out on your own what would happen if the firm continued to raise the fixed fee beyond $W - 1$. 

Figure 3.3.2: 

\[ y \\ W \\ W - K \\ 1 \\ 0 \\
0 \quad n \quad \frac{1}{p} \quad n + \frac{W-K}{p} \quad x \]
Chapter 4

Best Reply Behavior

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4.1 Introduction

So far, we have concentrated on individual optimization. This unified way of thinking about individual behavior makes it possible to understand a number of interesting economic problems. Some of these will be discussed in coming chapters. However, the biggest insights in economics come from thinking about group behavior. Individual and group behavior are tied together in economics by best reply behavior.

As you have seen in prior readings, the solutions to individual maximization problems depend on characteristics of the environment. When many people make maximizing decisions at the same time, their choices may or may not be consistent with one another. Perhaps the biggest insight of all in economics is that consistency restrictions on peoples’ choices will impose restrictions on the set of environments that we should ever expect to see.

The oldest application of this idea is the one that you probably saw in operation in your first year economics course. Every consumer’s demand for every good depends on the price that she thinks prevails for that good. If consumers’ beliefs about these prices are completely arbitrary, then some goods will be in excess demand and some in excess supply. As a consequence, all consumers should have the same belief about the price at which they can buy each good, and this belief should be that the price will be the one at which aggregate demand and aggregate supply are equal. This is where the concept of a market comes from. A market isn’t a physical location where traders buy and sell, rather it is a state of beliefs that result in a kind of consistent and predictable behavior on the part of a large group of individual decision makers.
Prior to John Nash and game theory, economics couldn’t really get much beyond markets in thinking about group behavior. In cases where markets obviously couldn’t give a good description of behavior, very special models were proposed - for example, the monopoly model to deal with the obvious cases in which firms are not price takers. Nash showed that you can use the consistency of individual decisions in a much broader way to understand many non-market problems as well.

The basic idea behind Nash equilibrium is that instead of forming beliefs about prices, people form beliefs about what other people will do. Fixing these beliefs leads us back directly to the maximization framework that we have been studying so far. In order think about consistency among these beliefs, we have to extend the kind of reasoning we have been using to derive the demand curve to a much more flexible description of best reply behavior.

### 4.2 Consumption Theory with Externalities

To understand this method, let's modify our description of the individual choice problem slightly. As before, let's suppose that there are two goods. The price of good \( y \) will be 1, and the price of good \( x \) will be \( p \) throughout this argument. However, let's suppose that there are two consumers instead of just 1. Each consumer has \( W \) to spend on \( x \) and \( y \). The utility that each person gets for good \( x \) depends positively on the amount of good \( x \) that the other person chooses to consume. This is called a positive consumption externality. Such externalities are everywhere. Movies and novels are great examples - they are fine by themselves, but they are even more enjoyable if your friends have seen the same movie or read the same book, because then you can discuss it. On the margin you might be inclined to go to see a certain movie just because you know one of your friends has already seen it.

Let's call the consumers Alice and Bob, denoting Alice's consumption bundle by \((x_a, y_a)\) and Bob's by \((x_b, y_b)\). Alice's utility function is \( u(x_a, y_a, x_b) \) which says that her utility depends on the things that she chooses, \( x_a \) and \( y_a \) as well as the thing that Bob chooses, \( x_b \). When we say that her utility is affected, what we mean is that her preferences are affected by Bob's consumption. So when Bob changes his consumption, all of Alice's indifference curves will change shape. Suppose they change in such a way that the more Bob consumes, the more Alice is willing to pay for an extra unit of good \( x \). This kind of externality is sometimes referred to as a positive consumption externality.

More formally, this means that Alice's marginal rate of substitution of good \( y \) for good \( x \) increases as Bob increases his consumption. Recall that the marginal rate of substitution is given by the ratio of Alice's marginal utility for good \( x \) to her marginal utility for good \( y \). Recall as well, that the marginal rate of substitution is the same as the absolute value of the slope of Alice's indifference curve.

Now we are interested in the way that the solution to Alice's utility maximization problem changes as Bob varies his consumption. Let's analyze this a
couple of different ways.

4.2.1 The graphical Approach

There is nothing strange at all about Alice’s budget set. She can buy all the good \( x \) and \( y \) that she can afford at constant prices. So she is just going to choose the highest indifference curve that touches her budget set. The only thing that is new about this problem is the fact that the shape of Alice’s indifference curves depend on Bob’s consumption. We know how - as we just figured out, the amount of \( y \) she is willing to give up (or pay) for one unit of good \( x \) is increasing with Bob’s consumption. That means that her indifference curves will all get steeper when Bob’s consumption goes up. As a result, she will purchase more good \( x \) the more Bob purchases.

The first figure shows this simple logic.

![Figure 4.2.1](image)

Figure 4.2.1:

Start with a situation in which Bob’s consumption of good \( x \) is \( x_0 \). Alice’s preferences in this case lead her to choose the bundle \((x^*, y^*)\) which is the highest indifference curve touching her budget set. If Bob increases his consumption to \( x_0 + dx_0 \), then Alice’s indifference curves become steeper, like the red curves in the figure. The tangency then moves to the right to the point \((x', y')\). That’s about all there is to it.

The consumer’s demand function is a type of best reply function. It is common to think of the demand function in a graphical way, sloping downwards and eventually meeting some kind of supply function. We can do the same kind of thing for any best reply function. Here we don’t care so much about changes in price and income, but we do care about the impact of a change in Bob’s consumption.

The next Figure summarizes what we have figured out.
Figure 4.2.2:
The horizontal axis measures Bob's consumption of good $x$, the vertical axis measures Alice's consumption of good $x$. The thick upward sloping line records Alice's best reply to Bob's choice. When Bob chooses not to consume any $x$ at all, Alice still finds that the highest indifference curve touching her budget line touches at a point whose $x$ coordinate is $x^*(0)$. On the other hand, if Bob chooses the higher consumption $x^*_b$, then Alice finds that her tangency occurs at a new point given by $x^*(x^*_b)$ because her indifference curves have all become steeper.

We could do exactly the same sort of exercise for Bob, assuming that his preferences are also changed by Alice's consumption. The axes are already labeled the right way, just think of the curves the other way around. For instance, if Alice chooses to consume 0 units of good $x$, Bob is still likely to want some. The amount that he wants is determined by his best reply function which gives an $x$ value of $x^*_b(0)$.

With this observation, we can make the choices that Alice and Bob make consistent with one another. Alice is really guessing what consumption Bob will choose. Bob is doing the same. If Alice thinks that Bob isn't going to consume any $x$, her best choice is $x^*_a(0)$. But Bob never chooses 0 units of good $x$. You can see this by looking at the graph. No matter what consumption he thinks Alice will pick, he always chooses a positive consumption of good $x$. So Alice is going to see that she is wrong and start changing her beliefs about Bob. This in turn will cause her to change her consumption. It seems pretty compelling that things will settle down when Bob and Alice's beliefs about each other are correct. This happens when Alice consumes $x^*_a$ and Bob chooses $x^*_b$ - the point where their best reply functions meet.

This is a pretty big conceptual leap. We can't really predict what Alice is going to do in this case without knowing what Bob is going to do. We can't say anything about Bob without knowing what Alice is going to do. We seem to be at an impasse. What saves us is the best reply concept. The way to understand what Alice and Bob do is to think about the things they could have
done, but didn’t. In economic history, this idea is called *counter factual analysis*. Historians love to imagine what the Canadian Economy would look like if the rail line had never been built to the west coast, or what the American Economy would look like without the American civil war. We do the same - what would Alice do in all the counterfactual situations that she might encounter with Bob. Then we can consider each of these counterfactuals and try to think through whether they make sense.

The point of this section is no such much to show you how to do this. You will learn this method when you study game theory. The point here is to show how our utility theorem is making it possible to do this. Considering all possible counterfactuals is an impossible task using words and intuition - there is just too much to think about. The maximization approach suggested by the utility theorem allows us to reduce these counterfactuals to very simple geometrical and mathematical objects that we can manipulate.

### 4.2.2 An algebraic Treatment

An alternative approach to the one described above is to provide enough structure on the preferences of Alice and Bob that we can find their best reply functions explicitly. Suppose that Alice’s utility function is

\[ u(x_a, x_b, y) = x_a^\alpha(x_b) y^{1-\alpha(x_b)} \]

where we are going to let

\[ \alpha(x_b) \equiv \frac{px_b}{W} \left( \frac{3}{4} \right) + \frac{W - px_b}{W} \left( \frac{1}{4} \right) \tag{4.2.1} \]

because we need something concrete to find the best reply functions explicitly.

If you recall, the marginal rate of substitution of \( y \) for \( x \) is given by the ratio of the marginal utility of good \( x \) to the marginal utility of good \( y \). In words, this is the amount of good \( y \) the consumer would be willing to give up in order to get 1 additional unit of good \( x \). The MRS is then given by

\[ \frac{\alpha(x_b)}{1 - \alpha(x_b)} \frac{x_a^{\alpha(x_b)} y^{1-\alpha(x_b)}}{x_a^{\alpha(x_b)} y^{-\alpha(x_b)}} = \frac{\alpha(x_b) y}{(1 - \alpha(x_b)) x} \]

This is an increasing function of \( x_b \), because \( \frac{\alpha}{dx_b} = \frac{1}{2} \frac{p}{W} > 0 \).

From our previous work, we know that the \( x \) coordinate of the bundle that maximizes Alice’s utility is

\[ x_a^*(x_b) = \frac{\alpha(x_b) W}{p} = \frac{(W + 2px_b) W}{4W} = \frac{W + 2px_b}{4p} \tag{4.2.2} \]

Now we can put some more structure on the diagram that is given above. For example it is immediate from (4.2.2) that the intercept of Alice’s best reply function is \( x_a^*(0) = \frac{W}{4p} \). This function is linear and has slope \( \frac{1}{2} \).

If Bob has the same preferences, then his best reply function looks exactly the same. For their choices to be consistent with one another we want both Alice and Bob to guess each other’s actual consumption correctly. In other
words, Alice’s guess about what Bob would do, $x_b$, is equal to what Bob actually
does do when he correctly predicts Alice’s consumption, $x^*_a(x_b)$. That is

$$x_b = x^*_b[x^*_a(x_b)]$$

Notice how this works. We start with Bob’s consumption and use this to find
Alice’s consumption. We take the result and use it find Bob’s consumption. If we
get the consumption level we started with, then we have found the consumption
levels that are consistent with one another.

Lets try this in the previous diagram.

Figure 4.2.3:

Start with Bob’s consumption. In the diagram this is $x_0$ and is labelled on
the horizontal axis, where Bob’s consumption is measured. We need to compute
Alice’s consumption. From (4.2.2), this is

$$x^*_a(x_0) = \frac{W + 2px_0}{4p}$$

Using this, we can compute Bob’s consumption by measuring the (horizontal)
distance over to his best reply function. This would be

$$x^*_b\left(\frac{W + 2px_0}{4p}\right) = \frac{W + 2p\left(\frac{W + 2px_0}{4p}\right)}{4p} = x_1$$

It should be clear from the diagram that this won’t give us the right answer
unless we choose $x_0$ at the point where the best reply functions cross. From the
algebra above, this is simply the solution to

$$W + 2p\left(\frac{W + 2px}{4p}\right) = x$$

or

$$\frac{W}{2} = px$$

This is a simple Nash equilibrium.
Chapter 5

Uncertainty

In many problems in economics, people are forced to make decisions without knowing exactly what the consequences will be. For example, when you buy a lottery ticket, you don’t know whether or not you will win when you buy the ticket. There are many important problems like this. If you buy car insurance, you hope you won’t need it, but you aren’t sure. A politician who spends time and money running for public office is not sure whether or not they will be elected. A drug company that invests in developing a new drug is not sure whether or not it will actually work. A corporate insider who sells shares using inside information is never sure whether or not they will be caught.

One way to think about such problems is to use the concept of a lottery. A lottery is a pair of objects. The first, \( X \), is a list of possible consequences of a decision. The second is a list \( p = \{p_1, p_2, \ldots, p_n\} \) of probabilities with which you think that each of the consequences will occur. The number of probabilities you list, \( n \), is exactly equal to the number of consequences in \( X \). For example, if you buy a lottery ticket, the set of consequences consists of two things, you either win or lose. Each of these consequences occurs with probability \( \frac{1}{2} \). If you sell stock based on an insider’s tip, you either get away with it, or you don’t. More generally, there could be many consequences. If you open a new restaurant you might sell 1 or 2 or 3, or any number of meals per week. The set of consequences could be very large.

In this definition, the set of consequences could be very general. In particular, each of the consequences could be a lottery. A lottery over lotteries is called a compound lottery. Lotteries over anything else are called simple lotteries. As an
abstract example, consider the following game. I will flip a coin and if the coin comes up heads right away, I will give you $2. If it comes up tails, I will flip the coin again. If it comes up heads on the second flip, I will pay you $4. If it comes up tails, we stop and you get nothing. This is a simple example of a compound lottery. The first of the two lotteries has a single consequence, you receive $2 (of course you get this consequence with probability 1). The second lottery has two possibly consequences - either you get $4 or you get nothing. In this second lottery, each consequence occurs with equal probability. The compound lottery involves \( \frac{1}{2} \) chance that you will play the first lottery and get $2 for sure. Then there is \( \frac{1}{2} \) chance that you will play the second lottery and get either $4 or $0.

It should be clear to you that this compound lottery is pretty much the same thing as a simple lottery where the consequences are that you receive either $2, $4 or $0 with probabilities \( \frac{1}{2} \), \( \frac{1}{4} \), and \( \frac{1}{4} \). This type of lottery is sometimes referred to as the reduced lottery associated with the original compound lottery.

5.1.1 Monty Hall

We normally assume that compound lotteries and the reduced lotteries associated with them can be used interchangeably. This isn’t always as straightforward as in the example above. In the following problem, it is very easy to make a mistake calculating the reduced lottery. You are a contestant in a game show, and are given the choice of three doors. Behind one door is $1 million which you will win if you happen to open this door. There is nothing behind the other doors, and if you pick one of them you will win nothing. Once you choose one of the doors, the host will open one of the remaining doors and show you that it contains no prize. You are then given the option to change from the door you picked in the first place, to the remaining door. The problem is to decide whether or not to switch doors.

This is a compound lottery in which the prize is first placed randomly behind a door, then after observing where the prize is, the host randomly opens one of the remaining doors. It appears that the prize is equally likely to be behind either of the three doors, so it can’t matter whether you switch or not after the host opens the door. However, you will do better on average in this game if you always switch doors. You might be able to see this from the following casual reasoning - the prize could be behind any of the three doors. If it is behind the door that you chose, then it would, of course, be a mistake to switch. In either of the other alternatives, switching doors will win you the prize. To put it another way, the host will actually tell you which door contains the prize in two of the three situations you might face. Following his advice won’t always work, but it will most of the time.

Another way to think through the problem is to try to compute the reduced lottery that you actually face when you hold on to your original choice, and the one you face when you switch. The first part of the lottery involves the placement of the prize. With probability \( \frac{1}{3} \) it is placed behind either of the three doors. The second part of the lottery is the host’s announcement about the door that doesn’t contain the prize. We might as well assume that you pick
door A, since the thing works the same way no matter which door you pick. The lottery you get when you stick with your original choice is depicted in Figure 1.

Figure 5.1.1: You stick with your initial choice

The important part to understanding the correct strategy in this game is to observe that the outcome of the second lottery depends both on the outcome of the first lottery (the door where the prize is placed) and on your choice. In Figure 1, the first set of branches shows the various locations where the prize can be placed. The second set of branches shows the doors that the host can open. Notice that if your choice is A and the prize is actually there, then the host can choose completely randomly to open either door B or C. On the other hand, if the prize is behind doors B or C, then the host doesn’t have any choice and is forced to open a door that effectively reveals the location of the prize.

Figure 2 shows what the lottery looks like in the case where you switch doors.

Figure 5.1.2: You switch choices after the host opens a door

Then just glancing at the outcomes in these two figures, it should be clear that you will win two thirds of the time if you switch, but only one third of the time if you stick with your initial choice.

5.1.2 St. Petersburg

Here is a famous reduced lottery involving coins that actually provides most of the motivation for the approach that we currently use in economics. This
resembles the previous coin flipping problem. As before, if the coin comes up heads on the first flip, I give you $2, and if it comes up tails on the first flip, I flip again. If it comes up heads on the second flip I give you $4, otherwise I flip again. If it comes up heads on the third flip, I pay you $8, otherwise I flip again. We keep going until I flip a head, then if it takes me \( k \) flips to get the head, I pay you $2\(^k\). The set of consequences in this reduced lottery is the set \( \{1, 2, 3, \ldots, k, \ldots\} \).

It won’t take you too much thinking to see that the probabilities are

\[
\left\{ \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^k}, \ldots \right\}
\]

I am going to ask you how much you would be willing to pay me to play this game. You could refuse to play - then you would get nothing for sure. Or you could offer to pay me $2 or more (you can’t lose from this choice unless you pay more than $2). Both choices would involve different lotteries, though one of them (not playing) is sort of degenerate.

You might try to decide whether or not to play this game by figuring out how much you would win on average from the game. This calculation is straightforward. With probability \( \frac{1}{2} \) you win $2 right away, with probability \( \frac{1}{4} \) you win $4, with probability \( \frac{1}{8} \) you win $8, and so on. Averaging all these gives

\[
\sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} 1 = \infty
\]

You will never find anyone who is willing to pay an infinite amount of money to play this game.

### 5.2 Choosing among lotteries

In each of the problems above, you need to choose among lotteries. There is no obvious way to do this. However, as you actually make the choice, I am probably safe in thinking that you will be able to express a preference over any pair of lotteries, and that the preferences you express will be transitive (in other words, if you say that you prefer lottery \((X, p)\) to lottery \((X', p')\), and you say you like lottery \((X', p')\) more than lottery \((X'', p'')\), then I should be sure that you will prefer lottery \((X', p)\) to lottery \((X'', p'')\).

Now, if there is some set of lotteries \(L\) from which a choice is to be made, I can ask for pairwise comparisons between all the lotteries and eventually learn all of your preferences. To make the notation a little simpler, let me suppose that every lottery in my set of alternatives \(L\) has the same set \(X\) of consequences. Then I can think of a lottery as a simple list of probabilities with which these various outcomes occur. The outcomes don’t have to be amounts of money,
they can be anything imaginable, including lotteries as you have seen. Yet, as
with all choice problems I am probably not too far off the mark by assuming
you can express a preference between every pair of lotteries in \( L \), and that the
preferences you express will be transitive (which means that if \( p \succeq p' \), and
\( p' \succeq p'' \) then it must be that \( p \succeq p'' \)). If your preferences are also continuous in
an appropriate way, I will be able to represent with a utility function \( u \) in the
sense that \( p \succeq p' \) if and only if \( u(p) \geq u(p') \).

One of the more important discoveries in economics is that if your preferences
satisfy a third condition, referred to as the \textit{independence axiom} \(^{1} \), then this utility
function will, in fact, be linear in probabilities. The independence axiom say
this: suppose that for any three lotteries \( p, p', \) and \( p'' \), \( p \succeq p' \) implies that for
any \( \lambda \in [0,1] \)

\[
\lambda p + (1 - \lambda) p'' \succeq \\
\lambda p' + (1 - \lambda) p''.
\]

These last two objects are compound lotteries in which you are given lot-
ttery \( (X, p) \) (or \( (X, p') \)) with probability \( \lambda \) and lottery \( (X, p'') \) with probability
\( (1 - \lambda) \). If you can rank two lotteries, you will rank them the same way if they
are mixed with a common third lottery.

For the utility function to be linear in probabilities, it means that we will be
able to find numbers \( u_i \), one for each of the \( n \) outcomes in the set \( X \) such that

\[
u(X, p) = \sum_{i=1}^{n} u_i p_i
\]

Since there is one number \( u_i \) for each of the \( n \) outcomes in \( X \), it is convenient
to think of \( u_i \) as the utility value associated with outcome \( x_i \). If the outcomes
happened to be expressed in dollars, then we could think of the utility numbers
as representing some underlying utility for wealth. The point to be emphasized
is that the existence of the utility for wealth function is not an assumption, but
an implication of a certain method of ranking lotteries.

This is such an important idea that it is worthwhile to see how it works.
To see it, suppose that there is a \( b \in L \) (the best lottery) such that \( b \succeq p \)
for all \( p \in L \); a \( w \in L \) (the worst lottery) such that \( p \succeq w \) for all \( p \in L \).
We can now try to mimic the proof of the existence of a utility function that
we did previously. Recall that we used monotonicity of preferences and con-
 tinuity. So we will say preferences are monotonic if \( \lambda b + (1 - \lambda) w \succeq \lambda' b + \)
\( (1 - \lambda') w \) if and only if \( \lambda > \lambda' \). \(^{2} \) We will say that preferences are continuous
if the sets \( \{ \lambda \in [0,1] : \lambda b + (1 - \lambda w) \succeq p \} \) and \( \{ \lambda \in [0,1] : p \succeq \lambda b + (1 - \lambda) w \} \)
are both closed intervals.

Now we can state the result.

\(^{1}\)This is a bit of a misnomer. It isn’t really an axiom, since it is far from self evident. It is
more like an assumption.

\(^{2}\)The statement \( p \succ p' \) means that \( p \succeq p' \) but not \( p' \succeq p \). The notation \( p \sim p' \) means that
\( p \succeq p' \) and \( p' \succeq p \).
**Theorem 1** If preferences are complete, transitive, continuous, monotonic and satisfy the independence axiom, then there is a utility function $u$ representing preferences over lotteries in $\mathcal{L}$ that is linear in probabilities.

**Proof** The proof is constructive. Let’s start by creating the utility function. This is a function that assigns a real number to each lottery $p \in \mathcal{L}$. To do so set

$$u(p) = \{\lambda \in [0, 1] : \lambda b + (1 - \lambda) w \sim p\} \quad (5.2.1)$$

Notice that the $\lambda$ that satisfies this relation (warning - it is not an equation) always exists and is unique. To see why, observe that by completeness $\lambda' b + (1 - \lambda') w \succeq p$ or the reverse for every $\lambda' \in [0, 1]$. Then

$$\{\lambda \in [0, 1] : \lambda b + (1 - \lambda) w \geq p\} \cup \{\lambda \in [0, 1] : p \succeq \lambda b + (1 - \lambda) w\}$$

is all of the interval $[0, 1]$. Since both these sets are closed by the continuity of preferences, they must have at least one point in common. Since preferences are monotonic they can’t have more than one point in common (Prove this by contradiction.)

Next, we should show that the function $u(\cdot)$ as constructed above, actually represents preferences $\succeq$. This relies on the monotonicity of preferences and is left as an exercise.

Finally, we come to the important step - showing that the utility function defined in (5.2.1) above is linear in probabilities, i.e., that

$$u(\lambda p + (1 - \lambda) p') = \lambda u(p) + (1 - \lambda) u(p')$$

for all $\lambda$, $p$, and $p'$. I will not write down a series of relations - make sure that you don’t mistake them for equations. First observe that

$$\lambda p + (1 - \lambda) p' \sim$$

$$\lambda [u(p) b + (1 - u(p)) w] + (1 - \lambda) p'$$

This follows from the definition of the utility function $u(p)$ and the independence axiom (in the sense that we are mixing the lotteries $p$ and $u(p) b + (1 - u(p)) w$ together with the common third lottery $p'$). Do the same thing again to get

$$\lambda [u(p) b + (1 - u(p)) w] + (1 - \lambda) p' \sim$$

$$\lambda [u(p) b + (1 - u(p)) w] + (1 - \lambda) [u(p') b + (1 - u(p')) w]$$

This is a fairly complicated compound lottery (first you mix lotteries $b$ and $w$ together using $u(p)$ and $u(p')$. Then you mix the result using $\lambda$.) The reduced lottery associated with this is

$$[\lambda u(p) + (1 - \lambda) u(p')] b + [1 - \lambda u(p) - (1 - \lambda) u(p')] w$$

56
which, if you recall where we started, is then indifferent to $\lambda p + (1 - \lambda) p'$. If you glance back at (5.2.1), you will see that we have just discovered

$$u(\lambda p + (1 - \lambda) p') = \lambda u(p) + (1 - \lambda) u(p')$$

which is the linearity property we wanted.

\[\square\]

Modern economic theory is concerned largely with problems where there is some kind of risk or uncertainty about outcomes. In finance, this uncertainty arises from the inherent unpredictability of asset returns. In mechanism design (the theory of auctions and institutions), uncertainty arises because of the inability to know the tastes of others. In game theory, uncertainty arises because of the inability to predict exactly how another player will behave. Expected utility is the cornerstone of the modern approach to uncertainty, so it is probably one of the most useful ideas that you will encounter.

It has been challenged in a number of ways. The challenges reflect both the strength of the theory and its weakness. The strength of the theory lies in the fact that is lays out so precisely what can and cannot happen if the theory is true. The ‘can happen’ part is good, because theories are supposed to explain things we see. The ‘can’t happen’ part is also important since it shows what kinds of behavior would allow us to reject the theory.

You might wonder why we need a theorem like the one above connecting expected utility which is a model of the utility function, to the independence axiom, which you might think of as a restriction on the way people behave. Why couldn’t we just write down a specific utility function then somehow test that, instead of worrying about behavioral properties like the independence axiom.

There are two reasons. The first is, that provided you buy completeness, transitivity and continuity, the independence axiom and expected utility are equivalent. So the independence axiom provides all the behavioral restrictions that come from assuming the utility function is linear in probabilities. Assumptions about utility functions will typically make it easy to derive some restrictions on behavior, but not all of them. Knowing all of the implications makes it far easier to construct an effective empirical test.

The second reason is that it is quite possible to impose assumptions on utility that have no implications at all for behavior. A trivial example might involve assumptions about the utility value associated with the best or worst lotteries. In any case, theorems like the one above lay out very clearly what the additional implications of linearity are, and how they differ from the implications of other assumptions.

### 5.3 Empirical tests

I’ll provide an example of a challenge to expected utility that involves experimental tests (we already described how econometric tests could be used to test the implications of completeness and transitivity). To describe the test, let me simplify things a bit and suppose that the set of consequences consists of exactly
three things - i.e., $\mathcal{X} = \{x_1, x_2, x_3\}$. The set of lotteries, $\mathcal{L}$, is then just the set of triples of probabilities $q = \{q_1, q_2, q_3\}$. Since the probabilities have to sum to one, it is possible to depict $\mathcal{L}$ in a simple two dimensional diagram as in Figure 5.3.1.

Every point in the triangle with sides of length 1 in the diagram above is a lottery in $\mathcal{L}$. To see this, take a point like $q$. The coordinate on the horizontal axis, $q_1$ represents the probability with which consequence $x_1$ occurs. The coordinate on the vertical axis $q_2$ represents the probability with which consequence $x_2$ occurs under lottery $q$. The probability of consequence $x_3$ is just the remainder $1 - q_1 - q_2$. This is given by the length of the horizontal line segment from $q$ over to the hypotenuse of the triangle (the dashed line with the arrow at the end in the picture). Now the lottery $q'$ can be easily compared to $q$. $q'$ lies down, and to the right of $q$, and is closer to the hypotenuse of the triangle. So, it assigns lower probability to $x_1$, higher probability to $x_2$, and lower probability to $x_3$.

Each point in the triangle represents a different simple lottery in $\mathcal{L}$. Compound lotteries are lotteries over lotteries. For example, one might form a lottery with two consequences - consequence 1 is the lottery $q$ while consequence 2 is the lottery $q'$. Let $\lambda$ be the probability with which the first lottery $q$ is played. Then the reduced lottery associated with this compound lottery is $\lambda q + (1 - \lambda) q'$. This lottery is just the simple lottery that lies $\lambda/(1 - \lambda)$ of the way along the line segment between $q$ and $q'$. This point is illustrated in Figure 5.3.1.

Preferences over lotteries then consists of a family of indifference curves that look exactly like the ones you are used to using to think about preferences over commodity bundles. An indifference curve through the lottery $q$ is set of lotteries.
that have the same utility value as \(q\). Formally, an indifference curve is the set
\[
\{ \tilde{q} \in \mathcal{L} : u(\tilde{q}) = u(q) \}
\]
If we use the linearity property of preferences given in our theorem above, then the equation that defines this indifference curve is
\[
q_1 u_1 + q_2 u_2 + q_3 u_3 = u(q)
\]
Since \(q_3 = 1 - q_1 - q_2\), this becomes
\[
q_1 u_1 + q_2 u_2 + (1 - q_1 - q_2) u_3 = u(q)
\]
or
\[
q_2 = \frac{u(q) - qu_1 - (1 - q) u_3}{u_2 - u_3}
\]
This is a linear function of \(q_1\), which means that indifference curves are straight lines.

To see the argument another way, go back to the independence axiom, which states that for any three lotteries \(q\), \(q'\) and \(q''\) and any \(\lambda\), the lotteries \(\lambda q + (1 - \lambda) q''\) and \(\lambda q' + (1 - \lambda) q''\) must be ranked the same way as \(q\) and \(q'\). Since this must be true for any three lotteries, then it must be that we can use \(q'\) in place of \(q''\) in the argument above to conclude that \(\lambda q + (1 - \lambda) q'\) and \(\lambda q' + (1 - \lambda) q''\) must be ranked the same way as \(q\) and \(q'\). Then reducing the compound lottery, if \(q' \sim q'\) then \(\lambda q + (1 - \lambda) q'\sim q''\). Which means that every lottery on the line segment between \(q\) and \(q'\) must be indifferent to \(q\). That is just another way of saying that the indifference curves are straight lines.

It is a bit hard to deal with indifference. If you offer me \(q\) and \(q'\) in an experiment I will choose one of the them. I might strictly prefer the one I pick, or I might be indifferent between them. It would be hard to figure this out in practice. Fortunately, the independence axiom provides a much stronger condition that gets around this. You can see this condition in Figure 5.3.2.

In the Figure, \(q\) and \(q'\) are two lotteries on the same indifference curve. By the previous argument the indifference curve connecting them is a straight line. Choose some other lottery like \(p\) which strictly preferred to \(q\). We shall try to determine what the indifference curve looks like through the lottery \(p\).

Draw a line segment from \(q\) through \(p\) to some third lottery \(q''\) as in the figure. Suppose that \(\lambda\) is such that \(p = \lambda q + (1 - \lambda) q'\). Now since \(q \sim q'\), we have by the independence axiom that
\[
p = \lambda q + (1 - \lambda) q'' \sim \lambda q' + (1 - \lambda) q''
\]
Now the line segment from \(p\) to \(\lambda q' + (1 - \lambda) q''\) is evidently parallel to the line segment from \(q\) to \(q'\), so the indifference curve through \(p\) must be parallel to the indifference curve through \(q\). In other words, all the indifference curves are parallel to one another when the independence axiom holds.

Now this is something that can be tested with an experiment. Simply present an experimental subject a choice between two lotteries and observe their choice.
Once you see what their choice is, present them another two lotteries whose probabilities are scaled in such a way that knowing that indifference curves are all straight and parallel to one another will allow you to predict their choice in the second lottery.

This is the experiment that was suggested by Allais. The consequences are monetary with $x_1 = $100, $x_2 = $50 and $x_3 = 0$. The first pair of lotteries offered to the experimental subject are

$$q = \{0, 1, 0\}$$

and

$$q' = \{.1, .89, .01\}$$

In other words, you can have either a sure $50, or take a chance on getting $100 with a small chance that you will lose everything. Most people are inclined to take the sure $50 in this case.

The second pair of lotteries is

$$p = \{0, .11, .89\}$$

and

$$p' = \{.1, 0, .9\}$$

In this case, you probably won’t win anything with either lottery. Lottery $p$ offers you a small chance to earn $50. Lottery $p'$ gives you a slightly smaller chance of earning $100, but also increases the probability with which you won’t win anything. Most people are inclined to take the chance in this case and opt
for lottery $p'$ - perhaps because they are so unlikely to win anything they feel there is nothing to lose in going for the $100.

You should plot each of these four lotteries in a Figure like the one above. You will see that the line segment joining $q$ and $q'$ is parallel to the line segment joining $p$ and $p'$. If indifference curves are all straight lines, parallel to one another, anyone who chooses $q$ over $q'$ must also choose $p$ over $p'$ (just draw in the indifference curve that would induce them to choose $q$ over $q'$ then shift it down to see what they will do with $p$ and $p'$).
Chapter 6

Expected Utility and Risk Aversion

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6.1 Introduction

This reading describes how people’s aversion to risk affects the decisions they make about investment. Basically, the concepts used to do this analysis emerge naturally when people have expected utility preferences, but not otherwise. So, it illustrates one important way that expected utility is applied.

This analysis also makes it possible to illustrate how to do comparative statics. Comparative statics typically involve calculations designed to show the direction in which changes in the environment move peoples’ optimal decisions. Convincing comparative static results are ones that hold even if you only impose weak restrictions on preferences. So, for instance, to explain how an increase in price affects a buyer’s demand when he or she has Cobb-Douglas preferences is not very convincing because Cobb Douglas preferences are a very special case. In fact, as we have already shown, an (uncompensated) increase in price will only reduce demand under very special conditions. That is why the method for doing comparative statics tend to be a little more sophisticated, and the questions asked tend not to be the most obvious ones.

6.1.1 Lotteries with a Continuum of Outcomes

In portfolio theory, outcomes are all monetary. We looked at monetary lotteries in some of the example above. Yet it doesn’t make sense when thinking about stocks or bonds to restrict to only three or four outcomes, there are really an
infinity of possible outcomes. The way to think about this is to think of the set of potential outcomes \( \mathcal{X} \) as an interval of \( \mathbb{R} \). Of course, we can’t assign a positive probability to each of these outcomes the way we have so far. So instead we describe a lottery as a probability distribution function, \( F(x) \) that takes each possible value \( x \) in the set of potential outcomes and gives the probability that the actual outcome is less than or equal to \( x \).

We could actually describe the lotteries we have considered so far this way. For example, in the last chapter we thought about lotteries with three outcomes, in particular, the outcomes were \{1000, 500, 0\}, where each of these outcomes is supposed to be a monetary outcome. We assigned probability \( p_1 \) to the first outcomes, \( p_2 \) to the second and \( p_3 = 1 - p_1 - p_2 \) to the third.

Suppose we think of \( \mathcal{X} = [-100, +2000] \) as the set of possible outcomes. Then the following diagram shows you what the corresponding probability distribution function looks like for this lottery. The function \( F(x) \) starts out at 0 when \( x \) is equal to \(-100 \) which is the lowest possible monetary payoff.

For every value of \( x \) between \(-100 \) and 0 the probability that the outcome of the lottery is less than or equal to these values is constantly zero. However, the probability that the outcome of the lottery is less than or equal to 0 is exactly equal to the probability that 0 occurs, in other words, \( p_3 \), the probability that we have been assigning to the worst outcome. As the monetary payoff travels between 0 and 500, say a value like \( x \) in the diagram, the probability that the outcome is less than or equal to \( x \) is then constant at \( p_3 \). Suddenly it jumps up to \( p_2 + p_3 \) when \( x \) is 500, and so on, until it reaches 1 and stays there after \( x \) is 1000.

The previous chapter showed how the independence axiom makes it possible to compare different lotteries by computing the expected utility associated with those lotteries. The expected utility associated with the lottery \{\( p_1, p_2, p_3 \)\} for example, is given by

\[
p_1 u(1000) + p_2 u(500) + p_3 u(0)
\]

The lottery \{\( p_1, p_2, p_3 \)\} is the same as the probability distribution described in
Now suppose that we want to describe a lottery with an infinite number of outcomes, for example, a share that has a return that could be anything between $-100$ and $2000$. We can’t describe such a lottery by assigning probabilities to each possible outcome in this case. There are just too many of them. We could, however, still describe the lottery by giving the probability distribution function associated with it. An infinite lottery might have a probability distribution that looked like the one in the next figure.

This lottery (i.e., the probability distribution function associated with this lottery) has the same values at 0, 500, and 1000 as the previous lottery - $p_3$, $p_2 + p_3$, and 1. However, at a point like $x$, the probability that the outcome in this lottery is less than or equal to $x$ is strictly higher than $p_3$ which was its value in the previous lottery.

The expected utility theorem suggests that if the independence axiom holds for compound lotteries described as distribution functions, then we should be able to assign a utility value to all the outcomes between $-100$ and $2000$. Then if we could somehow multiply these utility values by something like probabilities associated with each outcome we would have a single utility value for the lottery. The way to do this is to integrate the utility values multiplied by the amount by which the probability distribution function increases at each possible outcome. This expression looks like

$$\int_{-100}^{2000} u(x)dF(x)$$

where $dF(x)$ means the change in the probability distribution function. For a lottery like the first of the two above, $dF$ is equal to zero at every point except for the three special outcomes, 0, 500, and 1000. Then, $dF$ takes the values $p_3$, $p_2$ and $p_1$ respectively. Then if we add up by integrating as above we just get the sum

$$p_1u(1000) + p_2u(500) + p_3u(0)$$

In a case like the second lottery, $dF$ is the derivative of the probability
distribution (sometimes called its density). In that case you would write $dF$ as $F'(x)dx$ where $F'(x) = \frac{dF(x)}{dx}$. That is the formulation you have probably seen before.

As an example, one special probability distribution function is the *uniform* distribution function. The idea behind it is that every possible outcome is equally likely. So if you take a point midway through the set of outcomes, i.e. 1050, then the probability the outcome is less than or equal to 1050 should be equal to $\frac{1}{2}$. The probability distribution that does this is

$$F(x) = \frac{x - (-100)}{2000 - (-100)}$$

The expected utility associated with this lottery is

$$\frac{1}{2100} \int u(x)dx$$

because of the fact that $F'(x) = \frac{1}{2100}$. Then using your high school algebra, the expected utility associated with the lottery whose probability distribution function is uniform is just the ratio of the area under the utility function between $-100$ and $2000$, to the area in the rectangle whose sides have length 1 and 2100.

### 6.1.2 Risk Aversion

As described above, the independence axiom implies that there is a *utility for wealth* function that people use to evaluate lotteries. One reasonable question to ask is whether we could guess something about the shape of this function from other properties of behavior. For example, the St. Petersburg bet that we discussed in the last chapter has an infinite expected value, but no one would pay an infinite amount to engage in the bet.

An even simpler example might be the following - I propose a new grading scheme for the exam. You can choose either of the following options - either take the mark you get, or I will flip a coin - heads I raise your mark by 10 points, tails I lower your mark by 10 points. Your expected grade is the same under either scheme. You probably wouldn’t want the second scheme because it is risky. An A grade could turn into a B, a pass could turn into a fail (or conversely).

The second grading scheme involves what is called a *fair* gamble in the sense that the expected gain to you of taking the bet is exactly zero. It seems plausible that most people simply wouldn’t be interested in taking a fair bet.\footnote{However, it should be added that the bets that people take in a casino are not fair. For example, if you fed coins into a slot machine for the rest of your life you would lose a lot of money. This doesn’t seem to stop people from going to casinos.}

This kind of behavior does have some implications for the shape of the utility function.

\footnote{The lotteries that you buy into at the corner store (like the Lotto 6-49) represent a strange variant on what happens in casinos. The way the lotteries work is roughly as follows - the lottery sells tickets. From the total revenue they earn by ticket sales, they take out a big chunk to cover operating costs and to give themselves some profit. The rest is put in a pool.}
You can what the implications are in the next figure.
Suppose our consumer has some initial level of wealth $W$. A fair bet is one whose expected return is zero. One example would be a bet that pays $x$ with probability $\frac{1}{2}$, but which costs you $x$ with the same probability. Doing nothing leaves you with $W$ for sure. The independence axiom implies that there is a utility level $u(W)$ associated with this outcome. The outcomes $W-x$ and $W+x$ also have associated utility values $u(W-x)$ and $u(W+x)$. These are marked in the diagram. The expected utility of the lottery we just described is then

$$\frac{1}{2}u(W-x) + \frac{1}{2}u(W+x)$$

In the diagram, this utility level is the distance from the point $W$ on the horizontal axis up to the point $c$ in the diagram. The assertion that our consumer would rather not engage in this bet means that the utility value of $W$ (for sure)

A number is then drawn randomly by dropping balls from an urn (or some other method that is independent of the number of tickets). If one or more people has the winning number they split all the money in the pool. The interesting event occurs when no one has the ticket. Then, when the lottery is run the next period, a portion of ticket sales is added to the pool which already contains the pool from the last lottery. If no one wins for a long time, the pool can become so large that the lottery ticket will be a more than fair bet. For example, as this is written the chance of winning the 6-49 is about 1 in 18 million, but the current pool of money to be won is about 40 million. So the lottery ticket appears to have an expected value of a little over $2. It only costs $2 to buy a ticket. So it appears that everyone should buy a ticket. Many people do, leading to enormous revenues and profits for the lottery corporation. This is a bit misleading because the odds of winning are independent of the number of bidders. That means that if many people bid, more than one person is likely to have a winning ticket. The average payout to a winner will be considerably smaller than $40$ million for this reason, which makes the expected value of the ticket smaller than $2.

To see this observe that the vertical distance between the points $d$ and $e$ is $u(W+x) - u(W-x)$. By similar triangles, the ratio of the vertical distance between $a$ and $c$ (call it $|c-a|$) to $u(W+x) - u(W-x)$ is the same as the ratio of the horizontal distance between $a$ and $d$ to $W+x-(W-x)$. This latter ratio is equal to $\frac{1}{2}$. Therefore

$$|c-a| = \frac{1}{2}(u(W+x) - u(W-x))$$
must be above c at a point like b. Since this preference not to take fair bets is likely to be true no matter what x is and no matter what W is, means that the utility for wealth function must lie everywhere above the line segment from d to e. In other words, the utility for wealth function must be concave.

6.1.3 Measuring Aversion to Risk

Stocks and bonds are more complex than lotteries because the monetary outcomes usually aren’t fair. Most traded stocks for example have a positive expected return. The thing that makes stock trading interesting is that it involves considerable risk. Whether or not an investor will want an initial public offering of some stock depends partly on the risk characteristics of the stock, and partly on the investors own attitudes toward risk. The economics literature has suggested some interesting ways of separating these two things.

Let F be the probability distribution over outcomes that is associated with some lottery. The expected payoff associated with the lottery is

\[ \mathbb{E}_x = \int x dF(x) \]

where \( dF(x) \) depends on the nature of the lottery as discussed above. Let’s assume for now that the probability distribution function F has a density so that we can write the expected payoff (and all the expected utility calculations) using the better known manner

\[ \mathbb{E}_x = \int x F'(x) dx \]

Let W be any level of wealth. Let’s try to calculate the maximum amount our consumer would be willing to pay to play this lottery F. Using the independence axiom we would find this price p by solving

\[ u(W) = \int u(W - p + x) F'(x) dx \]

It is going to tough to solve this exactly without more precise information about u so let’s solve it ’approximately’.

First take a second order Taylor approximation to the utility function inside the integral sign on the right hand side of the equation. A first order Taylor approximation won’t work very well here because it only gives a good approximation when x is small. However, there will typically be some risk that x could be quite large, so some correction for the curvature in u will help. The second order approximation is given by

\[
\int \left\{ u(W) + u'(W)(x-p) + u''(W) \frac{(x-p)^2}{2} \right\} F'(x) dx
\]

Now adding \( u(W - x) \) to both sides gives

\[ |c - d| + u(W - x) = \frac{1}{2} u(W - x) + \frac{1}{2} u(W + x) \]
If this is a good approximation, then it is approximately equal to \( U(W) \) when we choose the right \( p \), so simplify the equation

\[
\int \left\{ u(W) + u'(W)(x - p) + u''(W) \frac{(x - p)^2}{2} \right\} F'(x) \, dx = u(W)
\]
to get

\[
u'(W) \int (x - p)F'(x) \, dx + u''(W) \int \frac{(x - p)^2}{2} F'(x) \, dx = 0
\]
or

\[
p = \int xF'(x) \, dx + \frac{u''(w)}{u'(w)} \int \frac{(x - p)^2}{2} F'(x) \, dx
\]
The second term in this expression has a \( u''(w) \) which is negative if the utility for wealth function is concave. So the expression says that the maximum amount the consumer will be willing to pay for the lottery is equal to its mean return less a term consisting of two parts. One part is the integral which is related to the variance of the lottery or the degree to which it is spread out and unpredictable. The other part depends only on the consumer’s utility for wealth function. The larger in absolute value is the ratio \( \frac{u''(w)}{u'(w)} \), the less the consumer will be willing to pay.

This particular decomposition has proved to be very useful in thinking about portfolio theory. The term \( \frac{u''(W)}{u'(W)} \) is referred to as the Arrow-Pratt measure of absolute risk aversion.

We will shortly see how this measure can be used to get some insight into the way that investment decisions work. For the moment, the main thing to note is that the Arrow-Pratt measure is a conceptual device that emerges with the help of the expected utility theorem. One natural assumption would seem to be that the risk premium that a consumer would be willing to pay to avoid risk would be lower if the consumer were more wealthy. The formulation so far shows exactly how to formalize this idea - the function \( \frac{u''(W)}{u'(W)} \) should be a decreasing function. We will come back to this idea momentarily.

### 6.1.4 The Portfolio Problem

A fairly simple version of the portfolio problem can be studied by assuming that there are exactly two different securities that an investor can buy. A security is a lottery. The consequences of the lottery are possible rates of return on investment. If the consequence of the lottery is a rate of return \( s \), then the security pays \( 1 + s \) dollars tomorrow for each dollar that is invested in the lottery today. In the problem that we are going to analyze, one of the two securities is safe in the sense that the lottery that produces the rate of return gives a rate of return of zero for sure. So, each dollar invested today gives back
exactly one dollar tomorrow, no more and no less. There is also a risky security
where the probability distribution function for the random rate of return is \( F \).
Sometimes this rate of return will be positive and the security will pay back
more than one dollar for each dollar invested. Other times the rate of return
will actually be negative, and a dollar invested will return less than one dollar
tomorrow.

The investor has \( w \) dollars to invest in these two securities. His ex post
income (i.e., his income tomorrow after the rate of return on the lottery is
realized) when he invests \( i_s \) in the safe security and \( i_r \) in the risky security, is
given by

\[
i_s + i_r (1 + s)
\]

A pair \((i_s, i_r)\) is called a portfolio. Each portfolio generates a different lottery
over monetary outcomes. Then relying on the independence axiom, we can
invoke the expected utility theorem and conclude that there is a utility for
wealth function \( u \) such that one portfolio (and its associated lottery) \((i_s, i_r)\) is
preferred to another \((i'_s, i'_r)\) if

\[
\int u (i_s + i_r (1 + s)) F' (s) ds \geq \int u (i'_s + i'_r (1 + s)) F' (s) ds.
\]

If that is true, then the investor should choose the portfolio that maximizes

\[
\int u (i_s + i_r (1 + s)) F' (s) ds
\]

subject to the constraints that

\[
i_s + i_r \leq W \\
i_r \geq 0 \\
i_s \geq 0
\]

We could solve this using Lagrangian methods, recalling that before you do
so, you need to tease out some properties of the solution in order to guess which
constraints are likely to be important. To see a way to do this, simply imagine
that the integral above is just a particular function

\[
U (i_s, i_r) \equiv \int u (i_s + i_r (1 + s)) F' (s) ds
\]

and that the first constraint above is just a budget constraint where the prices
of both securities are just equal to 1. So, we should be able to find the optimal
portfolio by finding the highest indifference curve that touches the budget
constraint, as the indifference curve \(II\) does in Figure 1.
The slope of the indifference curve is given by the marginal utility of the good on the horizontal axis (i.e., $i_s$) divided by the marginal utility associated with the good on the vertical axis (i.e., $i_r$). Differentiating gives

$$\frac{\partial U(i_s, i_r)}{\partial i_s} = -\frac{\int u'(i_s + i_r (1 + s)) F'(s) ds}{\int u'(i_s + i_r (1 + s)) (1 + s) F'(s) ds}$$

Now, following our usual procedure, we can try to check the conditions under which we might find a solution at the corner where all wealth is invested in the riskless asset. This would occur if the indifference curve happened to be steeper than the budget line. So, let's evaluate the slope of the indifference curve at the bundle $(w, 0)$. Substituting into the formula above gives

$$-\frac{\int w' (w) F'(s) ds}{\int w' (w) (1 + s) F'(s) ds} = -\frac{1}{1 + \int s F'(s) ds}$$

This means that whether or not the investor optimally invests in the risky asset depends entirely on $\int s F'(s) ds$. If this is positive, the (absolute value of the) slope of the indifference curve is smaller than 1 and the tangency has to occur somewhere up the budget line to the left of $(w, 0)$. On the other hand, if the mean value of $s$ is less than zero, then the indifference curve will be steeper than the budget line, and the optimal solution will be at the corner of the budget set.

This is called the diversification theorem. If there is a risky asset whose expected return exceeds the expected return on the safe asset, then the optimal portfolio will always involve some investment in the risky asset.

**Comparative Statics and Wealth**

We might assume that wealthy people invest more in risky assets. This assertion seems that it must be true. Surprisingly, this is not always the case. It is difficult
to see why this might be by using intuition alone. As I have often mentioned, this is when mathematics can be very useful. Intuition is rarely wrong: it just doesn’t give the whole story. Math can often help you think out the parts that you tend to gloss over when you think intuitively. Often the biggest insights in economics come by understanding the complexities that intuition can’t see.

The problem that is discussed in this section also provides an opportunity to see the way that comparative statics is often done in applications. We are interested in what the effect of a change in wealth \( w \) will be on the optimal portfolio. One way to figure this out is to try to find out how a change in wealth will affect the position of the tangency between the indifference curve and budget line. To see how this line of argument works, start by substituting the fact that \( i_s = w - i_r \) into the tangency condition to get

\[
\frac{\int u'(w - i_r + i_r (1 + s)) F'(s) ds}{\int u''(w - i_r + i_r (1 + s))(1 + s) F''(s) ds} = 1
\]

Simplifying the arguments of the functions gives

\[
\frac{\int u'(w + i_r s) F'(s) ds}{\int u'(w + i_r s) (1 + s) F'(s) ds} = 1
\]

Multiplying both sides by the denominator on the left gives

\[
\int u'(w + i_r s) F'(s) ds - \int u'(w + i_r s) (1 + s) F'(s) ds = 0
\]

Canceling the common terms gives the equation

\[
\int u' (w + i_r [w] s) sF' (s) ds = 0 \quad (6.1.1)
\]

The trick at this point is to assume that \( i_r \) is actually a function of \( w \) that adjusts in such a way that the equation above is always satisfied. Then, in fact,

\[
\int u' (w + i_r [w] s) sF' (s) ds \equiv 0
\]

Since this holds uniformly under this definition, we can differentiate both sides of the expression with respect to \( w \) and the derivatives will also be equal. In other words

\[
\int u'' (w + i_r [w] s) sF'(s) ds + \frac{di_r [w]}{dw} \int u'' (w + i_r [w] s) s^2 F'(s) ds = 0
\]

Solving for the derivative gives

\[
\frac{di_r [w]}{dw} = -\frac{\int u'' (w + i_r [w] s) sF'(s) ds}{\int u'' (w + i_r [w] s) s^2 F'(s) ds}
\]

We want to know how an increase in \( w \) will change the amount invested in the risky asset. In other words, we want to know whether the derivative of the
optimal value of $i_r$ with respect to a change in wealth will be positive. This expression almost gives the answer. The denominator is an integral. Each term in the integrand is negative provided the investor is risk averse (which gives $u'' < 0$). There is a minus sign in front of the fraction, and minus times minus is positive. So, we could conclude that the derivative is positive if we could show that the numerator is positive. Unfortunately, this is not obvious if it is true. The derivative $u''$ is certainly negative, but $s$ can be either positive or negative. The sign of the integral will depend on how big $u''$ is when $s$ is negative compared with how big it is when $s$ is positive.

The lesson of the comparative statics has now been discovered. To conclude that increases in wealth raise investment, we need more information. Or, we need to restrict the set of preferences that we think are plausible. Fortunately, there is a fairly easy restriction that will do this trick. We have described the Arrow-Pratt measure of absolute risk aversion $\frac{u''(w)}{u'(w)}$. It is proportional to the size of the risk premium associated with fair gambles. Suppose we assumed that this measure of risk aversion is decreasing with wealth. That would mean that we would be assuming (or restricting attention to) investors whose risk premium falls as they become more wealthy. Surely these investors must raise investment as their wealth increases. So, let’s check this out.

Write out the Arrow-Pratt measure as it appears in our comparative static equation. Then we would have

$$-\frac{u''(w + i_r s)}{u'(w + i_r s)} \leq -\frac{u''(w)}{u'(w)}$$

whenever $s > 0$ by the assumption that the Arrow-Pratt measure is decreasing.

Well the whole problem arises from the fact that we don’t know that $s$ is positive. So, let’s try a trick. If $s$ is positive, it must also be true that

$$-\frac{u''(w + i_r s)}{u'(w + i_r s)} s \leq -\frac{u''(w)}{u'(w)} s$$

The nice thing about this expression is that if we change the sign of $s$ to negative (which of course means that $-\frac{u''(w + i_r s)}{u'(w + i_r s)} \geq -\frac{u''(w)}{u'(w)}$) then it would still be true that

$$-\frac{u''(w + i_r s)}{u'(w + i_r s)} s \leq -\frac{u''(w)}{u'(w)} s$$

So, this last expression is actually correct no matter what the sign of $s$. So, let’s multiply both sides of this inequality by $u'(w + i_r s)$ then integrate the result over $s$ to get

$$-\int u''(w + i_r s) s F'(s) ds \leq -\frac{u''(w)}{u'(w)} \int u'(w + i_r s) s F'(s) ds$$

If you look back, you will see that the right hand side of this equation is proportional to the left hand side of (6.1.1) which is zero. So, we have

$$-\int u''(w + i_r s) s F'(s) ds \leq 0$$
which is just the result we wanted (since it shows that \[ \int u''(w + is) sF'(s) ds \geq 0. \])

After all this work, what we have discovered is that an investor will increase his investment in the risky asset as his wealth rises provided his Arrow-Pratt measure of absolute risk aversion is decreasing as his wealth increases.
Chapter 7

Insurance

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7.1 Introduction

In this chapter, we study a very simple model of insurance using the ideas and concepts developed in the chapter on risk aversion. You may recall from the previous chapter, the concept of a lottery: a collection of outcomes that are each assigned a different probability of occurring. We are going to assume that the independence axiom holds, and that our consumers and decision makers have preferences that are linear in the probabilities with which these different outcomes occur.

Expected utility says that we should start with a list of all possible outcomes. Let’s just think of the set of outcomes as a finite set \( S \) (later on we will refer to them as states). Sometimes, we let \( S \) refer to the number of outcomes in this set. We let \( s \) refer to a particular outcome in this set. A lottery is a vector of probabilities \( p \) consisting of \( S \) non-negative components, the sum of which must equal 1. The expected utility theorem says that there is a vector of real numbers \( \{ u_1, \ldots, u_S \} \) such that two lotteries \( p \) and \( q \) can be ranked in the following way: \( p \succeq q \) if and only if

\[
\sum_{s=1}^{S} p_s u_s \geq \sum_{s=1}^{S} q_s u_s
\]

In most of what we do in economics, including the insurance problem, we fix the probabilities \( p \) then vary the numbers \( u_s \) in some fashion. This is exactly what we are going to do here with the insurance problem.

On the surface, the expected utility theorem doesn’t seem to support this approach. In fact, it is a very restrictive special case which you can understand
if you think of the set of outcomes as a continuum, and the set of lotteries as a set of probability measures on this continuum. At the moment, the details of this argument aren’t so important. All you need to remember is that we will use the expected utility theorem to support the arguments here.

Let’s apply these ideas to insurance. We start with a consumer who has an income $y$ but who expects to have an accident with some probability $p$. The accident has a known monetary cost $d$. The consumer deals with a competitive insurance company which sells policies to many similar consumers and knows the probability with which the consumer has an accident.

A good example might be a farmer whose fields produce a yearly income $y$ unless there is a late frost which kills off some of his crop. The probability of a late frost is $p$. A bad example would be car insurance. There are a couple of reasons why. The first is that the probability of an accident is something that a driver has some control over. If you are insured against the costs of an accident, then there is little incentive to take care. This is referred to as moral hazard. A farmer, on the other hand, has little control over the weather. Drivers also have a lot of information about their own accident probability that the insurance company doesn’t. The weather, on the other hand, is unpredictable, but in a way that everyone agrees.

The insurance company will sell a policy to the consumer. A policy is a premium $q$ and a net benefit $b$. The net benefit is the difference between the gross benefit $B$ and the premium $q$. The way the insurance policy works is that the consumer pays the insurance company $q$ up front. Then, if (and only if) the consumer has an accident, the insurance company pays the consumer $B$. Equivalently, if the consumer has an accident, the insurance company pays back the premium $q$ and gives the consumer an additional $b$ to make up for her loss.

The insurance company is doing this with many different consumers and is willing to offer the consumer any policy that breaks even in the sense that the company’s expected profit from the policy is zero. The expected profit to the company from a policy with premium $q$ and net benefit $b$ is

$$(1 - p)q - pb$$

Our consumer has expected utility preferences, so if she buys a policy with premium $q$ and net benefit $b$, her expected utility is

$$(1 - p)u(y - q) + qu(y - d + b)$$

The consumer’s problem is to buy a policy that maximizes her expected utility.

Let’s solve the problem two different ways. First, we use the graphical approach. To see this approach, we can convert the problem into something that looks exactly like the consumer problems we have already studied.
The horizontal axis in the diagram measures the amount of income the consumer enjoys in the event that she does not have an accident. If she doesn’t buy insurance, this will just be $y$. If she does buy insurance, it will be $y$ less the premium that she pays. The vertical axis measures the level of consumption in the event that our consumer does have an accident. In this diagram, the consumer’s *endowment* is the point $(y, y - d)$. The consumer can switch from her endowment to the consumption pair $(y - q, y - d + b)$ by buying a policy with premium $q$ and net benefit $b$.

In the diagram, I have drawn this as if it created an entire budget line for the consumer. The reasoning is as follows: the insurance company will presumably be willing to sell the consumer any policy that generates zero expected profit, i.e., any policy that satisfies $(1 - p)q - pb = 0$, or $b = q \frac{1 - p}{p}$. The set of all such policies generates a feasible set that passes through the endowment point (since the policy $(0, 0)$ obviously gives the insurance company zero expected profit) and has slope $-\frac{1 - p}{p}$.

The consumer’s indifference curves are made up of all the pairs $(c_n, c_a)$ that yield the same level of expected utility. So for example, the indifference curve through the endowment is given by the set of solutions to

\[(1 - p)u(c_n) + pu(c_a) = (1 - p)u(y) + pu(y - d) \tag{7.1.1}\]

Using the method of total differentials, the slope of the indifference curve can be derived by solving

\[(1 - p)u'(c_n)dc_n + pu'(c_a)dc_a = 0 \tag{7.1.2}\]

or

\[
\frac{dc_a}{dc_n} = -\frac{(1 - p)u'(c_n)}{pu'(c_a)} \tag{7.1.3}
\]

A very nice property of expected utility preferences is that if $c_a = c_n$, then $u'(c_a) = u'(c_n)$. If, in addition, the function $u$ is concave (has a decreasing first
derivative), then the indifference curves will appear as convex curves as in the following picture:

![Figure 7.1.2:](image)

Since the slope of every indifference curve is $-\frac{1-p}{p}$ on the 45° line, the highest indifference curve’s tangency point to the budget set will occur on the 45° line. The nice implication is that a risk averse consumer who can buy insurance at ‘actuarially fair’ prices (i.e., prices that give the insurance company zero expected profit) will buy insurance up to the point where his consumption is the same whether or not she has an accident.

To be complete, let’s solve this as well using Lagrangian methods because it proves insightful. First, the consumer’s problem (at least when dealing with a fair insurance company) is to maximize

$$(1-p)u(y-q) + pu(y-d+b) \quad (7.1.4)$$

by choosing a premium and benefit package $(q, b)$ subject to the constraint that

$$(1-p)q - pb = 0 \quad (7.1.5)$$

Notice that there is nothing here about the premium and benefit being positive. It is conceivable that the consumer might want to bet with another consumer that she would not have an accident. In that case, she would receive money from the other consumer when she didn’t have an accident, which coincides with a negative value for $q$. Then, to satisfy the constraint, she would have to pay out in the event that she did have an accident.

Since there are no inequality constraints, our Lagrangian theorem says that at the optimal solution, there will be a multiplier $\lambda$ such that

$$-(1-p)u'(y-q) + \lambda(1-p) = 0 \quad (7.1.6)$$

and

$$pu'(y-d+b) - \lambda p = 0 \quad (7.1.7)$$

The probabilities cancel in this expression, so we get $u'(y-q) = u'(y-d+b) = \lambda$. If we assume that the marginal utility of income is monotonically declining, then this can only occur when $y-q = y-d+b$. 

78
Chapter 8

First Welfare Theorem In Production Economies

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8.1 Profit Maximization

Firms transform goods from one thing into another. If there are two goods, $x$ and $y$, then a firm can transform $x$ into $y$ or $y$ into $x$ depending on what consumers want. The first figure below represents a simple production technology that the firms might use to do this. In the figure, the function $y = f(x)$ represents the feasible production combinations available to the firm. One option for the firm
is to use good \(x\) as an input and produce \(y\) as an output, as it does at the point \((x^1, y^1)\). At this point, the input \(x\) is negative and the output \(y\) is positive. On the other hand, the firm could as easily use \(y\) as an input (\(y\) is negative) and produce \(x\) as an output as it does at the point \((x^0, y^0)\).

Of course, the firm has to make some choice about which of the possible combinations of inputs and outputs that it makes. Economists like the following view: the firm, in fact, owns all of the endowments of good \(x\) and \(y\). Consumers, in turn, own the firm. If there are two consumers, 1 and 2, we could let \(\theta\) be the proportion of the firm owned by consumer 1 while \((1 - \theta)\) is the proportion owned by consumer 2. Let \(\omega_x\) and \(\omega_y\) be the total amounts of good \(x\) and \(y\) that are available to the firm. The firm then gets to choose, so to speak, the aggregate endowments of goods \(x\) and \(y\) available in the economy when it chooses it’s production.

Let \(z_x\) and \(z_y\) be the aggregate amounts of \(x\) and \(y\) that the firm chooses to make available in the economy. If the firm decides that it wants to produce good \(y\) from good \(x\), meaning that \(z_y > \omega_y\) and \(z_x < \omega_x\), then it needs to use up some of its endowment of good \(x\) and use it in production of good \(y\). So, \(z_y = \omega_y + f(z_x - \omega_x)\). (Remember that to get \(y\) out of the production process, we need a negative argument for good \(x\) the way \(f\) is drawn in Figure 1. On the other hand, if it wants to produce good \(x\) (that is \(z_x > \omega_x\)), then it has to use up some of its endowment of good \(y\) (so, \(z_y < \omega_y\)). Then

\[
z_y = \omega_y + f(z_x - \omega_x)
\]

where recall that \(f\) will be negative whenever the input to \(f\) is positive.

Now, imagine drawing a picture as in Figure 2 of the function

\[
z_y = \omega_y + f(z_x - \omega_x)
\]
as \(z_x\) varies between \(\omega_x\) and \(z : f(z - \omega_x) = \omega_y\). This function is usually referred to as the production possibilities frontier. It is given as the line segment \(CD\) in Figure 2.

Now, suppose that the prices for \(x\) and \(y\) are given by \(p\) and 1, respectively (I keep using 1 for the price of \(y\) because it is only the relative price of good \(x\) that makes a difference to the firm or the consumers). Whatever the firm chooses to produce it can sell to the consumers at prices \(p\) and 1. So, the profit, or revenue, of the firm is just \(pz_x + z_y\), if it produces \((z_x, z_y)\). An iso-profit locus is a collection of productions that give the same profit. The line segment \(AB\) in Figure 2 gives part of one such production locus. There are a family of such loci—all of the lines that are parallel to \(AB\). If the firm maximizes profits, it picks the highest iso-profit curve that touches its production possibilities frontier. This gives the production choice \((z_1, z_2)\) in Figure 2.

Once the firm has produced this output, it distributes its profits back to its shareholders: the fraction \(\theta\) goes to consumer 1, and \((1 - \theta)\) goes to consumer 2. Since the consumers use these profits to finance their purchases, both of them would prefer that the firm choose the aggregate production that maximizes its profits since that will always provide them with their highest income for
consumption. The income of consumer 1 is $\theta(pz_1 + z_2)$. This means that the budget line that consumer 1 faces is the one that intersects the $x$-axis at the point $\theta(pz_1 + z_2)/p$ (since that is the maximum quantity of good $x$ he would be able to purchase with that income).

The outcome at an arbitrary (i.e. not equilibrium) price pair $(p, 1)$ is given in Figure 2.

Each of the two consumers chooses the best point in his or her budget set (which occurs where their indifference curves are tangent to their corresponding budget lines). The choice by consumer 1 is the consumption bundle $(x_1, y_1)$ in the figure. Consumer 2’s choice should be read with respect to the coordinate system that starts at the point $(z_1, z_2)$. So, the quantity of good $x$ that consumer 2 wants is given by the horizontal distance between the point $x'$ and the point $z_1$. From this, you can see that the total demand for good $x$ (which is given by $x_1 + (z_1 - x')$) exceeds the total amount of good $x$ that is produced by the firm. So, the relative price of $x$ should rise.

As the relative price of $x$ rises, the family of iso-profit curves faced by the firm will all get steeper. This will cause the firm to choose a profit-maximizing level of output on its production possibilities frontier that involves more $x$ and less $y$. As one might expect, the rising price of good $x$ will cause both consumers to demand a little less $x$ and a little more $y$. Eventually, the increase in supply of $x$ and the reduction in demand will bring the market to a state of equilibrium.
8.2 Competitive (Walrasian) Equilibrium

A competitive (Walrasian) equilibrium is a pair of consumption choices \((x_1^*, y_1^*)\) for consumer 1 and \((x_2^*, y_2^*)\) for consumer 2, and a production plan \((z_1^*, z_2^*)\) for the firm such that there is a price \(p'\) for good \(x\) for which the following things are true:

1. \(x_1^* + x_2^* = z_1^*; y_1^* + y_2^* = z_2^*\) (the markets clear);
2. \(p'z_1^* + z_2^* \geq p'z_1 + z_2\) for any pair \((z_1, z_2)\) on the firm’s production possibilities frontier; and
3. \(u_1(x_1^*, y_1^*) \geq u_1(x_1, y_1)\) for all \((x_1, y_1) : p'x_1 + y_1 \leq \theta(p'z_1^* + z_2^*)\) and \(u_2(x_2^*, y_2^*) \geq u_2(x_2, y_2)\) for all \((x_2, y_2) : p'x_2 + y_2 \leq (1 - \theta)(p'z_1^* + z_2^*)\).

You can see what happens after the price of good \(x\) rises (to \(p'\)) in Figure 3. In the picture, consumer 1 now has income \(\theta(p'z_1^* + z_2^*)\) which he uses to buy \(x_1^*\) units of good \(x\). Now, consumer 2 chooses to buy \(z_1^* - x_1^*\) units of good \(x\), and the markets clear.

8.3 First Welfare Theorem

The first welfare theorem is one of the most important contributions of classical microeconomic theory. It says that no feasible allocation exists in which all
consumers are better off than they are in the competitive equilibrium. This is similar to the argument that we made for an exchange economy: consumer indifference curves must be tangent at any equilibrium. However, production adds another wrinkle. It might be true that both consumers could be made better off if the firm would just behave in a different fashion and pick some production plan that doesn’t necessarily maximize profits.

Actually we can argue that if we take any pair of consumption bundles where both consumers are better off, then there can be no production plan that makes this feasible. To see this, suppose that \((x'_1, y'_1)\) and \((x'_2, y'_2)\) are consumption bundles such that

\[
u_1 (x'_1, y'_1) > u_1 (x^*_1, y^*_1)
\]

and

\[
u_2 (x'_2, y'_2) > u_2 (x^*_2, y^*_2)
\]

One observation is immediate. Whenever this is true, it must be that

\[
p' x'_1 + y'_1 > \theta (p' z_1^* + z_2^*)
\]

and

\[
p' x'_2 + y'_2 > (1 - \theta) (p' z_1^* + z_2^*)
\]

The reason for this is that consumers choose the very best consumption bundles that they can afford with their income. If they could have afforded \((x'_1, y'_1)\) or \((x'_2, y'_2)\), then they certainly would have chosen them.

Now, if the firm is maximizing its profits

\[
p' z_1^* + z_2^* \geq p' z'_1 + z'_2
\]

for any \((z'_1, z'_2)\) along the firm’s production possibilities frontier. So, it must be that

\[
p' x'_1 + y'_1 > \theta (p' z'_1 + z'_2)
\]

and

\[
p' x'_2 + y'_2 > (1 - \theta) (p' z'_1 + z'_2)
\]

Then, if we add these two inequalities together, we get

\[
p' (x'_1 + x'_2 - z'_1) + (y'_1 + y'_2 - z'_2) > 0
\]

If prices are positive, then at least one of the two expressions \((x'_1 + x'_2 - z'_1)\) and \((y'_1 + y'_2 - z'_2)\) are strictly positive, which means the firm simply can’t produce enough to supply what consumers want.

So, it is good for firms to maximize profits in two senses. First, if the firm were to propose some alternate production plan which didn’t involve profit maximization, both consumers would expect their income to fall (notice that this is partly because they don’t expect the change in production plan to have any effect on prices). So, the shareholders of the firm would unanimously vote against such a change. Second, even if the firm could change its production plan, and even if prices do change, the alternate plan can’t possibly make both consumers better off. Notice that when we make either of these arguments, we don’t value profits of firms for their own sake. We are only concerned with the utility of consumers.
8.4 Distribution

One thing you should notice about this entire construction is that firms don’t, in any sense, create wealth or goods. The ability to create is embedded in the production possibilities frontier which is taken as a given.¹ All firms do is decide how to allocate this wealth. Many potential ways to choose among alternate production plans exist. For example, the government could choose the entire production plan. This was the model used in centrally-planned economies like the old Soviet Union. Alternatively, one could imagine a mixture of private, profit-maximizing firms and publicly-regulated companies, similar to what happens in most Western economies.

Profit maximization isn’t necessarily good for everyone. To see this, consider the following example in which one of the consumers (say consumer 1) simply has an endowment of the goods $x$ and $y$ but does not own shares. The other, consumer 2, owns a firm that can transform $x$ into $y$ (and conversely)—possibly by buying some of the endowment of consumer 1. The firm that is owned by consumer 2 starts with some endowment of goods.

To begin, suppose that the government declares that the firm is not allowed to produce anything and that it simply has to give its endowment to consumer 2 who can use the endowment to finance the best consumption plan possible. Restrictions like this are pretty common. An example might be a zoning restriction that prevents a homeowner from turning her house into an apartment building, or a farm owner who is not allowed to build housing on his farm land. A possible outcome is shown in Figure 4.

In Figure 4, consumer 1 starts with an endowment equal to $(\omega_1^x, \omega_1^y)$. Consumer 1 owns no shares of the firm. The firm owns an endowment $(\omega_2^x, \omega_2^y)$ and this firm is in turn owned by consumer 2. If the firm simply offers its endowment for sale on the market then the feasible set of trades is given by the wide flat box. Prices adjust until the relative price of $x$ is $p'$. In the associated exchange equilibrium, consumer 1 receives the allocation $(x_1^*, y_1^*)$. Consumer 2’s indifference curve is tangent to this point, so – conditional on the production decision of the firm – no allocation can make both consumer 1 and consumer 2 better off. Notice that in this equilibrium, consumer 1 is selling some of his endowment of good $y$ in order to acquire good $x$. Of course, consumer 2 is doing the opposite: selling off the good $x$ that the firm provides in order to acquire good $y$.

The iso-profit curves faced by the firm are all parallel straight lines whose slopes are equal to $-p'$. Plotting the aggregate endowment point on the production possibilities frontier shows that simply selling off the endowment does not maximize the firm’s profits. The iso-profit line (with slope $-p$) is flatter than the production possibilities frontier. Consumer 2 will do better if the firm alters its production plan to produce some additional $y$ from the endowment of good $x$ because this will increase the income that consumer 2 takes to the market. To put it in a slightly different way, observe that the steep production possibilities

¹The traditional theory of the firm has nothing to say about where this production possibilities frontier comes from. This makes it pretty useless in thinking about things like economic development, or economic growth.
curve means that consumer 2 can acquire good $y$ much more cheaply by having the firm produce it than she can by buying it from consumer 1.

If the firm is now free to raise its output of $y$, it will create an excess supply of good $y$ that will make $y$’s relative price fall, or make $x$’s relative price rise. Both these things are bad for consumer 1 who is buying $x$ and selling $y$. When the markets finally clear, the relative price of good $x$ will level off at $p$, and consumer 1 will end up at point $E$ where he is much worse off than he was in the original equilibrium.

Oddly enough, this new equilibrium is Pareto optimal. There is no way to make both of the consumers better off by changing the firm’s production plan. How does one reconcile this with the fact that consumer 1 was so much better off at the endowment point?

One possible answer that may have occurred to you is that the endowment point is also Pareto optimal. This is partly right and partly wrong. The slope of the production possibilities curve, at the point where the iso-profit curve $A'B'$ crosses it, is the marginal rate of transformation between $x$ and $y$. If you think of using one small unit $dx$ of $x$ as the input, then the slope gives the amount of $y$ that you get back out for each such unit. Think now about the point $(x_1^*, y_1^*)$ where the indifference curves for the consumers are tangent. The indifference curves both have slope $-p'$ at this point which is the same as the marginal rate of transformation along the production possibilities frontier.

So, what is the slope of consumers 1’s indifference curve? His marginal rate of substitution is the amount of good $y$ you would need to give him to
compensate him when you take away a little \((dx)\) of his good \(x\). Consumer 2’s indifference curve is tangent at this point. That means if you do a tiny transfer of good \(x\), say \(dx\), from consumer 1 to consumer 2, and consumer 2 compensates 1 by giving him \(dy\) in exchange, where \(dy\) is 1’s (and 2’s) marginal rate of substitution, neither 1 or 2 are any better off.

Instead of transferring good \(x\) from consumer 2 to consumer 1, suppose that 1 transfers a tiny bit of good \(x\) to 2 who uses it to produce additional \(y\) using the production function. The production possibilities frontier is steeper than 1’s indifference curve, so this will give 2 more than enough output to pay 1 his marginal rate of substitution and maintain his utility. But then, all the residual output will be left over for consumer 2 to enjoy. In other words, when the production possibilities frontier is steeper than both consumer’s indifference curves, 2 can take a little of 1’s good \(x\) and use it to produce \(y\), which he uses to pay 1 back for the good \(x\). Then, 2 will have some output left over for himself. The endowment point can’t be Pareto optimal.

On the other hand, if Pareto optimality is the only objective, it can be achieved at the initial endowment point simply by moving along the contract curve until the consumers’ marginal rates of substitution are equal to the marginal rate of transformation in production. Figure 5 shows how this might be accomplished.

Notice that at the point \(E'\) in Figure 5, the indifference curves are both tangent and have the same slope as the production possibilities curve at the endowment point. So, everything is Pareto optimal. This might be achieved
by having the government regulate the firm’s output choice, tax away some of the firm’s profit (a tax on dividends or capital gains) and then redistribute the proceeds to consumer 1. Consumer 1 will love this alternative plan, and consumer 2 will hate it, but it will produce a Pareto optimal outcome.

Pareto optimality is a perfectly sensible objective for economic policy to try to accomplish. Profit maximization by firms is one way to achieve this, but you need to remember that it is a means to a goal, not a goal in itself. You should also try to remember, as Figure 5 illustrates, that there are alternative ways of achieving Pareto optimality. Different methods lead to different distributional consequences - so, even though all consumers will agree that they want the outcome to be Pareto optimal, they may sensibly disagree about how this is accomplished.
Chapter 9

Public Goods

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9.1 Introduction

A public good is a good having the property that the total output of the good is enjoyed by everyone. In contrast, a private good has the property that if I consume it, you cannot. Some of the more important goods in the modern economics are actually public. For example, an idea or a bit of information is a public good. Whether or not you know the idea, or have the information does not impact on my ability to know the same thing or have the same information (though whether or not you have the same information as I do may determine whether or not I can make money off you). A music file shared on my computer is a public good – if you take it from me, my ability to enjoy it is not affected at all. A theorem, like the first welfare theorem that we studied last week, is a public good: I do not forget it when you learn it. Contrast this with my car or my lunch which are both private goods: if you take them from me I cannot enjoy them at all.

This note describes an equilibrium for the voluntary contribution game, which is a common way to think about how public goods are provided. It explains why the amount of the public good in the voluntary contribution game is underproduced (because the outcome is not Pareto optimal: there is another outcome that will make everyone better off). This note then explains how public goods can be priced to ensure Pareto optimal supply.
9.1.1 The voluntary contribution game

Suppose there are two goods, one public \((y)\) and one private \((x)\). Let \(f(x)\) denote the amount of the public good that can be produced from \(x\) units of the private good. Suppose there are two consumers with utility functions \(u_1(x, y)\) and \(u_2(x, y)\) respectively. Their endowments of the private good are \(\omega_1\) and \(\omega_2\). The set of points \(\{(x, y) : y = f(\omega_1 + \omega_2 - x)\}\) is the production possibilities frontier.

If the first consumer decides to consume \(x_1\) (and devote the rest of his endowment \(\omega_1\) to production of the public good) while consumer 2 decides to consume \(x_2\) the utilities of each of the consumers are given by

\[
\begin{align*}
u_1(x_1, f(\omega_1 + \omega_2 - x - x_2)) \\
u_2(x_2, f(\omega_1 + \omega_2 - x_1 - x_2))
\end{align*}
\]

for consumer 1 and

for consumer 2. The important point is that if consumer 1 say decides to consume a bit less of the private good and produce a bit more of the public good, then consumer 2 will enjoy the additional public good too without any cost at all.

The voluntary contribution game equilibrium is one way to predict how much of the private good each consumer will choose. Each of the consumers simply picks the amount of the private good they want on their own. This is a bit hard to do because the amount that each consumer will choose to contribute depends on how much they expect the other consumer to contribute. A good way to do this is to use a Nash equilibrium in place of a Walrasian equilibrium. Instead of taking prices to be fixed, each consumer takes the contribution of the other consumer to be fixed and chooses the contribution that maximizes his utility given this expectation. In a Walrasian equilibrium, when a consumer acts as if he believes that prices are fixed, he has to be physically able to purchase the bundle that maximizes his utility at these prices. That is why the price expectations have to be such that markets clear. Analogously, when the consumer chooses his optimal contribution given a fixed expectation about the contribution of the other consumer, he has to end up with exactly the amount of the public good that he expected to get.

Formally, a Nash equilibrium for the voluntary contribution game is a pair of private consumptions \(x_1^*\) and \(x_2^*\) such that

\[
\begin{align*}u_1(x_1^*, f(\omega_1 + \omega_2 - x_1 - x_2^*)) & \geq u_1(x', f(\omega_1 + \omega_2 - x' - x_2^*)) \\
\end{align*}
\]

for any alternative contribution \(x' \in [0, \omega_1]\) and

\[
\begin{align*}u_2(x_2^*, f(\omega_1 + \omega_2 - x_1^* - x_2^*)) & \geq u_2(x', f(\omega_1 + \omega_2 - x_1^* - x'))
\end{align*}
\]

for any alternative contribution \(x' \in [0, \omega_2]\).

One way to view the outcome of this game is given in Figure 1 where the two consumers choose \(x_1^*\) and \(x_2^*\). The ‘budget line’ that consumer 1 faces,
for example, when consumer 2 chooses consumption $x^*_2$ is the set of all pairs $\{(x_1, y) : y = f(\omega_1 + \omega_2 - x_1 - x^*_2)\}$. The slope of this is exactly the same as the slope of the production possibilities frontier at the point $(x^*_1 + x^*_2, y^*)$. The same is true for consumer 2. So, in the equilibrium of the voluntary contribution game, each consumer has the same marginal rate of substitution and the same marginal rate of transformation in production. With private goods, this is exactly what you want. Recall that, in the Edgeworth box, both consumers’ indifference curves were tangent and the common slope of their indifference curves was equal to the slope of the production possibilities frontier. With public goods, this is not the outcome that you want.

An alternative approach – that helps to explain how contributions are determined and why these contributions are not Pareto optimal – is to try find the best choice for consumer 1 to make for all the different possible contribution levels that consumer 2 might choose. This approach is more common in game theory and involves the construction of something called the best reply function. The best reply functions are put together to understand the final outcome.

In Figure 2, the various consumption choices that player 2 can make are given along the bottom axis. Each such choice implies a contribution to the production of the public good - just take the difference between the consumption level and the endowment to find it (this figure would look a bit different if player 2’s contributions to production of the public good were listed on the bottom axis – that approach is more common).

The green lines represent iso-utility curves for consumer 1. They are solutions to equations of the form

$$u_1(x_1, f(\omega_1 + \omega_2 - x_1 - x_2)) = K$$

where $K$ is some constant. Consumer 1 achieves higher utility (holding his own consumption level constant) the lower is the consumption level of consumer 2.
Lower consumption by consumer 2 means that consumer 2 is contributing more of her endowment to produce the public good.

In the Nash equilibrium, consumer 1 forms some belief about the consumption level that consumer 2 will pick. Suppose for the moment, that he believes that consumer 2 will choose consumption \( x'' \). Then he can attain any \((x_1, x_2)\) combination that lies on the vertical line through \( x'' \). The best such point is the one that lies on the highest iso-utility curve. That is the one through the point \((R_1[x'', x''])\) where an iso-utility curve is just tangent to this vertical line. (If he were to lower his planned consumption by moving down this vertical line, he would end up on a lower iso-utility curve like the one that lies just to the right of the point \((R_1[x'', x''])\).

There would be a different best choice like this for every different choice that consumer 2 makes. The picture shows the corresponding tangencies at point \( x' \) and \( x''' \). If you joined all these best choices together they would form a line (not necessarily straight as it is in the picture) called consumer 1’s reaction function. This is the line \( R_1R_1 \) in the picture. It explains what consumer 1 would choose to do for every possible different belief that he might have about the consumption choice of consumer 2.

Doing the same exercise for consumer 2 yields a similar curve, which is drawn in Figure 3 as \( R_2R_2 \). The point where these two curves intersect is the Nash equilibrium. Each consumer chooses his best consumption given what he expects the other consumer to choose; and, as it turns out, the other consumer always does exactly what he expects him to do.

When you try to construct the iso-utility curves for person 2, he will choose one that is tangent to the flat line through \( x_1^* \) since he expects consumer 1’s consumption choice to \( x_1^* \) no matter what he does. This means that the iso-utility curves for the two consumers must cut through each other as shown in the diagram. There must be a point like \( E \) where both of the consumers would be better off if they could jointly agree to move there. That would involve each
of them reducing their own consumption of the private good and using that to increase production of the public good.

9.2 Resolving the Public Good Problem

There is a rather surprising way to resolve this problem. The entire difficulty with public goods arises because a consumer who raises his or her contribution to the public good increases the utility of the other trader. To get consumers to choose the right amounts something has to be done to 'internalize' this externality. Here is one way to do it.

First declare that there are actually 3 goods, \( y_1, y_2, \) and \( x \). The first is public good for person 1, the second public good for person 2 and the third the private good. All production of these three goods will be undertaken by a single profit-maximizing firm whose production possibilities frontier is just

\[
\{(y_1, y_2, x) : y_1 = y_2 = f(\omega_1 + \omega_2 - x)\}
\]

All endowments are owned by the firm, but consumer 1 owns the share \( \omega_1 / (\omega_1 + \omega_2) \) of the firm and will receive that share of its profits. Consumer 2 will own the complementary share \( \omega_2 / (\omega_1 + \omega_2) \). We will make one big assumption, which is that this single firm is a price-taker.

Consumer 1 only cares about his consumption of the private good and his consumption of good 1; consumer 2 only cares about her consumption of the private good and good 2. Consumer 1 doesn’t care how much \( y_2 \) consumer 2 consumes, so there are no externalities in consumption. The firm ‘internalizes’ all the externalities associated with the public good. The physical connection between \( y_1 \) and \( y_2 \) is simply a part of its production process, so there are no production externalities. So, all we need to do is to find the Walrasian equilibrium of this economy with production and that will give us a Pareto optimal
allocation by the first welfare theorem that we studied last week. The solution is given in Figure 4.

Notice that, when the firm increases production of the public good, it receives revenue twice on each unit it produces. Consumer 1 pays $p_1$ for that unit, but consumer 2 also pays $p_2$ for it. So, the iso-profit curve for the profit maximizing firm that must be tangent to the production possibilities frontier has slope $\frac{1}{p_1+p_2}$. The firm earns its profits for its production decision then distributes these profits to its owners. Consumer 1 receives income

$$\frac{\omega_1}{\omega_1 + \omega_2}(p_1 y^* + p_2 y^* + x_1^* + x_2^*)$$

which is labelled on the horizontal axis in Figure 4. Consumer 2 receives

$$\frac{\omega_2}{\omega_1 + \omega_2}(p_1 y^* + p_2 y^* + x_1^* + x_2^*)$$

which is the point where 2’s budget line intersects the horizontal axis (that has not been labeled in the figure to keep things simpler).

Consumer 1 now faces a budget line (the blue line in the picture) along which he chooses his best consumption bundle. Notice that since consumer 1 only buys good $y_1$ at price $p_1$ – the slope of this budget line is $\frac{1}{p_1}$ not $\frac{1}{p_1+p_2}$. So in this equilibrium, consumers marginal rates of substitution will be different from the firm’s marginal rate of substitution and generally different from each other.

The market clearing conditions are twofold. First, consumers must both choose to purchase the common level of output of the public good that has been offered by the firm. Second, the sum of the private good demand of each consumer must be equal to the total amount of the private good that the firm has chosen to produce.

The prices that support this outcome are often know as Lindahl prices. The reciprocal of the slope of the production possibilities frontier is the marginal cost.
of producing one extra unit of the public good (expressed in terms of units of the private good). Since the iso-profit line must be tangent to the iso-profit curve, this marginal cost is equal to $p_1 + p_2$. The reciprocal of the slopes of the consumers' indifference curves are equal to their marginal willingness to pay for the public good (again expressed in terms of the private good). Since the indifference curves are tangent to the individual budget lines, these willingnesses to pay are $p_1$ and $p_2$ respectively. So, the Lindahl prices ensure that the marginal cost of producing the public good is exactly equal to the sum of the two consumers' willingness to pay.
Chapter 10

What Does Undergrad Micro Theory say about Music Downloading

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“Most of us would never even consider stealing something, say a picture or a piece of clothing from a friend’s house. Our sense of right and wrong keeps most of us from doing something so selfish and antisocial. Yet when it comes to stealing digital recordings of copyrighted music, people somehow seem to think the same rules don’t apply even though criminal penalties can be as high as five years in prison or $250,000 in fines. Contrary to popular opinion, illegally downloading or copying copyrighted music is the same as stealing; there is no difference.” - Christian Music Trade Association (www.cmta.com)

“No, I’d say that of the world’s economies, there’s more that believe in intellectual property today than ever. There are fewer communists in the world today than there were. There are some new modern-day sort of communists who want to get rid of the incentive for musicians and movie makers and software makers under various guises. They don’t think that those incentives should exist.” - Bill Gates Cnet interview (news.com.com/Gates taking a seat in your den/2008-1041 3-5514121.html)

“Because of the ad skips.... It’s theft. Your contract with the network when you get the show is you’re going to watch the spots.
Otherwise you couldn’t get the show on an ad-supported basis. Any time you skip a commercial or watch the button, you’re actually stealing the programming.” - Jaimie Kellner, CEO Turner Broadcasting

“Music is everybody’s possession. It’s only the publishers who think people own it.” - John Lennon

Despite the righteousness of the Christian Music Association, there is an important difference between a picture or a piece of clothing and a music file. When you steal a picture or a sweater of mine, I don’t have them anymore. If you download a file from my computer, you can’t diminish my enjoyment of the same song on the same file at all. Neither do I hurt you in any way if I copy a software program from your computer, or watch the same TV show as you. No matter how rhetorically attractive this sounds to a music company’s lawyers, you can’t steal digital music or a television show.

On the other hand, courts – especially in the United States – have supported large corporations who have tried to sue file swappers. This hasn’t apparently had much impact on file sharing in general, but has shutdown websites associated with file sharing; for example, Napster and, more recently, Lokitorrent were shutdown. The motivations of the Recording Industry Association of America (RIAA) are clear in trying to close these websites: they feel that most people who download music over the internet would pay the record company if they could not get the song for free. It seems the American courts agree with this and feel the recording companies are entitled to their money.

Not all countries agree on music downloading. For example, downloading music files from p2p networks in Canada for personal use is apparently legal (http://www.cb-cda.gc.ca/new-e.html). This isn’t a loophole, there is a specific “fair dealing” exemption that applies to personal use, nor does Canada have particularly weak copyright legislation. Canadian policy toward music downloading is simply different than it is in the US. Since Americans can pretty easily download music files from Canadian computers, you can see that this situation might create problems for the RIAA.

Since only communists – according to Mr. Gates – could possibly disagree with strong copyright legislation being applied to music and software, you might expect simple microeconomic theory to make all of this pretty easy to understand. In particular, basic micro theory – as it is currently taught in, say, second-year university courses – must show why copyright is such a fundamental part of capitalist economics. It would also be nice to have this explanation in a form that could be freely downloaded using bittorrent instead of paying a lot of money to a publisher. This short article is the result.

The bottom line is the same as it often is in economics. Strong copyright legislation is neither all good nor all bad. It benefits some people greatly, and hurts others. Since ‘strong copyright’ is a very crude approximation to the method that one is supposed to use to create incentives for artists and musicians, it can screw things up pretty badly for everyone. Finally, ‘strong copyright’
always comes with a cost. As it is implemented in the US, copyright implies monopoly. Monopoly power will typically be used to restrict output. In this sense, copyright must always work against the provision of music and software at the same time that it works for it (which is one reason why Canadian law differs from US law). Whether copyright is good or bad depends on whether the monopoly effect is bigger or smaller than any incentive effect. This will vary with the product involved, so copyright may be good for music and bad for software or conversely.

The first part of this reading simply shows the basic logic behind copyright as applied to music downloading. When a musician creates a digital recording of a performance, the digital recording is freely available to everyone with an internet connection. Basically, the act of making a recording conveys a very special kind of positive externality: everyone can enjoy the recording equally without diminishing the enjoyment of anyone else. This is what economists often call a public good. Competitive markets by themselves don’t create the right incentives for musicians to produce this public good. So, some kind of intervention is needed to make things work nicely. This note describes the Lindahl solution to the problem, which was actually invented in 1919 (any micro text should have this). This solution ignores some important problems, but it explains the basic rationale for copyright protection in the first place, and makes it possible to illustrate in a simply way why it might also fail.

From my perspective, the music downloading problem is a nice way to illustrate the principles behind public goods. The simple model described in this note also makes it possible to discuss some of the distributive consequences of the Lindahl solution. Left to their own devices, musicians won’t produce ‘enough’ music. The Lindahl solution will implement a Pareto optimal outcome. In other words, at the Lindahl allocation itself, it won’t be possible to do anything that makes everyone better off at the same time. Any change in policy must hurt someone. That doesn’t mean, though, that imposing a Lindahl solution on a market which is otherwise uncontrolled will make everyone better off. It may or may not. It is simple enough to illustrate situations where the imposition of the Lindahl solution makes non-musicians worse off. It is obvious why this should be – they initially get free music. After Lindahl, they have to pay, and the new music that is created may not make them any better off at all.¹

The theory and implementation of the Lindahl solution are two entirely different things. For example, for Lindahl to work, we need to know how much everyone is willing to pay for music. Obviously, we don’t have this information. Copyright legislation is the compromise – just treat everyone the same way (it seems fair as well). Since this way of trying to implement the Lindahl solution distorts it, this method can also distort the final equilibrium outcome, as well. The example below shows that it can distort the outcome in ways that actually make everyone worse off than they would have been if musicians had simply been left to their own devices without copyright protection.

¹Lovers of both classical and industrial music are united by the fact that they won’t be any better off if the music companies offer them more Britney Spears records.
The main problem with copyright legislation is that it grants monopoly powers to the copyright holder. If the copyright holder then tries to maximize profits, he or she will restrict output and raise price, which partly (or possibly completely) defeats the purpose of the copyright legislation in the first place. Of course, no musician really wants to restrict the number of people who listen to his music. Yet, he will typically sell the copyright to a corporation who has exactly this intention in mind. These costs associated with monopoly power must be weighed against the incentive effects of copyright before the public interest can be determined.

This exact trade off explains the differences between copyright law in Canada and the US. Canada uses alternative means to create incentives for musicians, which allows it to limit the monopoly costs associated with traditional copyright. We discuss this briefly at the end of the note.

### 10.1 What is Wrong with Leaving Things Alone

Let’s start with a situation where everyone is left to their own devices: musicians make music, everybody else does what they do. Musicians aren’t going to disappear because people download music files. Some people argue that musicians like file sharing because it gives more visibility to their music.

We will imagine that there are two consumers, only one of whom can produce music out of his or her endowment. Call her the musician. The music that she produces is a pure public good - her music is enjoyed equally by both of them. To make things transparent, assume that both consumers have an equal endowment $\omega$ of the private good and that the production technology is linear. Each unit of the private good that the musician gives up in order to create her music produces exactly $b$ units of music.

In this environment, the musician faces a ‘budget constraint’ that looks exactly like the one used in standard consumer theory. She can consume her entire endowment of $\omega$ units of the private good if she wants. Alternatively, she could convert her entire endowment into music (maybe spend all her time on her music). The production technology in this case would give her $b\omega$ units of music. Anything in between is also possible. All of these possibilities are summarized in a simple diagram in Figure 10.1.1 by the line connecting the point $\omega$ on the horizontal axis with the point $b\omega$ on the vertical axis.

The musician’s preferences are summarized by a family of indifference curves, which are convex toward the origin. She chooses the highest indifference curve that lies in her budget set and picks the combination $(x^*, y^*)$. The other consumer can’t do a lot about this. He cannot make music from his own endowment, so he has nothing to add to what the musician has already produced. On the other hand, the music that the musician has produced is available for download, and the non-musician can consume all of it if he wants to. His consumption bundle is then the point $(\omega, y^*)$. This outcome is usually referred to as the equilibrium of the voluntary contribution game. Everyone acts on their own and does the best they can with what is available to them.
The non-musician does considerably better than the musician in all of this, so you might not think that the outcome is ‘fair’ in some sense. The real problem with this outcome is not so much that it isn’t fair. The real problem is that it is possible to change things in such a way that makes both the musician and the non-musician better off. The basic idea is to have the non-musician pay the musician some of his endowment to make a little more music.

Just to make things work out in a provocative way, let’s suppose that both musicians and non-musicians have identical quasi-linear preferences. That means that each of them has a utility function like this:

$$U(x, y) = x + v(y)$$

where \(v\) is some concave increasing function representing utility for the public good. We can conveniently think of good \(x\) as just being some quantity of money. The interesting thing about quasi-linear utility functions is that the marginal rate of substitution between the public and private good depends only on consumption of the public good. No matter the output of the public good, both consumers will always have the same marginal rate of substitution, since they will always consume the same amount of the public good.

Then in Figure 10.1.1, the slope of the musician’s indifference curve at \((x^*, y^*)\) is exactly the same as the slope of the non-musician’s indifference curve at \((\omega, y^*)\). How are we going to go about making both of them better off? You have probably heard of schemes like the following: the non-musician promises to match dollar for dollar any further contribution that the musician makes. If the musician accepts this, and if she contributes another dollar to making music, then the non-musician will throw in another dollar that the musician can use. Next, each dollar that she contributes generates 2\(b\) units of music instead of only \(b\) units of music, assuming the musician doesn’t just run away with the money.
The non-musician still can’t make any music, but he has changed the musician’s incentives. The new ‘budget set’ the musician faces is shown in Figure 10.1.2. If the musician gives up an additional dollar of her endowment, the non-musician will match it and she will be able to produce $2b$ units of music instead of just $b$ units, as before, as long as she uses the extra ‘money’ to make music and not to increase her consumption of other goods. The ‘budget line’ that the musician faces is now twice as steep as it was before. This is the dashed line in Figure 10.1.2 that starts at the point $(x^*, y^*)$. Since this line is steeper than the original budget line, it is also steeper than the musician’s indifference curve. So, she can make herself better off by increasing the amount of music that she produces.

The non-musician will then give up some of his endowment of the consumption good and get more music in return. By the quasi-linearity assumption, this will make him better off. To see why, observe that the musician starts with consumption $(x^*, y^*)$ and ends up at the point $(x', y')$ where she is better off. This means that

$$x' + v(y') > x^* + v(y^*)$$

or

$$v(y') - v(y^*) > x^* - x'. \quad \text{The musician gives up } (x^* - x') \text{ units of the consumption good to get } y' - y^* \text{ additional units of music.}$$

This last inequality says that this makes the musician better off. Since the non-musician is matching any contributions that the musician makes, he also gives up $(x^* - x')$ units of the consumption good to get $y' - y^*$ units of music. Then

$$\omega - (x^* - x') + v(y') > \omega + v(y^*)$$

and the non-musician is also made strictly better off. This is all outlined in Figure 10.1.2.

Thus, the equilibrium of the voluntary contribution game is not Pareto optimal. Since you can make both musicians and non-musicians alike better off by
inducing the musician to make more music, it is legitimate to say that too little music is produced in this equilibrium. The same argument can be made for virtually any type of intellectual property: software, games, movies, academic research, scientific discoveries, and so on. This isn’t a problem that has arisen simply because of the internet.

You should also understand some of the immediate pitfalls in this argument. First, the non-musician in the example above is made better off if he matches the contributions of the musician. He should be happy to pay part of the cost of production. In this simple example above, as long as the non-musician gets some utility from the musician’s music, then there is some fee and increase in production of music that will make both better off. Finding this fee may be difficult.

For example, in the scheme above, the musician is allowed to choose whatever level of music she wishes to produce. If the non-musician doesn’t value music as much as the musician does, then the musician may overdo it. The non-musician may be forced to match payments he would rather that the musician never make. As an extreme, the non-musician may not value the musician’s music at all. If that were true, the musician shouldn’t be paid at all. This is a classic defence of file sharers who claim that they would never buy any of the tracks that they download anyway. If you are going to start making the payments dependent on how much the non-musicians like the music, this is problematic because all the non-musicians will want to tell you that they don’t like the music.

10.2 The Lindahl Solution

Let’s try to implement this solution with something that looks like a market. First, suppose we create a fictitious firm. We can give this firm each of the consumers’ endowments and all the decision-making power over how much music it produces with those endowments. The firm will use the musician to produce the music. It will give her, say, $x$ dollars and expect $bx$ units of music in return. The musician won’t be able to keep the music, however. By creating the music, it becomes the firm’s possession. File sharing will be completely illegal; so much so that the musician won’t even be able to share files on her own computer. If she wants to consume music, she will have to buy it back from the firm. To make things work out in a straightforward way, suppose we also require that the consumers buy the consumption good from the firm. To allow them to pay for this, we will create money. When the firm chooses how much music to produce, it will offer that music and any remaining private good to consumers at market prices. This will generate profits for the firm that it will distribute back to the consumers as money. The consumers use the money they get from the firm to buy the music and consumption goods they want.

This fiction is the ‘publisher’s world’ that John Lennon described above. How does it work? We will announce that the price that each consumer has to pay for each download (that is, each unit of music) is exactly \(\frac{1}{b}x\). The price for a unit of the private good is just 1. The firm then makes its decision to produce,
say, \( x'' \) units of the private good and \( y'' \) units of music. If the consumers want to buy all this, then the firm’s total profits are then \( \frac{1}{2b} y'' + \frac{1}{2b} y'' + x'' \). The first term is the money that the firm makes from selling music to the musician, the second term is the money from selling music to the non-musician, while the final term is the money earned by selling the private good to both of them.

If this production decision by the firm is feasible for it, then the firm’s profits are easy to figure out. They are

\[
\frac{1}{2b} y'' + \frac{1}{2b} y'' + x'' = 2y'' + x'' = \frac{b(2\omega - x'')}{b} + x'' = 2\omega
\]

That means that each of the two consumers has \( \omega \) (half the firm’s profit) to spend on music and consumption. Each faces a ‘budget line’ whose slope is \( 2b \) and which allows them to purchase back their endowment \( \omega \) if they want to.

All of this information is plotted in Figure 10.2.1. You should verify that the choice \((x'', y'')\) by the firm actually maximizes the firm’s profits given the price \( \frac{1}{b} \) being paid by each consumer (show that the slope of the firm’s iso-profit curves are all equal to \( b \)). Given this choice, the common budget line for each of the two consumers is the red line starting at \( \omega \) on the \( x \) axis and winding up at \( 2b\omega \) on the \( y \) axis. Since both consumers have the same preferences, then each of them will make the same choice along this budget line. This choice is given by the green point labeled \((x_1, y_1)\) in the picture. Notice that if both consumers make this choice, then each of them consumes \( y'' \) units of music, which is just the amount that the firm chooses to produce. From each consumer’s perspective, they simply paid the firm to download \( y'' \) songs. We really need them both to buy all the music that is produced, because if they don’t, one of them will quickly realize that they can download the rest of it for free on the internet anyway.

Observe, as well, that each consumer buys \( x'' \) units of the private good. From the geometry of the picture, it should be clear to you that \( 2x'' = x_1 \), so the consumers want exactly the same amount of the private good as the firm produced. Of course, since this is a private good, not a public good, when the musician buys \( x'' \) from the firm, there is only \( x'' \) left over for the non-musician.

This solution to the music downloading problem supports an outcome that is Pareto optimal: there is no way to change things in a way that will make both of them better off. Their marginal rates of substitution are the same as they would be in a competitive equilibrium with private goods. However, that isn’t what’s important. The important property here is that if you add up what each consumer is willing to pay to get one more music download, you get \( \frac{1}{b} \). If you calculate what it costs the firm to produce one more song to download, that is also \( \frac{1}{b} \). So, the sum of the consumers’ marginal willingness to pay is equal to the marginal cost.

This solution is nice in the sense that both the musician and the non-musician end up in exactly the same situation. They get the same consumption bundle, which seems fair somehow. Yet, notice something else about this solution. If you look back at the equilibrium of the voluntary contribution game, the musician does a lot better than she did in that case, but the non-musician is worse off.
than he was in the equilibrium of the voluntary contribution game. He originally was able to download music for free. Now, he has to pay. He gets more music, but that doesn’t compensate him for the loss of consumption that he incurs.

You should keep one observation in mind: a possible alternative exists that will make everyone better off than they are in the equilibrium of the voluntary contribution game. But you can’t do this by simply forcing people to pay for the music they download. The case illustrated here is a best-case situation in which the music firm actually takes the Lindahl price $\frac{1}{2b}$ to be fixed and beyond its control. As you will see below, this will not happen. But even if it did, the non-musician is made worse off. In this best of all possible cases, forcing people to pay for things they could get for free is largely a exercise in redistribution. It hardly seems like the non-musician needs to be a communist to oppose it.

### 10.3 Some potential problems

Before we get to the real problem with copyright, let’s consider a few complications. One reasonable change in this simple model would be to acknowledge the fact that non-musicians probably aren’t willing to pay quite as much for music downloads as musicians are. For example, the musician might have utility function

\[ x + v_m(y) \]

while the non-musician has utility function

\[ x + v_n(y) \]

where \( v'_n(y) < v'_m(y) \) (the marginal utility of music for the non-musician is smaller than the marginal utility of music for the musician) for all \( y \). It is
easy enough to capture this in the figure above by making the non-musicians indifference curve steeper than the musician’s indifference curve.\(^2\) If we did this, and continue to charge the price \(\frac{1}{2}b\) to each of them, then the non-musician would want more of the private good \(x\) and less music. This wouldn’t work because the firm isn’t actually producing enough of the private good to allow this. Thus, there would be an excess demand for the private good.

The Lindahl solution to this problem is to charge the non-musician a different, lower price for music (and a correspondingly higher price for food) than is charged to the musician. The solution is given in Figure 10.3.1. As reference points to show what is going on, the solutions to the voluntary contribution game from Figure 10.1.1 and the symmetric solution from the last Figure 10.2.1 are included in the diagram. Note that, because the outcome of the voluntary contribution game is determined entirely by the musician, the outcome of the voluntary contribution game is independent of what the non-musician’s preferences are. The dashed line through the bundle \((x_1, y_1)\) is the budget line that both of them face in the symmetric solution. To satisfy the non-musician’s preference for other consumption goods, the Lindahl price-setter would lower the price of music and raise the price of the private good until the non-musician faced the blue budget line. He would then pick the bundle of goods that maximizes his utility at the point where the blue indifference curve just touches this budget line. Once he has changed the non-musician’s price, the price-setter can’t leave the musician’s price at \(\frac{1}{2}b\). This would create two problems. First, the non-musician still wants more food and less music than he did in the symmetric solution. If nothing else happens, there is an excess demand for food. Also, the non-musician isn’t going to buy all the music that is produced. He will quickly realize that he can download the rest of it over the internet for free anyway and

\(^2\)The slope of the musician’s indifference curve is \(\frac{1}{v_m(y)}\).
he won’t end up on the blue budget line where we want him to be. The solution is to have the firm produce more food and less music, and to raise the price that the musician pays for music until she faces the red budget line. To keep the diagram a little less cluttered, the tangency for the musician isn’t labeled, but notice that she and the non-musician both choose the same amount of music. As mentioned above, there is really no alternative since, if they don’t, one of them will simply download, whatever they don’t, have freely on the internet.

A digression on how to compute the Lindahl prices in this example

You may be wondering exactly how I calculated the prices for each of the two consumers in Figure 10.3.1. I won’t do the calculation, but I’ll explain the procedure. Pick an arbitrary, but small value for \( y \), say \( y_0 \). Both consumers have quasi-linear preferences, so if we fix output of music at \( y_0 \), then no matter how much food either of them eat, the slope of their indifference curves will be exactly \( \frac{1}{v_m'(y_0)} \) for the musician and \( \frac{1}{v_n'(y_0)} \) for the non-musician. Starting at the point \( \omega \) on the horizontal axis, draw a budget line for the musician with slope \( \frac{1}{v_m'(y_0)} \) and for the non-musician with slope \( \frac{1}{v_n'(y_0)} \). Each consumer’s indifference curve will be tangent to this budget line at \( y_0 \). This is illustrated in Figure 10.3.2

Now, suppose we try to sell music to the musician at price \( v_m'(y_0) \) and to the non-musician at price \( v_n'(y_0) \). Each of them will choose \( y_0 \). The corresponding demands for food will be \( x_m^* \) and \( x_n^* \), both of which are plotted in Figure 10.3.2. The total amount of food that is available when we produce \( y_0 \) units of music is given by the distance from the vertical axis to the point \( A \) in Figure 10.3.2. If you add \( x_m^* \) and \( x_n^* \) together, you will get an \( x \) value given by the dark point just horizontally to the left of \( A \). So, some food will be left over that no one...
wants to buy at current prices. In other words, there will be an excess supply of food.

On the other hand, try doing the same exercise with a very high value for \( y \), say \( y_2 \), as in Figure 10.3.2. Compute \( v'_m(y_2) \) and \( v'_n(y_2) \), and charge these prices to the two consumers after doling out \( \omega \) to each of them. The budget lines they face are given by the red lines in the figure. In this case it is sort of obvious from the diagram that they will want far more food than is actually produced. The amount of music produced in this case is unfeasible anyway.

As we raise the amount of music we produce from \( y_0 \) to \( y_2 \), the marginal utilities of both consumers fall. So, the prices they are charged for music will fall, accordingly. Their demand for music will rise, and we will use up more food in production of music. Both consumers’ demand for food will also be falling as we lower the price of music, but total demand won’t be falling as fast as the amount of food being produced.

At some point in between, we will produce some amount of music equal to \( y^* \). Prices for music would be set to \( v'_m(y^*) \) and \( v'_n(y^*) \) for the musician and the non-musician respectively. They will each choose to buy \( y^* \) units of music and buy food with whatever is left over in their budget. Summing up the amounts of food they choose to buy, we will get exactly the amount of food left over after we produce \( y^* \) units of music.

What of the firm in this calculation? The firm receives \( v'_m(y^*) y^* \) from the musician, \( v'_n(y^*) y^* \) from the non-musician for the music it produces, and \( x^*_m \) and \( x^*_n \) for the food that it offers. So, its total profits are

\[
v'_m(y^*) y^* + v'_n(y^*) y^* + x^*_m + x^*_n
\]

Since this outcome is feasible, it must be that

\[
x^*_m + x^*_n = 2\omega - \frac{y^*}{b}
\]

since \( \frac{y^*}{b} \) is the amount of the endowment that is needed to produce \( y^* \) units of music.

Then, since both consumers choose bundles that lie on their respective indifference curves, and each is given income \( \omega \),

\[
x^*_m + x^*_n + v'_m(y^*) y^* + v'_n(y^*) y^* = 2\omega
\]

Subtracting this second equation from the first gives

\[
\frac{1}{b} = v'_m(y^*) + v'_n(y^*)
\]

The term on the left is the marginal cost of producing one more unit of the public good. The term on the right is the sum of the two consumers’ willingness to pay for an additional unit of the public good. If you ever take public finance, you will learn this as the Samuelson condition for efficient production of the public good. Taking the reciprocal of the equation gives

\[
b = \frac{1}{v'_m(y^*) + v'_n(y^*)}
\]
The right hand side of this last expression is the slope of the firm’s iso-profit line when prices for music are \( v'_m(y^*) \) and \( v'_n(y^*) \). The term on the left is the slope of the production possibilities frontier faced by the firm. So, the firm is maximizing profits at these prices by producing \( y^* \) units of music. As a result, this outcome is actually a competitive equilibrium with all consumers and the firm acting as price-takers, and demand equal to supply. It is Pareto optimal by the first welfare theorem.

### 10.3.1 A Second Problem

Stepping back a bit, the non-musician has a lower willingness to pay for the public good than the musician, which means that he pays a lower price. This doesn’t seem fair. In addition, there is no real way to know the non-musician’s willingness to pay. Obviously, he is going to claim that it is very low if he wants to buy music at a low price. This would be too complicated to work out in the real world where there are millions of different non-musicians whose tastes we have to try to figure out. We probably want a solution that charges everyone the same price. Without going into a lot of technical details, it is pretty clear that the musician and non-musician will want different amounts of music. If we can enforce this, say by forcing everyone to buy CD’s, so the non-musician can be prevented from consuming the freely available music, then the final outcome won’t be Pareto optimal. The non-musician could be made better off without hurting the musician by letting him download the rest of the music for free. We don’t want that, because that is the problem we were trying to resolve with our copyright legislation in the first place.

Even if we ignore the free music available to the non-musician, we will have problems with the final production of music. The outcome could in fact be worse than it was in the voluntary contribution game. The example discussed here is a bit extreme because the production possibilities frontier is linear. The profit maximizing firm will either be indifferent about how much music it produces or will want to produce at one of the extremes. This makes for an pretty unintuitive equilibrium in this case. Suppose we charge each consumer \( \frac{1}{2} b \) for music as we did in the symmetric Lindahl solution we discussed above. However, let’s assume now that the non-musician has a lower willingness to pay for music than the musician does. Along the budget line the non-musician faces when music costs \( \frac{1}{2} b \), he will want to consume more food and less music. But, there is no more food if the firm produces the music that the musician wants. Normally, an excess demand like this would cause us to want to raise the price of food and lower the price of music. Unlike a pure private good example, though, this won’t work because there is not an excess supply of the public good. Demand is exactly equal to supply (Walras Law fails) for the public good, given that we simply prevent the non-musician from listening to some of the music. If we lower the price of music, the musician will want more of it, but the firm will want to produce less of it. So, the usual remedy will simply make the problem worse.

In fact, the only equilibrium for this case (with quasi-linear preferences and
linear production technology) when the firm produces only music and the price of music is set equal to $v'_n(2b_\omega)$. This induces both of the consumers to buy only music. This outcome produces a kit more music than any other solution. However, this isn’t necessarily good. Looking back at Figure 10.2.1 or Figure 10.3.1, this will make both consumers worse off than they were in the voluntary contribution game as long as the indifference curve for the musician in the solution to the voluntary contribution game lies to the right of the point $(0, 2b_\omega)$, which is quite reasonable.³

10.4 The Monopoly Problem

In everything we have discussed in this note so far, we have assumed that the firm that is making all the decisions about music and food is a competitive firm. It doesn’t believe that it has any influence over price. The essence of copyright is monopoly – just as it is for patents (I’ll come back to explain why in a moment). A single firm that produces all of the output of a good will inevitably try to control (i.e. raise) its price. To see how this might work, let’s go back to our previous version of the model where the musician and the non-musician have exactly the same quasi-linear preferences. At the Lindahl solution, the firm takes prices as fixed. The competitive price for music is $\frac{1}{2b}$ and as we showed above, the quantity of music that both consumers demand is $y_1$. Totaling up profits for the firm at this price and output gives profits $2_\omega$. The firm distributes these profits equally between the musician and the non-musician which is where they get the $\omega$ dollars that they spend on music and food.

What happens if the firm decides on its own that it wants to charge a higher price for music, say $q$? Since preferences are quasi-linear, the demand for music will fall to some new level, let’s call it $y' < y_1$. The firm’s profits are easy to calculate. It earns $q$ dollars for each of the $y'$ songs that it sells to the musician. It also receives $q$ for each of these same songs from the non-musician. The firm’s costs of producing the songs is $\frac{y}{b}$ units of food, so it has $2_\omega - \frac{y}{b}$ units of food left over that it sells for a dollar per unit. So, total profits are

$$2qy' + 2_\omega - \frac{y'}{b} = 2_\omega + \left(2q - \frac{1}{b}\right)y'$$

From this, it is immediately apparent that if the firm sets a price for music higher than the Lindahl price $\frac{1}{2b}$, then it will make more profits than it does in the Lindahl solution. So, we aren’t going to get the Lindahl (Pareto optimal) solution by assigning copyright and creating a monopoly.

What price will we get? All we need to do at this point is maximize the firm’s profit by choosing the appropriate price. The first order condition is

$$2y' + \left(2q - \frac{1}{b}\right)\frac{dy'}{dq} = 0$$

³There are only two goods here, but this illustrates the fact that you don’t buy the latest CD by the Mad Caddies because you have to pay so much for a CD by U2.
The solution will depend on the nature of preferences. To see how, suppose that we set the price equal to \( \frac{1}{b} \), which is the implicit price in the voluntary contribution game. Then, we can try to evaluate how profits are changing at this point. The derivative (with quasi-linear preferences) above becomes

\[
2y^* + \frac{1}{b} \frac{dy'}{dq}
\]

because demand for music is equal to \( y^* \) at price \( \frac{1}{b} \) with quasi-linear preferences. Divide both sides by \( y^* \) and multiply both sides by \( b \) (both of which are positive) so that this becomes

\[
2b + \frac{dy'}{dq}
\]

This suggests something pretty remarkable: if the elasticity of demand for music \( -\frac{dy'}{dq} \) is smaller than \( 2b \) when demand for music is equal to \( y^* \), the profits will be strictly increasing in price at \( y^* \). This means that the profit maximizing firm will raise the price of music so much that there will actually be less music produced than there was in the voluntary contribution game. Strong copyright, because of the monopoly power it conveys, can easily have a very perverse effect. The tendency for the monopoly firm to restrict output can easily outweigh the benefits of double charging for the output to subsidize the production of music.

To make matters simple, let’s suppose that conditions are such that the firm simply decides to charge the price \( \frac{1}{b} \), which is the implicit price in the outcome of the voluntary contribution game. No more music is produced than would be the case if downloading were absolutely free, all the monopoly firm wants to do in this case is to charge non-musicians for the downloads that they were previously making for free. Where does this money go? In the example here, the firm is acting in a way that is in the interest of its shareholders. It produces the largest profits that it can possibly attain, then distributes them back equally to the musician and the non-musician. Since total profits have increased, the musician’s share of these profits rises and his budget line lies to the right of the one that he faced in the voluntary contribution game (free downloading outcome). Total production of music hasn’t changed, so total production of food is also the same. This means the musician gets more food in the new equilibrium. Correspondingly, the non-musician gets less.

So, as you may have guessed, strong copyright legislation has a redistributive impact (toward shareholders) even when it has very perverse effects on music production. It happens that musicians own a lot of shares in the example in this little note. Of course, they may not own so many shares.

### 10.5 Are there Any Alternatives?

Strong copyright is an imperfect solution to the problem of creating incentives for producers of public goods. Are there alternatives? The Canadian approach,
as I mentioned in the introduction, is different from the US approach. Private use (or private ‘study’ as it is called in the law) is allowed under Canadian copyright legislation. This doesn’t mean that Canadian law doesn’t want to provide incentives for musicians, these are just created in a different way.

First, there is a tax imposed on media that are used to record digital versions of music, tapes, CD’s, hard disks etc. These taxes are remitted to the Canadian Private Copying Collective (http://cpcc.ca), which is supposed to distribute the money. The proceeds of the tax were around $28 million in 2002/3.4 This money is passed on to organizations representing artists. For example, 18.9% of these revenues were to be distributed to performers, 15.1% to record companies, and the rest to authors and publishers. The ACTRA Performers’ Rights Society received about $7 million of this money to distribute to their members. For a variety of reasons, less than 1% of this money was actually paid out to performers and artists (the figures are given at http://www.actra.ca), nonetheless, the money is there in principle. At least, ACTRA tells you what they do with the money. I haven’t had any luck figuring out what the record companies do with the money they receive from CPCC – they no doubt pay it to their lawyers.

Second, Canada directly subsidizes music through programs like the Sound Recording Development Program (see http://www.canadianheritage.gc.ca for a description) and through CRTC Canadian Content Restrictions.

Copyright policy involves a balance between incentives and monopoly power. Strong copyright legislation by itself doesn’t necessarily balance these interests. Hopefully, some of the simple economics presented here make some of these issues clearer.

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4Total CD sales in Canada by comparison, were about $600 million (http://www.cria.ca).
Chapter 11

Problems

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11.1 Preferences

1. What is a preference relation? What is it defined over?

2. What are the two properties of a rational preference relation? For each of these properties, describe a realistic situation in which the property would fail.

3. What does it mean for a person to be ‘rational’ in a particular economic environment? What does it not mean?

4. Why is it “almost tautological” to “assume that a person chooses the alternative that he or she prefers most”? Given this, where does the predictive content of economic theory come from? (i.e. what is the testable hypothesis?)

5. Why is it “a waste of time to argue whether or not consumers are rational”? What is the real objection that one might make?

6. Explain why a rational preference relation implies that indifference curves can not have a point in common.

7. Consider a game of tennis. If you are standing to the right, I prefer to hit the ball to the left. But, if I hit the ball to the left, you prefer to go to the left. But, if you are standing to the left, I prefer to hit the ball to the right. Thus, a consequence of my preference to hitting the ball to the left is that I prefer to hit the ball to the right (and vice versa). Does this mean my preferences are intransitive?
8. What does it mean for a function, \( u \), to represent a preference relation on \( X \)? (i.e., what properties must an arbitrary function possess in order for it to be called a ‘utility’ function?)

9. What are the benefits of using a utility function, as opposed to working with its underlying preference relation?

10. What does it mean for a preference relation to be ‘monotonic’? Describe a scenario where the assumption of monotonicity would be reasonable, and one where it would not.

11. Show that a utility function can not exist if the preference relation is not transitive. \textit{Hint:} Try a ‘proof by contradiction.’

12. Pick an arbitrary consumption bundle \((x, y) \in \mathbb{R}_+^2\) and draw the sets \(B\) and \(W\) (in \(\mathbb{R}_+^2\)) for the following situations:

   (a) I like consuming \(x\), and the more the better. I am completely indifferent to the amount of \(y\) that I consume.

   (b) As long as my consumption of \(x\) is less than \(\bar{x}\), the more the better. After \(\bar{x}\) the less the better. I am completely indifferent to the amount of \(y\) that I consume.

   (c) I like both \(x\) and \(y\), but a unit of \(x\) must be paired with a unit of \(y\) in order for me to enjoy it (like left and right shoes).

   (d) I prefer the bundle \((\bar{x}, \bar{y})\) to all other bundles. If I have any other bundle, then I am indifferent between having this bundle and having nothing \((0,0)\).

13. This question is designed to take you through the steps of the proof of the Theorem presented in the text (the existence of a utility function) by using specific description of preferences.

   Garrett likes owning shoes, but only gets use out of them if they are in a pair. Let \(L\) be the number of left shoes he owns, and \(R\) be the number of right shoes he owns. Now consider adding more shoes to the bundle \((L, R)\). In particular, if Garrett were to add \(\ell \geq 0\) left shoes and \(r \geq 0\) right shoes, then he would (strictly) prefer the new bundle, \((L + \ell, R + r)\), to the original bundle, \((L, R)\), if and only if both \(\ell\) and \(r\) are strictly positive. For example, this implies that he is indifferent to having more left shoes but not more right shoes.

14. Are these preferences rational (complete and transitive)? Are they monotonic?

   (a) Draw (with \(L\) on the vertical axis, and \(R\) on the horizontal axis) the following bundles and sets.

      i. \(Z\).
ii. An arbitrary bundle, \((L, R) \in \mathbb{R}_+^2\).

iii. \(B\) and \(W\) (relative to the above bundle).

iv. \(P^+\) and \(P^-\) (again, relative to the above bundle).

v. The ‘indifference bundle’, \(z \in Z\).

(b) Construct the utility function as suggested in the proof (hint: Pythagoras’ theorem might come in handy).

(c) Can you think of a simpler utility function?

(d) Check that both utility functions (the suggested one, and the simpler one) actually represent preferences.

15. Repeat Question 13 using the following description of preferences.

Ken also like shoes, but unlike Garrett, he does not wear them. He makes ‘shoe sculptures’ out of them. As far as he is concerned, a left shoe is equally as good as a right shoe. That is, suppose that we were to add more shoes to Ken’s initial bundle of \((L, R)\). In particular, if were to add \(\ell \geq 0\) left shoes and \(r \geq 0\) right shoes, then he would (strictly) prefer the new bundle, \((L + \ell, R + r)\), to the original bundle, \((L, R)\), if and only if \(\text{either } \ell \text{ or } r \text{ is strictly positive}^1\)

16. These questions are designed to highlight the role of the various assumptions used in the proof.

(a) What part of the proof would fail if we removed each of the following assumptions?

i. Continuity.

ii. Monotonicity.

iii. Transitivity.

iv. Completeness.

Which of these are ‘crucial’ in the sense that a failure of the assumption does imply that a utility function will not exist?

(b) The proof was only concerned with consumption bundles in \(\mathbb{R}_+^2\) (two-good consumption bundles). What problems, if any, would be introduced if we considered consumption bundles in \(\mathbb{R}_+^k\) (\(k\)-good consumption bundles), where \(k > 2\)?

(c) The proof was by ‘construction:’ it showed that a utility function existed (under the stated assumptions) by actually building such a function. Is this suggested function unique? Prove your claim.

17. Draw a graph showing the indifference curve associated with each of the following utility functions when the level of utility is equal to 1

- \(u(x, y) = \max\{x, y\}\)

---

^1 This includes the case in which both are strictly positive.
18. Simone has an endowment of $20 in period 1 and $40 in period 2. Her utility is \( u(c_1, c_2) = c_1 + c_2 \). She can put money in a bank account in period 1, and get it back again in period 2 without interest. She can also borrow money from a loan shark. For each dollar she borrows in period 1, she must pay back $1.50 in period 2. She can also buy a chair in period 1 for $10 that she knows that she will be able to resell in period 2 for $15.

- Draw her budget line assuming that she does not buy the chair (labeling you diagram carefully).
- Draw her budget line if she does buy the chair.
- Should she buy the chair, or not? What consumption bundle should she pick in each case?
- Predict her consumption, and whether or not she will buy the chair if her utility function is instead given by \( u(x, y) = \min\{x, y\} \).

19. Solve the problem

\[
\max 2x^\frac{1}{2} + 2y^\frac{1}{2}
\]

subject to the constraints \( 3x + y \leq 10; \ x \geq 0; \ y \geq 0 \) using the method of Lagrangian multipliers. Write down the Lagrangian function, and each of the first order conditions before solving the problem.

20. Solve these problems:

(a) Solve for the optimal value for \( x \) and \( y \) in the problem \( \max u(xy) = \frac{x^2}{x+y} \) subject to \( px + y \leq W; \ x \geq 0, \ y \geq 0 \) using the method of Lagrangians. Hint: the derivative of \( u(x, y) \) with respect to \( x \) is \( \frac{y^2}{(x+y)^2} \).

(b) Find the demand function associated with utility function \( u(x, y) = x + xy \) (Hint: there are corner solutions to this problem - the Lagrangian method is too slow for this problem).

(c) Solve the problem maximize \( u(x, y) = x \) subject to \( 2x + y \leq 10, \ x \geq 0, \) and \( y \geq 0 \) using the method of Lagrangians (i.e., find the optimal value for \( x \) and \( y \) and all the multipliers).

21. A consumer has downward-sloping indifference curves for goods \( x \) and \( y \). Below the 45° line, the indifference curves are straight lines with slope \( -\frac{1}{2} \). When the curves hit the 45° line, they become steeper. Above the 45° line they are also straight lines but their slope is equal to \( -2 \). Can you use the method we used to prove the existence of a utility function to provide a utility function that represents these preferences?
22. Consider the following situation:

Geraldine prefers sports cars to hybrid cars because sports cars are faster and more powerful. She prefers SUVs to sports cars because SUVs are roomier. She prefers hybrid cars to SUVs, however, because they are more fuel-efficient and environmentally sustainable. *Honest Abe’s Used Cars* charges $1000 to trade in Geraldine’s vehicle each time to “help her make up her mind.”

Why is it that people “who exhibit intransitive preferences [...] quickly change their behavior when this is pointed out to them”?

11.2 Demand Theory

1. In what sense is the optimal consumption problem a special case of the more abstract choice problem introduced in the previous chapter? In particular, what is the special name given to the choice set $\mathcal{X}$? What is a typical element $x \in \mathcal{X}$?

2. Explain how the ‘existence of a utility function’ theorem allows us to make progress with understanding consumer demand.

3. In order to make sharper predictions about economic behaviour, we will often need to add assumptions (e.g. on preferences) to the more fundamental axioms. The collection of such assumptions are known as a model. What is the point of analysing a model, when a model is simply a set of assumptions?

4. What are the two ‘big’ assumptions made by Classical consumer theory? Think of a scenario in which each would be inappropriate.

5. Do the classical assumptions require that a consumer have an unchanging preference relation over $\mathbb{B}$? That is, suppose that we observed a consumer’s choice of consumption bundle at two points in time, $x^*_t$ and $x^*_{t'}$ (where both of these are in $\mathbb{R}^n$). Further, suppose that prices and incomes were identical in the two time periods. What predictions could we make about $x^*_t$ and $x^*_{t'}$ if we allowed for the possibility that the consumer’s preference relation over $\mathbb{B}$ changed over time? Could we ever find evidence that our model is wrong?

6. There are two goods, $x$ and $y$. Their prices are $p_x$ and $p_y$, and income is $W$. Draw $\mathbb{B}$, the budget set. Show (graphically) the effect of

   (a) An increase in $W$.
   (b) An increase in $p_x$.
   (c) All prices changing by a factor of $\alpha$.
   (d) All prices and income changing by a factor of $\alpha$. 

117
What is the implication of the last of these for the classical consumer theory assumptions?

7. A consumer is growing tomatoes and has $W$ kg available for consumption in period 1. Whatever she doesn’t eat in a period will grow by a factor of $R$ by the next period. Let $x_t$ represent the consumption of tomatoes in period $t$.

   (a) Draw the budget set for the case when there are only two periods.
   (b) Describe this budget set in formal notation.
   (c) Describe this budget set in formal notation for the case in which there are $T$ periods.

8. Derive the demand function when preferences are given by:

   (a) $u(x, y) = x^2 + y^2$. *Hint:* This highlights the fact that the FOCs are necessary but not sufficient - draw the indifference curves.
   (b) $u(x, y) = x + y$.

9. Suppose that a consumer’s preferences are described by a utility function, $u(x)$. Prove that it is also true that his preferences are represented by a new function, $h(x)$, which is defined as $h(x) = g(u(x))$, where $g$ is an increasing function (e.g. an exponential or logarithmic function).

10. Use the previous question to help you derive the demand function when preferences are given by

    $$ u(x, y) = Z f(x, y), \quad (11.2.1) $$

    where $Z > 1$ and

    $$ f(x, y) = A - \frac{1}{\alpha \ln(x) + (1 - \alpha) \ln(y)}. \quad (11.2.2) $$

11. A quick inspection of the properties of the utility function often makes it much easier to derive the demand functions since the effects of the constraints are more transparent. For each of the utility functions below, identify the conditions under which positive quantities of both goods will be consumed at the optimum (so that the non-negativity constraints can be ignored when such conditions are met):

    (a) $u(x, y) = x^\alpha y^\beta$, where $\alpha, \beta > 0$.
    (b) $u(x, y) = x + \ln(y)$.
    (c) $u(x, y) = x^\alpha + y^\alpha$, where $\alpha \in (0, 1)$.
    (d) $u(x, y) = x^\alpha + y^\alpha$, where $\alpha \in [1, \infty)$.
    (e) $u(x, y) = (x + y)(1 - x - y)$. 

118
(f) \( u(x, y) = (x^\alpha + y^\alpha)(1 - x^\alpha - y^\alpha) \), where \( \alpha \in (0, 1) \).

12. For each of the functional forms in the previous question, identify the conditions under which the income constraint will be binding.

13. Suppose we want to test the basic assumption that a consumer’s preferences are independent of the budget set. Describe the test proposed in the text. What is the extra assumption on preferences that was required? Can you think of another approach to testing this basic assumption?

14. It is often thought that econometricians assist theorists by collecting and analysing data in order to determine whether a model is a plausible description of the world. This question shows that the reverse is sometimes true too - theorists may assist econometricians in terms of how they construct their econometric model.

Elliot the eager econometrician knows that consumer theorists talk about demand functions, and are especially interested in the effects of income and prices. Elliot thinks that theorists are full of hot wind and wants to test certain aspects of demand theory. To do this, each day he carefully records the prices of the \( n \) goods that he consumes. Along with this, he also records daily his income and his chosen consumption bundle (a quantity for each of the \( n \) goods). Thus, after a year or so he has a data set that contains observations on prices, incomes and demands. To facilitate his many probing tests, he needs to determine how prices and income influence his demand. To keep things simple, he decides to focus on his demand for pocket protectors. The most basic specification that comes to mind is the following econometric model:

\[
x_{1t} = \beta_0 + \beta_w W_t + \alpha_1 p_{1t} + \sum_{i=2}^{n} \alpha_i p_{it} + \varepsilon_t
\]

where \( x_{1t} \) is his demand for pocket protectors on day \( t \), \( W_t \) is his income on day \( t \) and \( p_{it} \) is the price of good \( i \) on day \( t \) (pocket protectors are good 1).

15. While waiting for his expensive statistical package to load, Elliot asks your opinion on his specification of the demand function. Do you see any problems? In particular think about:

(a) If demand theory is taken seriously, what must \( x_{1t} \) be when \( W_t = 0 \)? How does this restrict the possible values of \( \beta_0 \) and \( \alpha_j \) (\( j = 1, ... , n \))? Does demand theory itself impose these latter restrictions on the partial effects of prices on demands?

(b) Again, if demand theory is taken seriously, then how must \( x_{1t} \) change when all prices and income are changed in the same proportion? How does this restrict what \( \beta_w \) and \( \alpha_j \) (\( j = 1, ... , n \)) can be? Does demand theory itself impose these latter restrictions on the partial effects of income and prices on demands?
16. The general mistake that Elliot has made was to forget that a demand function is derived from preferences. That is, he assumed a demand function (based on simplicity) rather than assuming a form on preferences. In other words, it may very well be the case that there are no preferences that would produce this particular demand function. Do you have any general advice to give Elliot before he decides on some other specification?

17. Use graphical methods to describe preferences (draw indifference curves) for goods $x$ and $y$ for the following scenarios.

(a) As income increases, the demand for good $x$ falls.
(b) As the price of good $x$ increases, so too does demand for $x$.
(c) As income changes, the ratio of the demand for $x$ to the demand for $y$ is constant.
(d) As the demand for $y$ is unaffected by the price of good $x$.

18. Use the above question to answer the following.

(a) Is it possible for the quantity demanded of $x$ to decrease as income increases, yet increase when it’s price increases?
(b) Is it possible for the quantity demanded of $x$ to increase as its price increases, yet increase when income increases?
(c) Is it possible for the quantity demanded of both $x$ and $y$ to fall as income increases?

11.3 Discontinuous Budget Sets

1. In the text, a non-linear pricing problem was analyzed in which the per unit price increased after some threshold demand was reached. In this problem, we do the same thing, but assume that the per unit price falls. Let the per-unit price of good $y$ be constant and equal to 1. The per unit price of good $x$ is assumed to be $p$ for each unit up to the $n^{th}$. Then each additional unit costs $p - dp$. In other words, each additional unit of output purchased above and beyond the first $n$ costs less than the first $n$ unit. Assume that preferences are given by

$$u(x, y) = x^\alpha y^{1-\alpha}$$

Repeat the exercise given in text, and show how demand for good $x$ varies with $\alpha$.

2. Tickets for the World Cup Quidditch match have been sold in bundles of three, and are now being resold by scalpers. The lowest priced scalper offers his three tickets at a unit price of $100. So if a consumer wants $x \leq 3$ tickets, the price she will pay is $100x$. The next lowest price scalper is offering her three tickets for $110, the cheapest after that is $120 and so on. A consumer has preferences given by $u(x, y) = x^\alpha y^{1-\alpha}$ where $x$ stands for the number of tickets
she purchases, and $y$ is the amount of money she has left over for other stuff. She has $3000 to spend on tickets and other stuff. Write a computer program that takes as its input the value of $\alpha$ and outputs the number of tickets that the consumer with that value of $\alpha$ will buy. You can assume that tickets are infinitely divisible.

Hint: If our consumer decides to buy $x$ tickets, then she will pay as follows: compute $x/3$ and take its integer part and call it $j(x)$ (e.g. the integer part of $22/3$ is $7$, so $j(22/3) = 7$); let $p_0 = 90$; compute

$$
3(p_0 + i10) + (p_0 + j(x)10)(x - j(x)3)
$$

This procedure gives you the slope and position of the budget line for different values of $x$. Now generalize the method in the text to solve the problem.

One way to do this would be to write a short computer program which takes as its input a value for $\alpha$ and outputs a demand and expenditure on tickets. Scripting on computers is a useful skill to develop as an economist. It doesn’t really matter which language you use to write scripts (c, java, perl, bash, php etc) since they all use similar principals which you can apply in a variety of contexts, for example, statistics and econometrics packages, or computer algebra programs.

3. In the text, the response of a consumer with quasi-linear preferences to the imposition of a fixed fee is analyzed. Preferences in this problem are given by $u(x, y) = y + \log(x)$. Consumer income is $W$ and the firm’s pricing scheme requires that the consumer pay a fixed fee $K$, which entitles her to $n$ ‘free’ units of good $x$. Each unit in addition to the first $n$ costs the consumer $p$. Hold the price $p$ and the initial number of units $n$ constant, and suppose that the firm using this scheme wants choose a fee $K$ to maximize the profit it receives from this consumer. What fee would it choose? If you were the competitor of this firm (i.e., you are the firm selling good $y$ what might you do to counteract this strategy by the firm?

3. Suppose that a consumer has $100 to spend during the week and likes to use some of this money to play tennis. She can buy time at the local court for $1 per hour, but the local court will only let her play for a maximum of 20 hours. If she wants to play for more than 20 hours, she has to pay a fee of $20 at the private tennis club. Then she can buy additional time for $2 per hour. Suppose that her preferences are represented by a Cobb-Douglas utility function $u(x, y) = x^\alpha y^{1-\alpha}$ where $x$ is the number of hours of tennis she plays and $y$ is the money she has left over for other stuff. Draw a picture to describe the budget line she faces. Label each of the important points in the diagram. Give a formula to show how much time she will spend playing tennis at the local court and how much time she will spend at the private court as a function of the number $\alpha$.

4. Our consumer has $20 per month to spend on her cell phone and comic books. Comic books cost $1 each. The phone company has offered her two different deals - for $6 per month she can have 100 ‘free’ minutes on the phone.
Each minute thereafter will cost her 10 cents. Or, if she wants, she can take the deluxe plan, $10 per month which entitles her to 150 free minutes. As a bonus, extra minutes are then charged at the lower rate of 5 cents. Draw the consumer’s budget set in this case. Suppose her preferences are given by $u(x, y) = x^\alpha y^{1-\alpha}$. Explain how her choice of plan and telephone use should depend on $\alpha$.

11.4 Best Reply Behavior

1. Alice and Bob both produce Caesar salad dressing in the same kitchen. Each of them has an endowment of one unit of labor that can be used as effort in producing the dressing, or can be used to play video games. Both Alice and Bob have identical preferences given by:

$$U(x, y) = \ln(x) + y,$$

where $x$ is the hours of TV watched and $y$ is the output of Caesar salad dressing.

If an amount of effort, $z_i$, is used in Caesar salad dressing production by person $i$, then the amount of dressing produced is given by

$$y_i = f_i(z_i) = S_i z_i,$$

for $i \in \{\text{alice, bob}\}$. Note that $S_i$ is the marginal product of production effort for person $i$. An hour of video game playing is produced one-for-one from the labor endowment. That is (by substituting in the resource constraint), $x_i = 1 - z_i$.

The thing is that making the dressing is hard when the other person is not in the kitchen very much (because it is time-consuming to look for the spoons etc, and it gets very boring and lonely). In particular, it turns out that

$$S_a = cz_b, \quad \text{and} \quad S_b = cz_a,$$

where $c > 2$.

Suppose that Alice guesses that Bob will spend $z_b^*$ hours in the kitchen. Write out Alice’s optimization problem.

2. What is her best response number of hours in the kitchen, $z_a(z_b^*)$? Graph this best response in $(z_a, z_b)$ space.

   (a) Do the same exercise for Bob, but place his best response on the same graph that you have just drawn. *Hint:* The two are identical, so the problem (and graph) should be very similar.

   (b) Graphically identify the Nash equilibrium levels of production effort, $(z_a^*, z_b^*)$. Explicitly solve for the solution when $c = 3$. *Hint:* You will need to use the quadratic formula.
3. Consider the following (extreme form of) consumption externality. Alice is going to Bob’s house for dinner, and has agreed to bring something to drink. After arriving at the store, she realizes that she has no idea what Bob is planning to cook. Neither she or Bob cares much what they eat. If Bob is cooking pizza, she (and Bob) would prefer that she bring beer. If Bob is cooking pasta, then she (and Bob) would prefer that she bring wine. Bob has a similar problem - what he wants to cook will depend on what he thinks Alice will bring. Which outcomes (choices of a food and a drink) seem reasonable? Does it seem possible to predict exactly what kind of drink Alice will buy?

### 11.5 Expected Utility

1. What are the two objects that define a lottery? Define the lottery associated with rolling a fair die.

2. How is a simple lottery different from a compound lottery? What is a reduced lottery?

3. Write the reduced lottery associated with the following (compound) lotteries.

   (a) A coin is tossed. If it comes up heads, you roll a fair die and get an amount of money equal to the number that turns up (e.g. if you roll a four, you get $4). If it comes up tails, you toss the coin six more times and get an amount of money equal to the number of tails that comes up in the six tosses (e.g. if four of the six were tails, you get $4).

   (b) A coin is tossed. If it comes up heads, you roll a fair die and get an amount of money equal to the number that turns up (e.g. if you roll a four, you get $4). If it comes up tails, you start the process over again.

   (c) A coin is tossed. If it comes up heads, you toss a second coin. If this second coin comes up heads, you get $10 today. If the second coin comes up tails, you get $10 in one years time. If the first coin instead comes up tails, you get $10 in two years time.

4. Consider the following compound lottery, with probability \( \frac{1}{4} \) you will play the lottery \( \{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \} \), with probability \( \frac{2}{5} \) you play the lottery \( \{ 0, 1, 0 \} \), while with probability \( \frac{3}{8} \) you play the lottery \( \{ 0, 0, 1 \} \) where each element in the list is the probability with which you receive payoffs \( \{ 1000, 500, 0 \} \) respectively. What is the reduced lottery associated with this compound lottery.

5. You tell your friend the Monty Hall problem (game show host revealing doors etc.). They are convinced that it is irrelevant whether or not you
switch doors based on the following argument. “No matter what door I choose, I will end up in a situation in which I am choosing between two doors, one of which will have the prize behind it. Since the prize is equally likely to be behind all doors, the chance that it is behind the door I chose is 1/2, which is the same as the chance of it being behind the other door. Therefore, it doesn’t make a difference if I change doors”. Explain where your friend is making an error in their reasoning.

6. Consider again the Monty Hall problem. At first glance, what did you think the probability of winning the prize was? What is the actual probability of winning the prize (assuming you switch doors as suggested)?

7. We know that it is always better to switch doors in the 3-door version of the Monty Hall problem. Will this remain true for n doors? If no, at what value of n does it stop being worthwhile to switch doors? If yes, does the benefit to switching increase or decrease as n increases?

8. Change the Monty Hall problem so that the prize is initially placed behind door A with probability 1/2 (instead of probability 1/3 as in the problem we discussed in class). The prize is placed behind doors B and C with equal probability 1/4. Suppose you choose door A. Monty then opens one of the other doors and shows you there is nothing behind it. You are offered the chance to switch doors. What is the probability that you will find the prize if you keep door A? What is the probability if you choose to switch to the other door? Do you think you could increase your chances of finding the prize by choosing a door other than door A to start with?

9. Consider the St. Petersberg Paradox. When considering how much to pay for this lottery, it was argued that you could not lose by paying $2 or less since you are guaranteed to win at least this amount. On the other hand, there is no limit to how much you could win. However, if you win after t rounds the amount that you win will be finite. That is, if you win in round t, you get 1/2^t which is a finite number if t is finite. That is, the most you can win is strictly less than infinity. However, it was shown that the expected value of the lottery (and therefore how much a risk-neutral individual would be willing to pay) is infinite. How could it be that someone is willing to pay an amount for a lottery that is greater than the highest possible amount they could win?

10. Suggest two reasons why no individual would be willing to pay an infinite amount for the lottery described in the St. Petersberg Paradox (besides things like no-one has that much money).

11. A simplifying assumption used throughout the chapter was that all the lotteries in \( \mathcal{L} \) had the same set of possible outcomes, \( \mathcal{X} \). In what way does this assumption simplify the analysis? Is the assumption restrictive? Why or why not?
12. State the independence axiom both formally and intuitively. Can you think of a scenario where this axiom is likely to fail?

13. The independence axiom is concerned with ‘mixing’ or ‘combining’ lotteries with a third lottery. Care must be taken in understanding exactly what it means to ‘mix’ two lotteries. The question is designed to highlight a common error.

Suppose there are two possible and equally-likely states of the world: I toss a coin, and it either comes up heads or comes up tails. Lottery 1 gives you $2 if it is heads, and $0 if it is tails. Lottery 2 is the reverse - you get $2 if it is tails, and $0 if it is heads.

14. These two lotteries are clearly different (by name if by nothing else). Are the differences meaningful in terms of the treatment of lotteries given in the text? The following questions help with this issue.

(a) What is the difference between the $X$ associated with these two lotteries?
(b) What is the difference between the $p$ associated with these two lotteries?
(c) Given your answers to the above two questions, what is the difference (if any) between the lotteries?

15. What are the three assumptions on the preference relation over $L$? Explain what each of these means. For each assumption think of a realistic situation in which the assumption would be inappropriate.

16. In the text, the set of feasible lotteries over three possible outcomes is drawn as a triangle in two dimensions. What would the set of feasible lotteries look like if there were only two outcomes? Can you imagine how to draw the set of feasible outcomes when there are 4 outcomes? How many dimensions are needed in order to draw the set of feasible lotteries when there are $k$ outcomes? Why?

17. Consider the following lotteries $q = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, $q' = \{\frac{1}{10}, \frac{3}{10}, \frac{3}{10}\}$ and $q'' = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\}$. Create a compound lottery over these three lotteries in which each of them is run with equal probability. What is the reduced lottery associated with this compound lottery?

18. A coin is flipped at most three times. If it comes up heads on the first try, the game ends and you win $2. If it comes up tails, it is flipped again. If it comes up heads on the second try, you win $2. If it comes up tails, it is flipped again. If it comes up heads on the third try, you lose $2. If it comes up tails, the game is over an no one wins anything. Draw a tree depicting the compound lottery over compound lotteries that is involved in this process. Compute the reduced lottery associated with it. What is the expected value of this lottery?
19. In the Monty Hall problem, use Bayes rule to compute the posterior probability that the prize is behind door A once you have chosen A and the show host opens door C. (Bayes rule - \( \Pr \{ A|C \} = \frac{\Pr \{ A \cap C \}}{\Pr \{ C \}} \).

In words, the probability of event A conditional on event C is the probability that both A and C occur, divided by the probability that event C occurs. Draw yourself a picture to see if you can understand why this should be so).

20. Suppose that the set of lotteries is equal to all lotteries over three outcomes, \( x_1, x_2 \) and \( x_3 \) as above, with \( x_1 \) being the best outcome, and \( x_3 \) the worst. What are the best and worst lotteries \( b \) and \( w \) used in the in the expected utility theorem in this case? What are the utility values assigned to these lotteries by the construction in the theorem?

21. Show that the utility function constructed in the expected utility theorem above actually represents preferences in the sense that if \( u(p) > u(p') \) then \( p \succ p' \) and conversely.

22. Draw the lotteries that Allais suggested in the last section above in a diagram and show that anyone who prefers \( q \) to \( q' \) must also prefer \( p \) to \( p' \) if they have expected utility preferences.

### 11.6 Risk Aversion

1. What is a probability distribution function?

2. Suppose that the lottery \( (p, X) \) can represented by a probability distribution function, \( F \).

   (a) Does this impose any restriction on the set of objects in the set \( X \)?

   For example, could \( X = \{ \text{red ball, green ball, blue ball} \} \)?

   (b) Suppose in addition to representing \( (p, X) \), \( F \) happens to be such that it has a density, in other words, it has a derivative. What additional properties does \( X \) have to possess for this to be true? For example, could \( X = \{-2, -1, 0, 1, 2\} \)?

3. If \( F \) does have a density (derivative) and that \( X = [0, 1] \). What does \( \int_0^1 F'(x')dx' \) equal?

4. Again, using the assumption that \( X = [0, 1] \), consider a degenerate lottery that gives \$2\ for sure. Write the \( F \) that characterizes this lottery.

5. Write the \( F \) that characterizes the following lottery. A coin is tossed, and you get \$1 if it comes up heads and -$1 if it comes up tails.

6. Intuitively explain what a ‘risk premium’ is. Explain how to calculate it.
7. What is a ‘fair bet’? A person is said to be risk averse if they have a positive risk premium when faced with a fair bet. Suppose that I tell you that a person associates a risk premium of -$5 with some lottery \( L \) (not necessarily a fair lottery). What can we say about the person’s attitude toward risk?

8. Is it possible that an individual has a different risk premium associated with different lotteries? Explain.

9. Is it possible that two individuals with the exact same preferences over all lotteries have a different risk premium when faced with identical lotteries? If not, explain why not. If so, what must be different across individuals?

10. Daron is unsure how much he will get through a student loan this year. There is a 50% chance that he will get $2000 and 50% chance that he will get $3000. He already has $1000 saved.

   (a) Express the lottery that Daron faces, \( L \), in terms of \((p, X)\). Now write it in terms of \((W, F)\).

   (b) What allows us to express his relative preference for the lottery \( L \) in the form:

   \[ U(L) = p_1 u(x_1) + p_2 u(x_2) \]  

   (11.6.1)

   (c) Suggest an interpretation of the function \( u(x) \). Using your answer to the first part, fill in what \((p_1, p_2, x_1, x_2)\) are in this case.

   (d) Write an expression for the risk premium associated with \( L \). Is this positive or negative? Does this make sense in terms of your intuitive understanding of what a risk premium represents?

   (e) Calculate the risk premium if we assumed that \( u(x) = \sqrt{x} \).

11. Explain carefully how the expected utility theorem is useful in the process of calculating a risk premium.

12. What is the Arrow-Pratt measure of risk aversion? What is the relationship between the risk premium and the Arrow-Pratt measure of risk aversion? What simplifications were made in deriving this relationship? Under what cases would the simplifications be inappropriate?

13. What is the ‘Portfolio Problem’? In what way does each choice of the \((i_s, i_r)\) pair generate a lottery? Write this lottery in terms of a probability distribution function, \( F \).

14. What does the Diversification Theorem say? Does it only apply to those investors that are not too risk averse? What was the intuitive argument based on? Describe an aspect of the real world that might cause individuals to behave in a manner that is seemingly inconsistent with the theorem.
15. The *Diversification Theorem* was derived without actually ever solving for the optimal portfolio. This question verifies the Theorem by solving for the optimal portfolio.

We are going to (implicitly) find the optimal investment in the risky asset, \( i_r \), under the assumption of a specific form of \( u(x) \).

16. Argue that the wealth constraint \( i_r + i_s \leq W \) can safely be assumed to hold with equality (given the existence of a risk-free asset).

   (a) Use this to write the objective function in the form:

   \[ \int u(W + i_r s) F'(s) ds \]  

   (11.6.2)

   (b) Show that if the agent is risk-neutral (i.e. \( u(x) = x \)), the objective function can be written in the form:

   \[ \alpha_1 + \alpha_2 \cdot i_r, \]  

   where \( \alpha_1 \) and \( \alpha_2 \) are constants that are specific to \( W \) and \( F \).

   (c) Remaining in the risk-neutral case, what condition on \( F \) will ensure that \( i_r \) is positive? Does the level of \( W \) matter (apart from \( W > 0 \))? Is this general, or a feature of risk-neutrality?

   (d) Now suppose that there is some risk-aversion. In particular, suppose that \( u(x) = x^\alpha \), where \( \alpha \in (0, 1) \). Suppose that the optimal portfolio had the property that \( i_r = 0 \). Use the FOC to derive the property of the lottery that must have induced this. Does this make sense?

17. Comparative static techniques were used to show that if an investor’s preferences are such that the Arrow-Pratt measure of (absolute) risk-aversion is decreasing in wealth, then the investor will invest more in the risky asset as their wealth increases.

   (a) The fact that an investor invests more as wealth increases might have nothing to do with risk aversion, but rather simply that they have more available wealth. Does the amount invested in the risky asset increase with wealth for investors that have an Arrow-Pratt measure of risk aversion that is *constant* over wealth levels?

   (b) What happens to the *proportion* of wealth allocated to the risky asset as wealth increases (for an investor with a decreasing Arrow-Pratt measure of risk aversion)? That is, calculate the comparative static \( \partial (i_r(w)/w)/\partial w \).

18. Suppose it was observed that wealthier individuals tended to invest more in risky assets. Mike claims that this is evidence of investors having preferences that are adequately described by a utility function that exhibits a decreasing level of risk aversion (as measured by the Arrow-Pratt measure). Eric disagrees. He claims that wealthier people invest more because they are less affected by transactions costs.
(a) For Eric’s claim to make sense, what kind of transactions costs must he be thinking of? For example, would his argument apply if one had to pay $t for every dollar invested? What about if it cost $t per transaction (regardless of size)?

(b) Describe how you go about determining who (Mike or Eric) had the better interpretation of the observation that wealthier individuals invest more in the risky asset.

19. Compute the Arrow Pratt Measure of Absolute Risk Aversion for the following utility functions

- $u(w) = \ln(w)$
- $u(w) = a + bw - cw^2$ (For what values of $b$ and $c$ is this concave? What happens to the Arrow Pratt Measure outside this region?)
- $u(w) = -w^{-\beta}; \beta > 0$
- $u(w) = -e^{-w}$

20. Evaluate

\[ \int u'(w + irs) sF'(s) ds \]

and

\[ \int u''(w + irs) sF''(s) ds \]

for each of the utility functions above assume that $s$ is distributed uniformly on the interval $[-1, 3]$ (which means that $F(s) = \frac{s + 1}{4}$).

21. A more interesting question than the one in the reading is to ask whether an individual invests a higher proportion of his or her wealth in the risky asset as their wealth rises. Use the method in the reading to prove that this is true provided the Arrow Pratt measure of relative risk aversion is decreasing in wealth where the Arrow Pratt measure of relative risk aversion is given by

\[ -\frac{u''(w)}{u'(w)} w \]

11.7 Insurance Theory

1. Write an expression for the profits of the insurance company in terms of $p$, $q$ and $b$, and derive the actuarially fair premium, $q^*$, as a function of $p$ and $b$.

(a) Draw the 'budget set' for the consumer when the firm charges $q^*$ per unit of net payout, $b$.

(b) Show how this set would change when the firm charges a premium higher than $q^*$. 129
(c) How does this increase in the premium affect the consumer’s optimal insurance policy choice?

2. Martha is considering holding an outdoor cooking show for which she will get ticket sales of $y$ (Martha has no other wealth - due to recent kind donations to the Expensive Lawyers Foundation). However, there is some chance that it will rain, in which case she has to partially refund the tickets to the show. In total, she will have to refund $d$. Martha is risk averse and has preferences represented by \( u(\cdot) \).

She is considering buying insurance against the possibility of rain. She discovers that all insurance firms agree that the chance of rain on the day of her show is \( p \), and as such are willing to offer any contract that will pay a net amount of \( b \) in the event of rain, in exchange for a premium of \( q \) such that their profits are zero.

On a recent holiday to Jailstown, Martha met a very clever meteorologist that specialized in rain predictions. Martha phones this guy to get his inside opinion on what the chance of rain is. He says that his extensive and top-secret analysis reveals that the chance of rain is \( p' \).

3. Based on the insurance firms’ belief that the probability of rain is \( p \), write out the zero-profit relationship between \( q \), \( b \), and \( p \).

   (a) Martha trusts the meteorologist, and believes that the probability of rain is not \( p \), but in fact \( p' \). Does Martha still buy full insurance? How does this depend on the advice, \( p' \)?

4. Let the utility for wealth function be given by:

\[
u(x) = (\alpha + x)^\beta,
\]

where \( \alpha, \beta \geq 0 \) are parameters.

   (a) For what values of \( \beta \) is \( u \) a concave function? Does this depend on \( \alpha \)?

   (b) Suppose that \( \beta < 1 \). For what values of \( \alpha \) is \( u'(x)_{x=0} = \infty \)?

   (c) Using the standard notation from the text, suppose that \( d = y \) (so that in the accident state the consumer has zero wealth). Use the first two parts to determine the parameter values such that a consumer with preferences that are given by \( u \) will buy some insurance regardless of \( y \) and \( q/B \) (as long as they are both positive). If you are not able to provide an intuitive answer, use Lagrangian techniques.

5. Ben loves hockey. He really enjoys it when his favorite team, the Calgary Flames, are playing well. He has an income of \( y \), and gets a utility of \( u(y) \) when the Flames win. However, when the team loses, he gets upset.

\footnote{The opinion is “inside” in the sense that the insurance firms do not know it.}
and says that he would give up $d$ dollars in order for the Flames to have played better and won (so that his utility when the team loses is $u(y - d)$). Ben is risk averse, so that $u''(\cdot) < 0$ (and $u'(\cdot) > 0$).

A betting agency offers the following deal: you can pay $q$, and if the Flames lose you get a net payout of $b$ (i.e. $b$ is in addition to getting your $q$ bet back). There is competition among betting agencies which implies that their profits are zero. Everyone (Ben, the betting agencies, and everyone else) evaluates the probability that the Flames will win at $1 - p$.

6. Write out the betting agency’s profit function, and write the zero-profit relationship between $q$ and $b$. How does $q/b$ change as the Flames become more likely to win (i.e. as $p$ decreases)?

(a) Does Ben find it worthwhile to take the bet? That is, solve Ben’s optimal choice of $b$ and $q$ (subject to the zero-profit condition) and determine whether $b^* > 0$.

(b) Does the answer to the last part surprise you? In particular, how would you reconcile the fact that Ben is betting on his favorite team losing? What about the fact that Ben is risk-averse?

(c) What would $p$ have to be in order for Ben to not make any bets ($b^* = 0$)?

7. In what ways is a standard health insurance policy really insurance in the sense introduced in the text? In what ways is it not?

8. Suppose that instead of there being just two states (accident or no accident), there are three states. There is a no-accident state ($S_1$), an insurable-accident state $S_2$, and a non-insurable accident state, $S_3$. These states occur with probability $p_1$, $p_2$, and $1 - p_1 - p_2$ respectively. The consumer loses $d_2$ in state 2, and loses $d_3$ in state 3.

An insurance firm offers a policy which pays net benefits of $b$ in the event that the state is $S_2$, and collects a premium of $q$ in both of the other two states. Using Lagrangian methods, describe how the consumer’s demand for insurance is affected by the introduction of the uninsurable-accident state.

11.8 First Welfare Theorem

1. The function that describes how a firm transforms $x$ into $y$ (and vice versa) was left quite general in the text. Name some sensible properties of $f$. For example, (in terms of the first figure) which quadrant(s) should the function never pass through? Why not?
2. Draw the \( f \) function associated with the following scenario. The two goods are bread \((x)\) and toast \((y)\). By exposing slices of bread to a fire, a firm is able to transform one slice of bread into one piece of toast. However, once turned to toast, the firm is not able to ‘un-expose’ the piece of toast so that it turns back into a slice of bread.

3. If there are \( \omega_x \) units of \( x \) before production, and \( z_x \leq \omega_x \) units after production, how many units of \( x \) were used as inputs in the firm’s production process? How many units of \( y \) are produced using this level of input? Therefore, how many units of \( y \) are there after production (that is, what must \( z_y \) be)?

4. The previous question demonstrated that we can determine \( z_y \) if we know \( z_x \) (and, of course, we must know the endowment, \((\omega_y, \omega_x)\), and production function, \( f \)). To be sure, explicitly write \( z_y \) as a function of \( z_x \). What is this function commonly known as?

5. Must the endowment \((\omega_x, \omega_y)\) lie on the production possibilities frontier? If not, what ‘unusual’ property must \( f \) possess at either \( z_x = \omega_x \) or at \( z_x : f(z_x - \omega_x) = 0 \)?

6. In reality, there are two main benefits to being a shareholder in a firm. First, you get a share of the firm’s profits. Second, you get to vote on how the firm operates.

   Under these maintained assumptions, we claimed that the second benefit is illusory because all shareholders would like the firm to maximize profits.

7. Explain the logic behind why all investors would vote to maximize profits.

   (a) What are the precise assumptions (there are at least three) that are creating this divergence between reality and the model?

   \textit{Hint:} Think of real-world motivations for individuals to hold voting rights in a firm, and then think about the assumptions used that would remove such motivations.

8. Carefully define a Walrasian equilibrium in a production economy. In particular, describe the objects (e.g. a consumption choice for each consumer) and the properties of these objects (e.g. the consumption plan is affordable). What is the main difference between a production economy and an exchange economy?

9. Briefly outline the argument explaining why equilibrium in an exchange economy is Pareto efficient (the first welfare theorem).

   Since a production economy is basically an exchange economy in which these things called ‘firms’ effectively determine the ‘endowments’ that are to be exchanged, what must be true if the first welfare theorem still holds in a production economy?
10. Prove that the first welfare theorem (Walrasian equilibrium is Pareto efficient) is still true in a production economy.

*Hint:* Try a proof by contradiction: derive a contradiction that follows from the supposition that there exists alternative consumption plans such that these new plans make at least one consumer better off without making any other worse off.

11. Can a consumer in a production economy be made worse off than she would have been if she had refused to trade and instead consumed her endowment? Can a consumer be made worse off in a production economy relative to the welfare she experiences in the corresponding exchange economy?

12. Consider some equilibrium in an exchange economy. Now, suppose that a firm arises and has the technology to transform the goods into each other according to a production function $f$.

(a) Is it possible that all consumers have a lower utility after the firm arises?

(b) Is it possible for some consumer to have a lower utility after the firm arises? Does this depend on whether the consumer owns shares in the firm?

(c) Suppose that some consumer is made worse off by the introduction of the firm, and that this consumer owns some shares in the firm. Surely, the consumer that is made worse off would not vote for the firm to pursue profit maximization (since this outcome makes him worse off by construction). In particular, he would surely rather try to convince the firm to produce something close to the exchange economy allocations (since he was better off in that equilibrium). What assumption makes this reasoning invalid?

(d) What do you find appealing about a Pareto efficient allocation? Why might a Pareto efficient allocation be undesirable?

11.9 Public Goods

1. Define a *public good*, and give three examples. Is noise pollution a public good?

2. Do public goods tend to be over- or under-provided? Explain, and describe one possible solution.

3. Which assumption underlying the first welfare theorem is violated when studying an economy with a public good?

4. Consider the voluntary contribution game. Describe how agent 2’s consumption of the private good affects agent 1’s utility (i.e. write $u_1(x_1, y)$
in the form $u_i(x_1, x_2, \cdot)$. Is agent 1’s utility increasing or decreasing in $x_2$? Explain.

5. Carefully define a Nash Equilibrium. How is it different from a Walrasian equilibrium?

6. There are two agents with identical utility functions given by

$$u_i(x_i, y) = x_i^\beta + y, \quad (11.9.1)$$

where $x_i$ is the level of consumption of the private good for agent $i$, $y$ is the consumption of the public good, and $\beta \in (0, 1)$. The public good is produced using the production function:

$$y = f \left( \frac{\text{1's contribution}}{\omega_1 - x_1} + \frac{\text{2's contribution}}{\omega_2 - x_2} \right) = (\omega_1 - x_1 + \omega_2 - x_2)^\beta, \quad (11.9.2)$$

where $\omega_i$ is agent $i$’s endowment of the private good, and $\beta \in (0, 1)$.

(a) Write agent 1’s utility in terms of $x_1$ and $x_2$ (and $\omega_1$ and $\omega_2$). In $(x_1, x_2)$ space, draw (roughly) a family of indifference curves for agent 1. What general shape do they have?

(b) Suppose that agent 1 thinks that agent 2 is going to consume $\bar{x}_2$ units of the private good. What is the optimal choice of $x_1$ (in terms of $\bar{x}_2$)?

i. How is this optimal choice affected by $\bar{x}_2$? Intuitively, what is going on?

ii. How is this optimal choice affected by $\omega_1$? $\omega_2$?

iii. Draw a graph in $x_1, x_2$ space that shows agent 1’s optimal choice of $x_1$ for any given level of $\bar{x}_2$.

(c) Suppose that $\omega_1 = \omega_2$ so that there is no quantitative difference between the two agents. Given this symmetry, what would you expect the relationship between the optimal $x_1$ and the optimal $x_2$ to be? Use this to calculate the Nash equilibrium private consumption pair $(x_1^*, x_2^*)$.

(d) Suppose that a planner suspected that the Nash equilibrium was inefficient. The planner would like to set the private consumption levels so that it maximizes the sum of the two agents’ utilities. Set up the planner’s problem, and solve for the socially optimal private consumption levels $(x_1^{**}, x_2^{**})$.

(e) Compare the socially optimal private consumption levels to the Nash equilibrium levels:

i. How does $\beta$ affect the Nash equilibrium private consumption levels? How does it affect the socially optimal levels?
ii. Is the Nash equilibrium outcome inefficient (i.e., not Pareto efficient)? Is there too much or too little private consumption? Intuitively explain this result.

iii. How is the inefficiency affected by $\beta$? That is, do the Nash equilibrium consumption levels get closer to the optimal levels when $\beta$ approaches zero, or when $\beta$ approaches one? Any intuition for why this is so?

(f) Suppose that instead of assuming that each agent has a personal endowment, $\omega_i$, we instead assumed that the pair were given an endowment of $\omega$ to 'share'. That is, each agent is free to consume from $\omega$ but must take into account that whatever was left over (after both had consumed) was to be used in the production of the public good. Would this assumption change any of the above results? Why or why not?

(g) Suppose that some outside body (like a government) wanted to help these agents out by donating $z$ units of the private good (as an endowment that can be allocated to the production of the public good as desired). Due to some kind of favoritism, the total of $z$ is split as follows: agent 1 is to get $\lambda z$ and agent 2 is to get $(1 - \lambda)z$, where $\lambda \in [0, 1]$.

i. Suppose that there was no public good at all. Would agent 1 care about what $\lambda$ was? Which would he most prefer?

ii. Now introduce the public good in question. Now does agent 1 care? Why is this so?

iii. Suggest a slight change to the model that would result in agent 1 caring about $\lambda$ in the presence of a public good.

7. What happens to the Nash equilibrium of the voluntary contribution game as the number of agents increases? In particular, does efficiency get improved or worsened?

8. What is a Lindahl price? Describe how such prices are used to resolve the public goods problem.

9. How do you think agent 1’s Lindahl price is affected by her endowment (keeping agent 2’s endowment fixed)? What is your reasoning?

To derive this relationship in a particular economy, consider the following scenario. There are two agents, each with Cobb-Douglas preferences:

$$U(x_i, y_i) = x_i^\alpha y_i^{1-\alpha},$$

(11.9.3)

for $i = 1, 2$. Each agent is endowed with $\omega_i$ units of the private good which is then given to the firm in exchange for shares which gives the agent the rights to a proportion, $\omega_i/(\omega_1 + \omega_2)$ of the firm’s profits.
The firm transforms the private good into the public good with the simple linear technology (recall that $y_1 = y_2 = y$):

$$y = \omega_1 + \omega_2 - x_1 - x_2. \quad (11.9.4)$$

Taking the price of the public good as the numeraire WHAT???, the firm’s profits are given by:

$$\pi = x_1 + x_2 + (p_1 + p_2)y. \quad (11.9.5)$$

10. Write the firm’s constrained maximization problem.

(a) Rather than using a Lagrangian approach, try substituting the constraint in directly. That is, write the firm’s profit function in terms of endowments, prices, and $(x_1, x_2)$.

(b) If a Walrasian equilibrium were to exist in this economy, one requirement is that the firm is choosing $(x_1, x_2)$ to maximize profits. Another is that $(x_1, x_2)$ must be feasible (i.e. $x_1 + x_2 \in [0, \omega_1 + \omega_2]$). The latter implies the sensible restriction that the optimal production of the private goods can not be infinity or negative infinity.

Using this information, and the profit function derived in (10a), argue that the Lindahl prices must satisfy the following:

$$p_1 + p_2 = 1. \quad (11.9.6)$$

What does this imply that equilibrium profits will be?

(c) Leaving the firm’s production problem for now, let’s look at the agents’ consumption problem. Let $M_i$ denote the income of agent $i$. Set up agent $i$’s constrained maximization problem, and derive the demand functions (for $x_i$ and $y_i$).

(d) Now that we have the demands for $y_1$ and $y_2$ in terms of incomes and their (Lindahl) prices, we are able to derive another property of the Lindahl prices. What is the equilibrium relationship between $y_1$ and $y_2$? What does this relationship imply about how $p_1$ and $p_2$ are related to $M_1$ and $M_2$? Use the fact that $M_i$ equals the proportion of firms profits that agent $i$ has rights over to show that

$$\frac{p_1}{p_2} = \frac{\omega_1}{\omega_2}. \quad (11.9.7)$$

(e) Use the two above restrictions on $p_1$ and $p_2$ to derive $p_1$ as a function of $\omega_1$ and $\omega_2$. How is $p_1$ affected by an increase in $\omega_1$? How does this compare with your initial reasoning?

11. How do you think the Lindahl price facing agent 1 is affected by agent 2’s relative desire for the public good (relative to her desire for the private good)? Use the following modified structure provided in Question 9 to derive this.
(a) Suppose that instead of $\alpha$ being constant in the utility functions, that agent $i$ has a preference parameter of $\alpha_i$. Re-derive the demand functions under this change.

(b) Exploit the equilibrium relationship between $y_1$ and $y_2$ to derive a relationship between $p_1/p_2$, and $(\alpha_1, \alpha_2, M_1, M_2)$.

(c) Use the fact that $M_i$ equals the proportion of firms profits that agent $i$ has rights over to show that

$$\frac{p_1}{p_2} = \frac{(1 - \alpha_1)\omega_1}{(1 - \alpha_2)\omega_2}. \quad (11.9.8)$$

(d) Has anything changed on the production side? Re-iterate the arguments that can be used to show that

$$p_1 + p_2 = 1. \quad (11.9.9)$$

(e) Use the above two conditions to derive $p_1$ as a function of $(\omega_1, \omega_2, \alpha_1, \alpha_2)$. How is $p_1$ affected by $\alpha_2$? How does this compare to your initial reasoning?

12. We can verify that the Lindahl prices produce optimal choices. Consider the framework introduced in Question 9. For ease of calculation, suppose that $\omega_1 = \omega_2$ (and retain the assumption that the preference parameter, $\alpha$, is the same across agents), so that the agents are completely symmetric.

(a) Set up the planner’s problem (do not solve yet). What does the planner maximize? Which variables does she choose? What are the constraints she faces?

(b) Rather than setting up the Lagrangian, try substituting the public good production function in directly so that the objective function is completely in terms of $x_1$ and $x_2$ (and endowments of course).

(c) Since the agent’s are identical in every way (e.g. in preferences and endowments), what would you expect the relationship between the socially optimal $x_1$ and $x_2$? Use this relationship to simplify the objective function.

(d) Solve for the optimal level of private consumptions, $x_1$ and $x_2$.

(e) Compare this with the $x_1$ and $x_2$ induced by the Lindahl prices. Are they the same? Does this suggest an alternative method by which to calculate the Lindahl prices?

(f) Does the fact that the levels of private consumption are efficient imply that the level of the public good is efficient? Either make an argument for or against this, or manually verify whether it is true.
13. Informally speaking, Lindahl prices are a means by which to distort agents’ incentives to contribute to the public good. The most obvious way to do this is to make the private good relatively more expensive. Consider again the framework in Question 9, and recall that \( p_1 \) is the (Lindahl) price of agent 1’s public good when the private good has price 1.

(a) If the problem is that agents do not contribute enough to the public good, then given the above argument, would you expect \( p_1 \) to be greater than or less than 1? What about \( p_2 \)? Was this the case?

If we let \( M_1 \) be agent 1’s income (his share of profits), then you should have calculated his demand for the private good to be

\[
x_1^* = \alpha M_1.
\]

This demand does not seem to be a function of \( p_1 \) - a feature of the Cobb-Douglas specification.

14. How do you reconcile the fact that the Lindahl prices induced the efficient demand for the private good when the demand function depends only on income, and not prices directly?

15. Name two problems in using Lindahl prices as a solution to the public goods problem. In particular, think about problems that are likely to arise when trying to implement the mechanism. Can you think of a better solution to the public goods problem?

### 11.10 Music Downloading

1. Consider an economy with 2 goods and two consumers. Good \( x \) is a private good. Each consumer owns exactly one unit of good \( x \). The other good is a public good which is produced using good \( x \) as an input. There is no endowment of the public good, however each unit of good \( x \) can be used to produce exactly one unit of the public good. The complication is that only consumer 1 is able to produce the public good - consumer 2 can produce nothing on his own. The consumers have identical quasi linear utility functions given by \( U(x, y) = \ln(y) + x \). Negative consumption of good \( x \) is okay (think of it as borrowing).

(a) What is the equilibrium of the voluntary contribution game.

(b) Starting from the equilibrium of the voluntary contribution game, suppose that consumer 2 (the non-producer) proposes to match any additional contributions that consumer 1 makes to production of the public good one for one, provided that consumer 1 agrees to use these matching grants to produce the public good. Show that as long as consumer 2 can limit consumer 1’s contributions to the public good, then this scheme can be used to make both consumers better off.
(c) What is the Lindahl equilibrium price for the public good. How much public good is produced in the Lindahl equilibrium.

(d) Interpret the Lindahl equilibrium as one in which a single price taking competitive firm is given the exclusive right to sell both public and private good. What happens if the firm instead acts as a monopolist? In particular, show that a monopoly firm (who maximizes its own profit) will produce less of the public good than the producer does in the voluntary contribution game.

11.11 Constrained Optimization

1. As described in class, let the utility function for good $x$ and $y$ be $y + \ln(x)$ (natural logarithm). Write out the Lagrangian function and first order conditions. Find the demand function. Find the income and substitution effects (in derivative form) of an increase in the price of good $x$, and $y$.

2. Solve for the Cobb Douglas demand functions for utility function $u(x, y) = x^\alpha y^\beta$ (in this case let $\alpha + \beta > 1$). Also suppose that the consumer has an endowment of $x_0$ units of good $x$ and $y_0$ units of good $y$ (so the consumer maximizes utility subject to the constraint that the value of chosen consumption is less than or equal to $px_0 + qy_0$. Find the income and substitution effects in derivative form.

3. For utility function $y + bx$ find the demand function when $p < b$ and $q = 1$. Can you find the Lagrangian multipliers for this case?